Week 4

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Mathematics I

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4-8 October, 2021

Operations with limits

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *be two sequences with* $s_n \to \ell$ *and* $q_n \to \ell'$, $\ell, \ell' \in \mathbb{R}$ *(finite numbers). Then:*

- $s_n + q_n \to \ell + \ell'$
- $s_n - q_n \to \ell - \ell'$
- $s_n \cdot q_n \to \ell \cdot \ell'$.
- *if* $\ell' \neq 0$, *then* $\frac{s_n}{q_n} \to \frac{\ell}{\ell'}$.

Moreover,

- *If* $s_n \to +\infty$ *and* $q_n \to +\infty$, *then* $s_n + q_n \to +\infty$ *and* $s_n \cdot q_n \to +\infty$.
- *If* $s_n \to -\infty$ *and* $q_n \to -\infty$, *then* $s_n + q_n \to -\infty$ *and* $s_n \cdot q_n \to +\infty$.
- *If* $s_n \to +\infty$ *and* $q_n \to -\infty$, *then* $s_n \cdot q_n \to -\infty$.

"Practical" rules

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *be two sequences. The following relations hold:*

- *If* $s_n \to +\infty$ *and* $q_n \to +\infty$ *then* $s_n \cdot q_n \to +\infty$
- *If* $s_n \to +\infty$ *and* $q_n \to +\infty$ *then* $s_n + q_n \to +\infty$
- *If* $s_n \to -\infty$ *and* $q_n \to -\infty$ *then* $s_n \cdot q_n \to +\infty$
- *If* $s_n \to -\infty$ *and* $q_n \to -\infty$ *then* $s_n + q_n \to -\infty$
- *If* $s_n \to +\infty$ *and* $q_n \to -\infty$ *then* $s_n \cdot q_n \to -\infty$
- *If* $s_n \to \ell$, $\ell \neq 0$ *and* $q_n \to +\infty$ *then* $s_n \cdot q_n \to \begin{cases} +\infty & \text{if } \ell > 0 \\ -\infty & \text{if } \ell < 0 \end{cases}$
- *If* $s_n \to \ell$, $\ell \neq 0$ *and* $q_n \to -\infty$ *then* $s_n \cdot q_n \to \begin{cases} -\infty & \text{if } \ell > 0 \\ +\infty & \text{if } \ell < 0 \end{cases}$

Undetermined forms

There are cases that are not covered by these rules.
These cases are called **Undetermined forms**

Definition

We call undetermined forms expressions of type

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty,$$
$$\infty^0, \quad 0^0, \quad 1^\infty, \quad \frac{0}{0}$$

The result of an undetermined form cannot be guesses a-priori, but can be computed by applying

- suitable operations
- notable limits

Undetermined forms

Compute the following limit

$$\lim_{n\to\infty}(n^2 - n)$$

Solution

We have an indeterminate form $+\infty - \infty$.

To find the limit, we put in evidence the highest power:

$$\lim_{n\to\infty} n^2 - n \quad = \quad \lim_{n\to\infty} \underbrace{n^2}_{\to+\infty} \overbrace{\left(1 - \frac{1}{n}\right)}^{\to 1} = +\infty$$

Undetermined forms

Compute the following limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$$

Solution

We have $\sqrt{n^2+n}\to+\infty$ so the limit is of the form $+\infty-\infty$ and is undetermined.

To solve it, we use the following trick:

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right) &= \lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}\\
=\lim_{n\to\infty}\tfrac{n^2+n-n^2}{\sqrt{n^2+n}+n} &= \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}\\
=\lim_{n\to\infty}\tfrac{n}{n\left(\sqrt{1+\frac{1}{n}}+1\right)} &= \lim_{n\to\infty}\frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}=\frac{1}{2}
\end{aligned}$$

Undetermined forms

Compute the following limit

$$\lim_{n\to\infty} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7}$$

Solution

Putting in evidence the highest power at the numerator and at the denominator:

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} &= \lim_{n\to\infty} \frac{n^7\left(3 - 8n^{-1} + 15n^{-4} - 10n^{-6}\right)}{n^7\left(12n^{-6} - 4\right)} \\
&= \lim_{n\to\infty} \frac{\left(3 - 8\left(\frac{1}{n}\right) + 15\left(\frac{1}{n}\right)^4 - 10\left(\frac{1}{n}\right)^6\right)}{\left(12\left(\frac{1}{n}\right)^6 - 4\right)}
\end{aligned}$$

The terms in $1/n$ tend to zero. Thus, we have:

$$\lim_{n\to\infty} \frac{3n^7 - 8n^6 + 15n^3 - 10n}{12n - 4n^7} = -\frac{3}{4}$$

Notable limits

Theorem

Let $a \in \mathbb{R}$, $a \neq 1$. Then:

$$\lim_{n \to \infty} a^n = \begin{cases} 0 & \textit{if } |a| < 1 \\ +\infty & \textit{if } a > 1 \\ \nexists & \textit{if } a \leq -1 \end{cases}$$

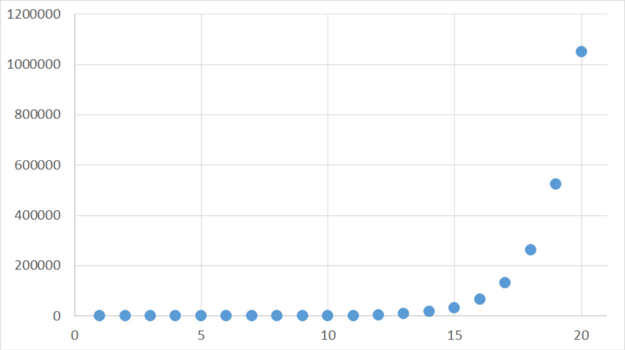
Notice that for the case $a = -1$ we obtain the well known limit

$$\lim_{n \to +\infty} (-1)^n$$

which does not exist!

Examples

Consider the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = 2^n$.



Then,

$$\lim_{n\to\infty} 2^n = +\infty$$

Exercise Prove the above limit using the definition.

Examples

Consider the sequences $(s_n)_{n\in\mathbb{N}}$ with $s_n = \left(\frac{1}{2}\right)^n$ and $(p_n)_{n\in\mathbb{N}}$ with $p_n = \left(-\frac{1}{2}\right)^n$.

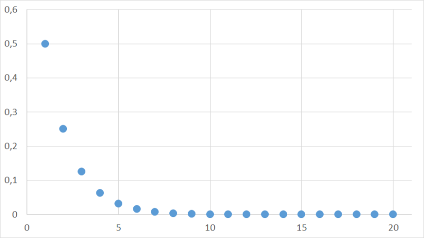


Figure: $s_n = \left(\frac{1}{2}\right)^n$

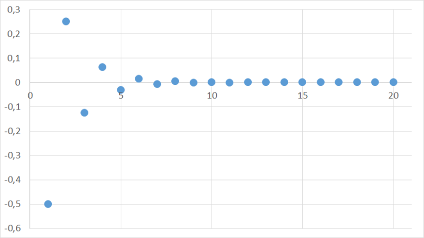


Figure: $p_n = \left(-\frac{1}{2}\right)^n$

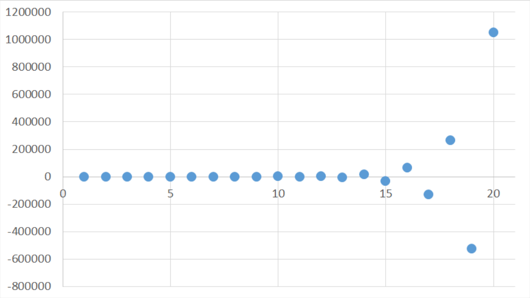
Then, we get that

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$$

Exercise Prove the first limit using the definition and the second limit using the Absolute Value Theorem.

Examples

Consider the sequences $(s_n)_{n\in\mathbb{N}}$ with $s_n = (-2)^n$.



n	s_n
1	-2
2	4
3	-8
4	16
5	-32
6	64
7	-128
8	256
9	-512
10	1024
11	-2048
12	4096
13	-8192
14	16384
15	-32768
16	65536
17	-131072
18	262144
19	-524288
20	1048576
21	-2097152

Then, we get that

$$\nexists \lim_{n\to\infty} (-2)^n$$

Exercise Prove that the above limit does not exist using the Theorem of Subsequences.

The factorial

Definition

Let $n \in \mathbb{N}$. The factorial is defined as:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

and $0! = 1$

Examples

- $1! = 1$
- $2! = 1 \cdot 2 = 2$
- $3! = 1 \cdot 2 \cdot 3 = 6$
- $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
- $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

Notable limits, cont'd

The following notable limits hold:

- $\forall b > 0,\ a > 1\ \lim_{n\to\infty} \frac{log_a(n)}{n^b} = 0$
- $\forall b > 0$ and $a > 1$, $\lim_{n\to\infty} \frac{n^b}{a^n} = 0$
- $\forall a > 1$, $\lim_{n\to\infty} \frac{a^n}{n!} = 0$
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Note that the above results imply:

- $\forall b > 0,\ a > 1\ \lim_{n\to\infty} \frac{n^b}{log_a(n)} = \lim_{n\to\infty} \frac{1}{\frac{log_a(n)}{n^b}} = +\infty$
- $\forall b > 0$ and $a > 1$, $\lim_{n\to\infty} \frac{a^n}{n^b} = \lim_{n\to\infty} \frac{1}{\frac{n^b}{a^n}} = +\infty$
- $\forall a > 1$, $\lim_{n\to\infty} \frac{n!}{a^n} = \lim_{n\to\infty} \frac{1}{\frac{a^n}{n!}} = +\infty$
- $\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{1}{\frac{n!}{n^n}} = +\infty$

Notable limits, cont'd

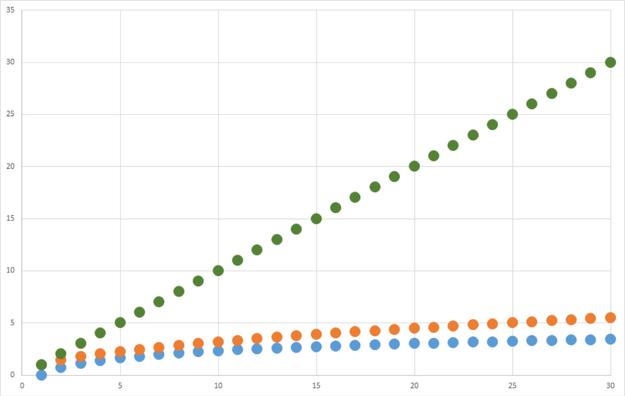
The above results can be interpreted in terms of "speed" of divergence:

- $\lim_{n\to\infty} \frac{log_a(n)}{n^b} = 0 \quad \Rightarrow$ The power diverges faster than the logarithm
- $\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \quad \Rightarrow$ The exponential diverges faster than the power
- $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad \Rightarrow$ The factorial diverges faster than the exponential
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0 \quad \Rightarrow$ n^n diverges faster than the factorial

At $+\infty$ we have the following "hierarchy of infinity"

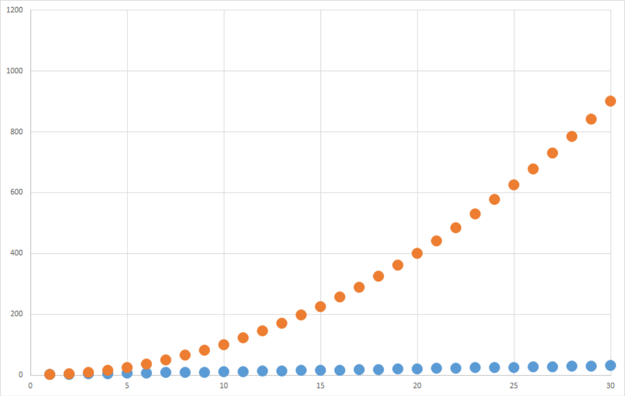
$$\log(n) < \sqrt[3]{n} < \sqrt{n} < n < n^2 < n^3 < \cdots < 2^n < e^n < 3^n < \cdots < n! < n^n$$

Examples



- $a_n = \log(n)$
- $b_n = \sqrt{n}$
- $c_n = n$

Examples



- $a_n = n$
- $b_n = n^2$

$\log(n)$	$\sqrt{n}$	n	n^2	2^n	$n!$
0	1	1	1	2	1
0,69	1,41	2	4	4	2
1,09	1,73	3	9	8	6
1,38	2	4	16	16	24
1,60	2,23	5	25	32	120
1,79	2,44	6	36	64	720
1,94	2,64	7	49	128	5040
2,07	2,82	8	64	256	40320
2,19	3	9	81	512	362880
2,30	3,16	10	100	1024	3628800
2,39	3,31	11	121	2048	39916800
2,48	3,46	12	144	4096	4,79E+08
2,56	3,60	13	169	8192	6,23E+09
2,63	3,74	14	196	16384	8,72E+10
2,70	3,87	15	225	32768	1,31E+12
2,77	4	16	256	65536	2,09E+13
2,83	4,12	17	289	131072	3,56E+14
2,89	4,24	18	324	262144	6,4E+15
2,94	4,35	19	361	524288	1,22E+17
2,99	4,47	20	400	1048576	2,43E+18

Examples, cont'd

Examples

$$\lim_{n\to\infty} \frac{3^n + \log_2(n)}{n!} = \lim_{n\to\infty} \frac{3^n}{n!} \left(1 + \overbrace{\frac{\log_2(n)}{3^n}}^{\to 0} \right) = 0$$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{10}}}{\log_2(n^{100})} = \lim_{n\to\infty} \frac{1}{100} \overbrace{\frac{n^{\frac{1}{10}}}{\log_2(n)}}^{\to+\infty} = +\infty$$

$$\lim_{n\to\infty} \frac{n! + n^{200000}}{n^n} = \frac{n!}{n^n} \left(1 + \overbrace{\frac{n^{200000}}{n!}}^{\to 0} \right) = 0$$

Put always in evidence the term diverging faster!

The notable limit: $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

Theorem

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$$

Proof The proof is divided in two parts:

- Part 1: we use the **key inequality** and the comparison theorem to prove show that $\lim\limits_{n\to\infty} n \sin\left(\dfrac{1}{n}\right) = 1$
- Part 2: We prove the **key inequality**

Part 1: the Comparison Theorem

Let $s_n = n \sin\left(\frac{1}{n}\right)$. By the key inequality we get that

$$\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$$

(See also the plot in the next slide.)

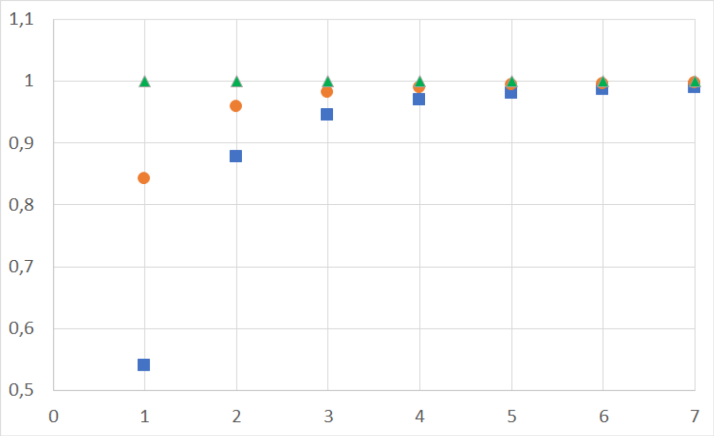
Now, let $a_n = \cos\left(\frac{1}{n}\right)$ and $b_n = 1$. Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

$$\lim_{n\to\infty} b_n = 1$$

Then by the Comparison Theorem, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$

Part 1: the Comparison Theorem

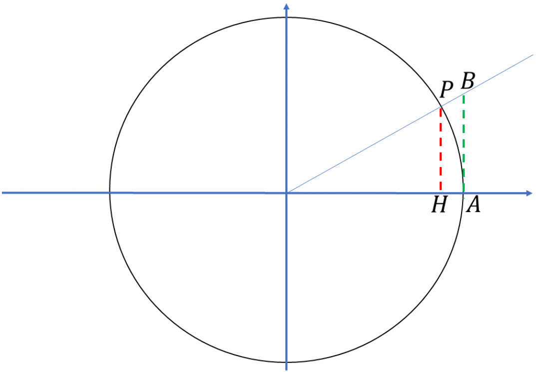


- Green sequence $b_n = 1$
- Orange sequence $s_n = n \sin\left(\frac{1}{n}\right)$
- Blue sequence $a_n = \cos\left(\frac{1}{n}\right)$

Part 2: The key inequality $\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$

For small angles θ, the length of the arc $\overset{\frown}{PA}$ is between the length of segments $\overline{PH}$ and $\overline{BA}$:

$$\overline{PH} \leq \overset{\frown}{PA} \leq \overline{BA}$$
$$\sin(\theta) \leq \theta \leq \tan(\theta)$$



Part 2: The key inequality $\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$

Dividing all members by $\sin(\theta)$ leads to

$$1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

Taking the reciprocals we obtain

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

Finally, we set $\theta = \frac{1}{n}$ and get

$$\cos\left(\frac{1}{n}\right) \leq n \sin\left(\frac{1}{n}\right) \leq 1$$

This concludes the proof.

Notable limits that follow from $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

- $\lim\limits_{n\to\infty} n \sin\left(\dfrac{1}{n}\right) = 1$
- $\lim\limits_{n\to\infty} n^2 \left(1 - \cos\left(\dfrac{1}{n}\right)\right) = \dfrac{1}{2}$
- If $s_n \to +\infty$, then $\lim\limits_{n\to\infty} s_n \sin\left(\dfrac{1}{s_n}\right) = 1$
- If $s_n \to +\infty$, then $\lim\limits_{n\to\infty} (s_n)^2 \left(1 - \cos\left(\dfrac{1}{s_n}\right)\right) = \dfrac{1}{2}$

Examples

$$\lim_{n\to\infty} n \sin\left(\frac{5}{n}\right)$$

Let $a_n = \frac{n}{5}$. Then $a_n \to +\infty$. Therefore we can multiply/divide by 5 the whole sequence and get

$$\lim_{n\to\infty} n \sin\left(\frac{5}{n}\right) = \lim_{n\to\infty} 5\frac{n}{5}\sin\left(\frac{1}{\frac{n}{5}}\right) = 5\cdot 1 = 5$$

Examples

$$\lim_{n\to\infty} n^2 \sin\left(\frac{1}{n^2+n}\right)$$

Let $a_n = n^2 + n$. Then $a_n \to +\infty$. Therefore we can multiply/divide by $n^2 + n$ the whole sequence and get

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n^2+n}\right) = \lim_{n\to\infty} \underbrace{\frac{n^2}{n^2+n}}_{1} \overbrace{(n^2+n)\sin\left(\frac{1}{n^2+n}\right)}^{1} = 1$$

Examples

$$\lim_{n\to\infty} n^3 \left(1 - \cos\left(\frac{2}{n^2}\right)\right)$$

Let $a_n = \frac{n^2}{2}$. Then $a_n \to +\infty$. Therefore we can multiply/divide by $(a_n)^2 = \frac{n^4}{4}$ the whole sequence and get

$$\lim_{n\to\infty} n^3 \left(1 - \cos\left(\frac{1}{\frac{n^2}{2}}\right)\right) = \lim_{n\to\infty} \underbrace{\frac{4n^3}{n^4}}_{0} \overbrace{\color{red}{\frac{n^4}{4}\left(1 - \cos\left(\frac{1}{\frac{n^2}{2}}\right)\right)}}^{\frac{1}{2}} = 0$$

Increasing/Decreasing sequences

Definition

A sequence $(s_n)_{n\in\mathbb{N}}$ is said to be

- **strictly increasing** if $s_n < s_{n+1}$ for all n.
- **increasing** if $s_n \leq s_{n+1}$ for all n.
- **strictly decreasing** if $s_n > s_{n+1}$ for all n.
- **decreasing** if $s_n \geq s_{n+1}$ for all n.

Theorem

(i) *Let $(s_n)_{n\in\mathbb{N}}$ be increasing. Then $s_n \to \ell$ if and only if $\exists A \in \mathbb{R}$ such that $s_n \leq A$, for all n.*

(ii) *Let $(s_n)_{n\in\mathbb{N}}$ be decreasing. Then $s_n \to \ell$ if and only if $\exists B \in \mathbb{R}$ such that $s_n \geq B$, for all n.*

Exercise

Exercise 1

The sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = n^3 + 3n$ is strictly increasing.

Solution: We will show that $s_{n+1} > s_n$ for all n

$$s_{n+1} = (n+1)^3 + 3(n+1) = \underbrace{n^3 + 3n}_{s_n} + 3n^2 + 1 + 3n + 3$$
$$= s_n + 3n^2 + 3n + 4 > s_n$$

Exercise 2

The sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = e^{-n+1}$ is strictly decreasing.

Solution: We will show that $s_{n+1} < s_n$ for all n

$$s_{n+1} = e^{-(n+1)+1} = \underbrace{e^{-n+1}}_{s_n} \cdot e^{-1}$$
$$= \frac{s_n}{e} < s_n$$

The Euler sequence

Definition

The sequence $(s_n)_{n\in\mathbb{N}}$ with

$$s_n = \left(1 + \frac{1}{n}\right)^n$$

is called the Euler sequence.

Theorem

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e,$$

where e is the Euler's (or Neper's) number.

The Euler sequence

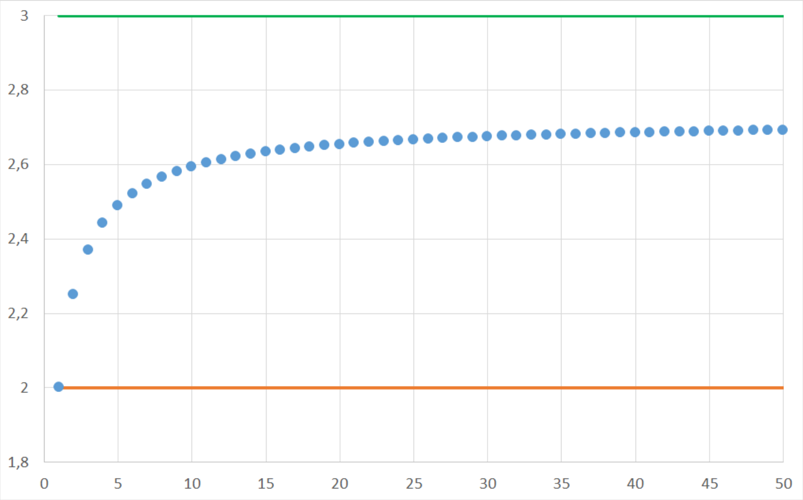


Figure: Euler Sequence (blue), boundary values: lower bound (orange) and upper bound (green)

Sequences: the Euler sequence

Intuitive proof

1. $s_1 = 2$
2. $s_n < s_{n+1}$ for all n, i.e. Euler sequence is (strictly) **increasing**
   (see the plot in the previous slide. For curiosity: a rigorous proof is given in the Lecture Notes).
3. $s_n < 3$ for all n, i.e. Euler sequence is bounded from **above**

By properties 2 and 3, the sequence is

$$\text{Increasing and bounded from above} \;\Rightarrow\; \text{There exists} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We call $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Moreover, since the sequence is strictly increasing and its first value is 2 (see property 1), then we get that $2 < e < 3$.

The number e is called the **Euler's number** or **Neper's number**.

Notable limits that derive from the Euler sequence

- $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{n \to \infty} n \log \left(1 + \frac{1}{n}\right) = 1$
- $\lim_{n \to \infty} n \left(e^{\frac{1}{n}} - 1\right)$
- If $s_n \to +\infty$, then $\lim_{n \to \infty} \left(1 + \frac{1}{s_n}\right)^{s_n} = e$
- If $s_n \to +\infty$, then $\lim_{n \to \infty} s_n \log \left(1 + \frac{1}{s_n}\right) = 1$
- $\lim_{n \to \infty} s_n \left(e^{\frac{1}{s_n}} - 1\right) = 1$

The Euler sequence

Exercise Compute

$$\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2}.$$

Solution

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2} &= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{7}}\right)^{n^2} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2}{7}}\right)^{\frac{n^2}{7}7} \\
&= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n^2}{7}}\right)^{\frac{n^2}{7}}}_{e}\right]^7 \\
&= e^7
\end{aligned}$$

The Euler sequence

Exercise Compute

$$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n.$$

Solution

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n &= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n-1}{2}}\right)^{n\cdot\frac{n-1}{2}\,\frac{2}{n-1}} \\
&= \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{1}{\frac{n-1}{2}}\right)^{\frac{n-1}{2}}}_{e}\right]^{\overbrace{\frac{2n}{n-1}}^{2}} \\
&= e^2
\end{aligned}$$

The Euler sequence

Exercise Compute

$$\lim_{n\to\infty}(n-1)\log\left(\frac{2n+7}{2n+1}\right).$$

Solution

$$\begin{aligned}
\lim_{n\to\infty}(n-1)\log\left(\frac{2n+7}{2n+1}\right) &= \lim_{n\to\infty}(n-1)\log\left(1+\frac{6}{2n+1}\right)\\
&= \lim_{n\to\infty}(n-1)\log\left(1+\frac{1}{\frac{2n+1}{6}}\right)\\
&= \lim_{n\to\infty}(n-1)\frac{2n+1}{6}\,\frac{6}{2n+1}\log\left(1+\frac{1}{\frac{2n+1}{6}}\right)\\
&= \lim_{n\to\infty}\overbrace{\frac{6(n-1)}{2n+1}}^{3}\,\underbrace{\frac{2n+1}{6}\log\left(1+\frac{1}{\frac{2n+1}{6}}\right)}_{1}\\
&= 3
\end{aligned}$$

The Geometric sum

Let $x \in \mathbb{R}$ and consider the following sum:

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

$S_n(x)$ can be re-written using the "summation" symbol:

$$S_n(x) = \sum_{k=0}^{n} x^k$$

The above sum is called **Geometric sum**.

Examples

- $S_{10}\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{10}$
- $S_5(3) = 1 + 3 + 3^2 + 3^3 + \cdots + 3^5$

Question: What is the result of this sum?

The Geometric sum

Theorem

For all $x \neq 1$

$$S_n(x) = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

For $x = 1$ *we get that* $S_n(1) = n + 1$

The Geometric sum

Proof If $x = 1$, then we get that $S_n(1) = \underbrace{1 + 1 + 1 + \cdots + 1}_{(n+1) \text{ times}} = n + 1$

Now, suppose that $x \neq 1$. Recall that:

$$S_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

and consider also the quantity $xS_n(x)$:

$$xS_n(x) = x + x^2 + x^3 + x^4 + \cdots + x^{n+1}$$

Now, take the difference $S_n(x) - xS_n(x)$:

$$\begin{aligned} xS_n(x) - xS_n(x) &= 1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots + \cancel{x^n} - \cancel{x} - \cancel{x^2} - \cancel{x^3} \cdots - \cancel{x^n} - x^{n+1} \\ &= 1 - x^{n+1} \end{aligned}$$

Therefore:

$$S_n(x)(1 - x) = 1 - x^{n+1} \;\Rightarrow\; S_n(x) = \frac{1 - x^{n+1}}{1 - x},$$

which concludes the proof.

The Geometric sum: examples

Examples

- Compute:

$$S_{10}\left(\frac{1}{2}\right) = \sum_{k=0}^{10}\left(\frac{1}{2}\right)^k$$

Solution In this case, $x = \frac{1}{2}$ and $n = 10$. Thus:

$$S_{10}\left(\frac{1}{2}\right) = \frac{1-\left(\frac{1}{2}\right)^{10+1}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{11}\right) \sim 1.9990234375$$

- Compute:

$$S_5(3) = \sum_{k=0}^{5} 3^k$$

Solution In this case, $x = 3$ and $n = 5$. Thus:

$$S_5(3) = \frac{1-3^{5+1}}{1-3} = \frac{1-3^6}{-2} = \frac{3^6-1}{2} = 364$$

The Geometric series

Definition

If $\lim_{n\to\infty} S_n(x)$ exists and it is finite we call

$$S(x) = \lim_{n\to\infty} S_n(x) = \sum_{k=0}^{\infty} x^k.$$

The quantity $S(x)$ is called the **Geometric series**.

Theorem

$$S(x) = \sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ +\infty & \text{if } x \geq 1 \\ \nexists & \text{if } x \leq -1 \end{cases}$$

The Geometric series

Proof

Using the result on the geometric sum we have:

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \lim_{n\to\infty} \frac{1-x^n x}{1-x}$$

To compute this limit we recall that $x \in \mathbb{R}$, $x \neq 1$,

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases}$$

Then plugging this result into the limit on the top line we get

$$\lim_{n\to\infty} \frac{1-x^n x}{1-x} = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases}$$

Finally, if $x = 1$, $S(1) = \lim_{n\to\infty} S_n(1) = \lim_{n\to\infty} n+1 = +\infty$
This completes the proof.

The Geometric series: examples

Examples

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

$$\sum_{k=0}^{\infty} 2^k = +\infty$$

$$\sum_{k=0}^{\infty} (-2)^k = \nexists$$