MATHEMATICS 1
ADDITIONAL EXERCISES N. 3

KATIA COLANERI

Notation: log stands for the natural logatithm (i.e. the logarithm with the basis e)

1. Inverse function, composite functions and plots

(1) For each of the following functions, say if they are invertible and if so, compute the inverse.

$$(a) \quad f:\mathbb{R}\to\mathbb{R} \quad f(x)=4-3x$$

This functions represents a line. It is strictly decreasing for all $x\in\mathbb{R}$ (You must prove that it is strictly decreasing!) and its range is $R_f=\mathbb{R}$, hence by the theorem on invertibility of monotonic functions it is invertible and its inverse is $f^{-1}:\mathbb{R}\to\mathbb{R}$, $f^{-1}(y)=\dfrac{4-y}{3}$

$$(b) \quad f:\mathbb{R}\to\mathbb{R} \quad f(x)=4-x^2$$

this function is not injective, hence not bijective. Therefore it is not invertible.

$$(c) \quad f:[0,+\infty)\to(-\infty,4] \quad f(x)=4-x^2$$

This functions represents a branch of a parabola. It is strictly decreasing for all $x\in[0,+\infty)$ (You must prove that it is strictly decreasing!) and its range is $R_f=(-\infty,4]$, hence by the theorem on invertibility of monotonic functions it is invertible and its inverse is $f^{-1}:(-\infty,4]\to[0,+\infty)$, $f^{-1}(y)=\sqrt{4-y}$

$$(d) \quad f:\mathbb{R}\setminus\{-1\}\to\mathbb{R}\setminus\{0\} \quad f(x)=\frac{1}{x+1}$$

This function is bijective (You must prove that it is bijective!), hence invertible. Its inverse is $f^{-1}:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{-1\}$, $f^-1(y)=\frac{1-y}{y}$.

Notice that this function is not monotonic on $\mathbb{R}\setminus\{-1\}$, hence the theorem on invertibility of monotonic functions cannot be applied.

$$(e) \quad f:\mathbb{R}\to[5,+\infty) \quad f(x)=x^2+5$$

This function is not injective, hence not bijective. Then it is not invertible.

Katia Colaneri, Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, 00133 Rome, Italy.

E-mail address: katia.colaneri@uniroma2.it.

$$(f) \quad f:(-\infty,4]\to[0,+\infty) \quad f(x)=\sqrt{8-2x}$$

This function is monotonic strictly decreasing (You must prove that it is strictly decreasing!) for all $x\in(-\infty,4]$ and its range is $R_f=[0,+\infty)$, hence by the theorem on invertibility of monotonic functions it is invertible. The inverse is $f^{-1}:[0,+\infty)\to(-\infty,4]$, $f^{-1}(y)=\frac{8-y^2}{2}$.

$$(g) \quad f:\mathbb{R}\to(0,+\infty) \quad f(x)=e^{2x+1}$$

This function is monotonic strictly increasing for all $x\in\mathbb{R}$ (You must prove that it is strictly increasing!) and its range is $R_f=(0,+\infty)$, hence by the theorem on invertibility of monotonic functions it is invertible. The inverse is $f^{-1}:(0,+\infty)\to\mathbb{R}$, $f^{-1}(y)=\frac{\log(y)-1}{2}$.

$$(h) \quad f:\mathbb{R}\setminus\{2\}\to\mathbb{R}\setminus\{1\} \quad f(x)=\frac{x+1}{x-2}$$

This function is bijective (You must prove that it is bijective!), hence invertible. Its inverse is $f^{-1}:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{2\}$, $f^-1(y)=\frac{1+2y}{y-1}$.

Notice that this function is not monotonic on $\mathbb{R}\setminus\{-1\}$, hence the theorem on invertibility of monotonic functions cannot be applied.

$$(i) \quad f:(3,+\infty)\to\mathbb{R} \quad f(x)=\log(x-3)$$

This function is strictly increasing for all $x\in(3,+\infty)$ (You must prove that it is strictly increasing!) and the range is $R_f=\mathbb{R}$. Hence by the theorem on invertibility of monotonic functions it is invertible. The inverse is $f^{-1}:\mathbb{R}\to(3,+\infty)$, $f^{-1}(y)=e^y+3$.

(2) Given the following plots of functions $g(x)$, draw, if possible:
- the inverse function
- $|g(x)|$ (red line)
- $g(x+2)$ (green line)
- $g(x)-3$ (black line)

(3) For each of the following pair of functions f and g, compute $f(g(x))$ and $g(f(x))$ and specify their domain and range

$$(a) \quad f:\mathbb{R}\to\mathbb{R} \quad f(x)=2x+1, \quad g:\mathbb{R}\to[0,+\infty) \quad g(x)=x^2$$

$$f(g(x))=2x^2+1, \quad f\circ g:\mathbb{R}\to[1,+\infty)$$
$$g(f(x))=(2x+1)^2, \quad g\circ f:\mathbb{R}\to[0,+\infty)$$

$$(b) \quad f:\mathbb{R}\to\mathbb{R} \quad f(x)=x^3, \quad g:\mathbb{R}\setminus 1\to\mathbb{R}\setminus\{0\} \quad g(x)=$$

$$f(g(x)) = \left(\frac{1}{x-1}\right)^3, \quad f \circ g : \mathbb{R} \setminus 1 \to \mathbb{R} \setminus \{0\}$$
$$g(f(x)) = \frac{1}{x^3-1}, \quad g \circ f : \mathbb{R} \setminus 1 \to \mathbb{R} \setminus \{0\}$$

$$(c) \quad f : \mathbb{R} \to (0, +\infty) \quad f(x) = e^x, \quad g : \mathbb{R} \to \mathbb{R} \quad g(x) = 3x + 5$$

$$f(g(x)) = e^{3x+5}, \quad f \circ g : \mathbb{R} \to (0, +\infty)$$
$$g(f(x)) = 3e^x + 5, \quad g \circ f : \mathbb{R} \to (5, +\infty)$$

$$(d) \quad f : (0, +\infty) \to \mathbb{R} \quad f(x) = \log(x), \quad g : \mathbb{R} \to [-1, +\infty) \quad g(x) = x^2 - 1$$

$$f(g(x)) = \log(x^2 - 1), \quad f \circ g : (-\infty, -1) \to (1, +\infty)$$
$$g(f(x)) = \log^2(x) - 1, \quad g \circ f : (0, +\infty) \to [-1, +\infty)$$

$$(e) \quad f : [-1, 1] \to [0, 1] \quad f(x) = \sqrt{1 - x^2}, \quad g : (-1, +\infty) \to \mathbb{R} \quad g(x) = \log(x + 1)$$

$$f(g(x)) = \sqrt{1 - \log^2(x + 1)}, \quad f \circ g : (e^{-1} - 1, e - 1) \to [0, 1]$$
$$g(f(x)) = \log(\sqrt{1 - x^2} + 1), \quad g \circ f : [-1, 1] \to [0, \log(2)]$$

2. Sequences

(1) Prove the following limits using the definition.

$$(a) \quad \lim_{n \to \infty} \frac{n + 5}{n^2} = 0$$

We will show that

$$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left| \frac{n + 5}{n^2} \right| < \varepsilon$$

To do this we consider the inequality $\left|\dfrac{n+5}{n^2}\right| < \varepsilon$ and we solve it with respect to n: Recall that $n \in \mathbb{N}$, and hence $n > 0$, then we have

$$\begin{aligned}
&\left|\frac{n+5}{n^2}\right| < \varepsilon \\
&\frac{n+5}{n^2} < \varepsilon \\
&-\varepsilon n^2 + n + 5 < 0 \\
&n > \frac{1+\sqrt{1+20\varepsilon}}{2\varepsilon}
\end{aligned}$$

Then we set $n^* = \left[\frac{1+\sqrt{1+20\varepsilon}}{2\varepsilon}\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = [104.7723] = 104$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{1+\sqrt{1+20\varepsilon}}{2\varepsilon}\right]$, then $\left|\dfrac{n+5}{n^2}\right| < \varepsilon$, and hence the definition is verified.

$$(b) \qquad \lim_{n\to\infty} \frac{2n+1}{n} = 2$$

We will show that

$$\forall \varepsilon > 0 \; \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|\frac{2n+1}{n} - 2\right| < \varepsilon$$

To do this we consider the inequality $\left|\dfrac{2n+1}{n} - 2\right| < \varepsilon$ and we solve it with respect to n: Recall that $n \in \mathbb{N}$, and hence $n > 0$, then we have

$$\begin{aligned}
&\left|\frac{2n+1}{n} - 2\right| < \varepsilon \\
&\left|\frac{1}{n}\right| < \varepsilon \\
&\frac{1}{n} < \varepsilon \\
&n > \frac{1}{\varepsilon}
\end{aligned}$$

Then we set $n^* = \left[\frac{1}{\varepsilon}\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = 100$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{1}{\varepsilon}\right]$, then $\left|\dfrac{2n+1}{n} - 2\right| < \varepsilon$, and hence the definition is verified.

$$(c) \qquad \lim_{n\to\infty} \frac{n^2+n+1}{n} = +\infty$$

We will show that

$$\forall M > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \frac{n^2+n+1}{n} > M$$

To do this we consider the inequality $\dfrac{n^2+n+1}{n} > M$ and we solve it with respect to n:

$$\begin{aligned}
&\frac{n^2+n+1}{n} > M \\
&n^2-(M-1)n+1 > 0 \\
&n > \frac{M-1+\sqrt{(M-1)^2-4}}{2}
\end{aligned}$$

Then we set $n^* = \left[\frac{M-1+\sqrt{(M-1)^2-4}}{2}\right]$. Notice that if M is large (for example $M = 1000$), then n^* is large (in case $M = 1000$, we get that $n^* = [998.999] = 998$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{M-1+\sqrt{(M-1)^2-4}}{2}\right]$, then $\dfrac{n^2+n+1}{n} > M$, and hence the definition is verified.

$$(d) \quad \lim_{n\to\infty} \sqrt{n} = +\infty$$

We will show that

$$\forall M > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \sqrt{n} > M$$

To do this we consider the inequality $\sqrt{n} > M$ and we solve it with respect to n:

$$\begin{aligned}
&\sqrt{n} > M \\
&n > M^2
\end{aligned}$$

Then we set $n^* = [M^2]$. Notice that if M is large (for example $M = 100$), then n^* is large (in case $M = 100$, we get that $n^* = 10000$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = [M^2]$, then $\sqrt{n} > M$, and hence the definition is verified.

$$(e) \quad \lim_{n\to\infty} \frac{1-n^2}{n^2} = -1$$

We will show that

$$\forall \varepsilon > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|\frac{1-n^2}{n^2}+1\right| < \varepsilon$$

To do this we consider the inequality $\left|\dfrac{1-n^2}{n^2}+1\right| < \varepsilon$ and we solve it with respect to n: Recall that $n \in \mathbb{N}$, and hence $n > 0$, then we have

$$\begin{aligned} &\left|\frac{1-n^2}{n^2}+1\right| < \varepsilon \\ &\left|\frac{1}{n^2}\right| < \varepsilon \\ &\frac{1}{n^2} < \varepsilon \\ &n > \frac{1}{\sqrt{\varepsilon}} \end{aligned}$$

Then we set $n^* = \left[\frac{1}{\sqrt{\varepsilon}}\right]$. Notice that if ε is small (for example $\varepsilon = 0.0001$), then n^* is large (in case $\varepsilon = 0.0001$, we get that $n^* = 100$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{1}{\sqrt{\varepsilon}}\right]$, then $\left|\dfrac{1-n^2}{n^2}+1\right| < \varepsilon$, and hence the definition is verified.

$$(f) \qquad \lim_{n\to\infty} 1 - n^3 = -\infty$$

We will show that

$$\forall M > 0 \; \exists n^* \in \mathbb{N} : n > n^* \Rightarrow |1-n^3| > M$$

To do this we consider the inequality $|1-n^3| > M$ and we solve it with respect to n: Notice that $n \geq 1$, then $1 - n^3 \leq 0$ and hence $|1-n^3| = n^3 - 1$

$$\begin{aligned} &|1-n^3| > M \\ &n^3 - 1 > M \\ &n > \sqrt[3]{M+1} \end{aligned}$$

Then we set $n^* = \left[\sqrt[3]{M+1}\right]$. Notice that if M is large (for example $M = 10000$), then n^* is large (in case $M = 10000$, we get that $n^* = 21$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\sqrt[3]{M+1}\right]$, then $|1-n^3| > M$, and hence the definition is verified.

$$(g) \qquad \lim_{n\to\infty} \frac{5}{n+3} = 0$$

We will show that

$$\forall \varepsilon > 0 \; \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|\frac{5}{n+3}\right| < \varepsilon$$

To do this we consider the inequality $\left|\frac{5}{n+3}\right| < \varepsilon$ and we solve it with respect to n: Recall that $n \in \mathbb{N}$, and hence $n > 0$, then we have

$$\begin{aligned} \left|\frac{5}{n+3}\right| &< \varepsilon \\ \frac{5}{n+3} &< \varepsilon \\ n &> \frac{5}{\varepsilon} - 3 \end{aligned}$$

Then we set $n^* = \left[\frac{5}{\varepsilon} - 3\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = 497$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{5}{\varepsilon} - 3\right]$, then $\left|\frac{5}{n+3}\right| < \varepsilon$, and hence the definition is verified.

$$(h) \qquad \lim_{n\to\infty} \frac{n^2+2n}{3n^2+1} = \frac{1}{3}$$

We will show that

$$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|\frac{n^2+2n}{3n^2+1} - \frac{1}{3}\right| < \varepsilon$$

To do this we consider the inequality $\left|\frac{n^2+2n}{3n^2+1} - \frac{1}{3}\right| < \varepsilon$ and we solve it with respect to n: Recall that $n \in \mathbb{N}$, and hence $n > 0$, then we have

$$\begin{aligned} \left|\frac{n^2+2n}{3n^2+1} - \frac{1}{3}\right| &< \varepsilon \\ \left|\frac{6n-1}{3(3n^2+1)}\right| &< \varepsilon \\ \frac{6n-1}{3(3n^2+1)} &< \varepsilon \\ n &> \frac{3+\sqrt{9-9\varepsilon(1+3\varepsilon)}}{9\varepsilon} \end{aligned}$$

Then we set $n^* = \left[\frac{3+\sqrt{9-9\varepsilon(1+3\varepsilon)}}{9\varepsilon}\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = 66$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{3+\sqrt{9-9\varepsilon(1+3\varepsilon)}}{9\varepsilon}\right]$, then $\left|\frac{n^2+2n}{3n^2+1} - \frac{1}{3}\right| < \varepsilon$, and hence the definition is verified.

$$(i) \qquad \lim_{n\to\infty} e^n = +\infty$$

We will show that

$$\forall M > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow e^n > M$$

To do this we consider the inequality $e^n > M$ and we solve it with respect to n:

$$\begin{aligned} & e^n > M \\ & n > \log(M) \end{aligned}$$

Then we set $n^* = [\log(M)]$. Notice that if M is large (for example $M = 100000$), then n^* is large (in case $M = 100000$, we get that $n^* = 11$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = [\log(M)]$, then $e^n > M$, and hence the definition is verified.

$$(j) \quad \lim_{n\to\infty} e^{-n} = 0$$

We will show that

$$\forall \varepsilon > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|e^{-n}\right| < \varepsilon$$

To do this we consider the inequality $\left|e^{-n}\right| < \varepsilon$ and we solve it with respect to n:

$$\begin{aligned} & \left|e^{-n}\right| < \varepsilon \\ & e^{-n} < \varepsilon \\ & -n < \log(\varepsilon) \\ & n > \log\left(\frac{1}{\varepsilon}\right) \end{aligned}$$

Then we set $n^* = \left[\log\left(\frac{1}{\varepsilon}\right)\right]$. Notice that if ε is small (for example $\varepsilon = 0.00001$), then n^* is large (in case $\varepsilon = 0.00001$, we get that $n^* = 11$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\log\left(\frac{1}{\varepsilon}\right)\right]$, then $\left|e^{-n}\right| < \varepsilon$, and hence the definition is verified.

$$(k) \quad \lim_{n\to\infty} \log(n+1) = +\infty$$

We will show that

$$\forall M > 0\ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \log(n+1) > M$$

To do this we consider the inequality $\log(n+1) > M$ and we solve it with respect to n:

$$\begin{aligned} & \log(n+1) > M \\ & n+1 > e^M \\ & n > e^M - 1 \end{aligned}$$

Then we set $n^* = \left[e^M - 1\right]$. Notice that if M is large, then n^* is large.

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[e^M - 1\right]$, then $\log(n+1) > M$, and hence the definition is verified.

$$(l) \qquad \lim_{n\to\infty} \log\left(\frac{n}{n+1}\right) = 0$$

We will show that

$$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|\log\left(\frac{n}{n+1}\right)\right| < \varepsilon$$

To do this we consider the inequality $\left|\log\left(\frac{n}{n+1}\right)\right| < \varepsilon$ and we solve it with respect to n: notice first that $\frac{n}{n+1} < 1$, and hence $\log\left(\frac{n}{n+1}\right) < 0$, therefore

$$\begin{aligned}
&\left|\log\left(\frac{n}{n+1}\right)\right| < \varepsilon \\
&-\log\left(\frac{n}{n+1}\right) < \varepsilon \\
&\log\left(\frac{n}{n+1}\right) > -\varepsilon \\
&\frac{n}{n+1} > e^{-\varepsilon} \\
&n > \frac{e^{-\varepsilon}}{1 - e^{-\varepsilon}}
\end{aligned}$$

Then we set $n^* = \left[\frac{e^{-\varepsilon}}{1-e^{-\varepsilon}}\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = 99$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{e^{-\varepsilon}}{1-e^{-\varepsilon}}\right]$, then $\left|\log\left(\frac{n}{n+1}\right)\right| < \varepsilon$, and hence the definition is verified.

$$(m) \qquad \lim_{n\to\infty} e^{\frac{n}{n+1}} = e$$

We will show that

$$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} : n > n^* \Rightarrow \left|e^{\frac{n}{n+1}} - e\right| < \varepsilon$$

To do this we consider the inequality $\left|e^{\frac{n}{n+1}} - e\right| < \varepsilon$ and we solve it with respect to n: notice first that $\frac{n}{n+1} < 1$, and hence $e^{\frac{n}{n+1}} - e < 0$, therefore

$$\begin{aligned}
&\left|e^{\frac{n}{n+1}} - e\right| < \varepsilon \\
&e - e^{\frac{n}{n+1}} < \varepsilon \\
&e^{\frac{n}{n+1}} > e - \varepsilon \\
&\frac{n}{n+1} > \log(e-\varepsilon) \\
&n > \frac{\log(e-\varepsilon)}{1-\log(e-\varepsilon)}
\end{aligned}$$

Then we set $n^* = \left[\frac{\log(e-\varepsilon)}{1-\log(e-\varepsilon)}\right]$. Notice that if ε is small (for example $\varepsilon = 0.01$), then n^* is large (in case $\varepsilon = 0.01$, we get that $n^* = 270$).

By reading bottom-up the chain of inequalities above we get that for all $n > n^* = \left[\frac{\log(e-\varepsilon)}{1-\log(e-\varepsilon)}\right]$, then $\left|e^{\frac{n}{n+1}} - e\right| < \varepsilon$, and hence the definition is verified.

(2) Show that the following limits do not exist

$$(a) \quad \lim_{n\to\infty} \cos(n\pi)$$

Let $s_n = \cos(n\pi)$, and consider the even subsequence $s_{2n} = \cos(2n\pi) = 1$ for all n and the odd subsequence $s_{2n+1} = \cos((2n+1)\pi) = -1$ for all n. Since $\lim_{n\to\infty} s_{2n} = 1$ and $\lim_{n\to\infty} s_{2n+1} = -1$ (that is they are different), by the theorem on subsequences we get that $\lim_{n\to\infty} \cos(n\pi)$ does not exist.

$$(b) \quad \lim_{n\to\infty} \frac{(-1)^n n}{n+1}$$

Let $s_n = \frac{(-1)^n n}{n+1}$, and consider the even subsequence $s_{2n} = \frac{2n}{2n+1}$ for all n and the odd subsequence $s_{2n+1} = \frac{-2n-1}{2n+2}$ for all n. Since $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} \frac{2n}{2n+1} = 1$ and $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{-2n-1}{2n+2} = -1$ (that is they are different), by the theorem on subsequences we get that $\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$ does not exist.

$$(c) \quad \lim_{n\to\infty} n^{(-1)^n}$$

Let $s_n = n^{(-1)^n}$, and consider the even subsequence $s_{2n} = (2n)^{(-1)^{2n}} = 2n$ for all n and the odd subsequence $s_{2n+1} = (2n+1)^{(-1)^{2n+1n}} = (\frac{1}{2n+1})$ for all n. Since $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} 2n = +\infty$ and $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$ (that is they are different), by the theorem on subsequences we get that $\lim_{n\to\infty} n^{(-1)^n}$ does not exist.

$$(d) \quad \lim_{n\to\infty} (-2)^n$$

Let $s_n = (-2)^n$, and consider the even subsequence $s_{2n} = (-2)^{2n} = 2^{2n}$ for all n and the odd subsequence $s_{2n+1} = (-2)^{2n+1} = -2 \cdot 2^{2n}$ for all n. Since $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} 2^{2n} = +\infty$ and $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} -2\cdot 2^{2n} = -\infty$ (that is they are different), by the theorem on subsequences we get that $\lim_{n\to\infty}(-2)^n$ does not exist.

$$(e) \quad \lim_{n\to\infty} (-n)^n$$

Let $s_n = (-2)^n$, and consider the even subsequence $s_{2n} = (-2n)^{2n} = (2n)^{2n}$ for all n and the odd subsequence $s_{2n+1} = (-2n-1)^{2n+1} = (-2n-1) \cdot (2n+1)^{2n}$ for all n. Since $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty}(2n)^{2n} = +\infty$ and $\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(-2n-1)\cdot(2n+1)^{2n} = -\infty$ (that is they are different), by the theorem on subsequences we get that $\lim_{n\to\infty}(-n)^n$ does not exist.

(3) Compute the following limits using the Absolute value Theorem or the Comparison Theorem.

$$(a) \quad \lim_{n\to\infty} \frac{\cos(n)}{n^2}$$

Let $s_n = \frac{\cos(n)}{n^2}$ and observe that

$$-1 \leq \cos(n) \leq 1$$
$$\frac{-1}{n^2} \leq \frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$$

Let $a_n = \frac{-1}{n^2}$ and $b_n = \frac{1}{n^2}$. Since $\lim_{n\to\infty} \frac{-1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, by the Comparison Theorem we also get that $\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0$.

$$(b) \quad \lim_{n\to\infty} \frac{(-1)^n n + 1}{1 - n^2}$$

Let $s_n = \frac{(-1)^n n+1}{1-n^2}$ and observe that

$$\frac{-n+1}{1-n^2} \leq \frac{(-1)^n n + 1}{1-n^2} \leq \frac{n+1}{1-n^2}$$

Let $a_n = \frac{-n+1}{1-n^2}$ and $b_n = \frac{n+1}{1-n^2}$. Since $\lim_{n\to\infty} \frac{-n+1}{1-n^2} = 0$ and $\lim_{n\to\infty} \frac{n+1}{1-n^2} = 0$, by the Comparison Theorem we also get that $\lim_{n\to\infty} \frac{(-1)^n n+1}{1-n^2} = 0$.

$$(c) \quad \lim_{n\to\infty} \frac{3 + \sin(n)}{n+4}$$

Let $s_n = \frac{3+\sin(n)}{n+4}$ and observe that

$$-1 \leq \sin(n) \leq 1$$
$$\frac{3-1}{n+4} \leq \frac{3+\sin(n)}{n+4} \leq \frac{3+1}{n+4}$$

Let $a_n = \frac{2}{n+4}$ and $b_n = \frac{4}{n+4}$. Since $\lim_{n\to\infty} \frac{2}{n+4} = 0$ and $\lim_{n\to\infty} \frac{4}{n+4} = 0$, by the Comparison Theorem we also get that $\lim_{n\to\infty} \frac{3+\sin(n)}{n+4} = 0$.

$$(d) \quad \lim_{n\to\infty} \frac{3}{n\cos(n\pi)}$$

Let $s_n = \frac{3}{n\cos(n\pi)}$ and observe that $|s_n| = \left|\frac{3}{n\cos(n\pi)}\right| = \frac{3}{n}$. Since $\lim_{n\to\infty} |s_n| = \lim_{n\to\infty} \frac{3}{n} = 0$, by the Absolute Value Theorem we get that $\lim_{n\to\infty} \frac{3}{n\cos(n\pi)}$

$$(e) \quad \lim_{n\to\infty} \frac{3}{n\cos(n)+2n}$$

Let $s_n = \frac{3}{n\cos(n)+2n}$ and observe that

$$-1 \leq \cos(n) \leq 1$$
$$\frac{3}{n+2n} \leq \frac{3}{n\cos(n)+2n} \leq \frac{3}{n}$$

Let $a_n = \frac{1}{n}$ and $b_n = \frac{3}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{3}{n} = 0$, by the Comparison Theorem we also get that $\lim_{n\to\infty} \frac{3}{n\cos(n)+2n} = 0$.

$$(f) \quad \lim_{n\to\infty} \frac{n+n(-1)^n}{n^2}$$

Let $s_n = \frac{n+n(-1)^n}{n^2}$ and observe that

$$0 \leq \frac{n+n(-1)^n}{n^2} \leq \frac{2n}{n^2}$$

Let $a_n = 0$ and $b_n = \frac{2}{n}$. Since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \frac{2}{n} = 0$, by the Comparison Theorem we also get that $\lim_{n\to\infty} \frac{n+n(-1)^n}{n^2} = 0$.

$$(g) \quad \lim_{n\to\infty} \frac{2n\cos(n\pi)}{n+n^2}$$

Let $s_n = \frac{2n\cos(n\pi)}{n+n^2}$ and observe that $|s_n| = \left|\frac{2n\cos(n\pi)}{n+n^2}\right| = \frac{2n}{n(n+1)} = \frac{2}{n+1}$. Since $\lim_{n\to\infty} |s_n| = \lim_{n\to\infty} \frac{2}{n+1} = 0$, by the Absolute Value Theorem we get that $\lim_{n\to\infty} \frac{2n\cos(n\pi)}{n+n^2} = 0$