

Mathematics I - Practice 4

1

1. Use the definition of limit to prove the following statements

DEF. s_n converges $\lim_{n \rightarrow +\infty} s_n = l \quad l \in \mathbb{R} \quad l$ is finite
if: $\forall \varepsilon > 0 \quad \exists n^* \in \mathbb{N} : \forall n > n^* \Rightarrow |s_n - l| < \varepsilon$
↑ "For every" ↑ "Exists" ↑ "Such that"

a) $\lim_{n \rightarrow +\infty} \frac{3}{n} = 0$

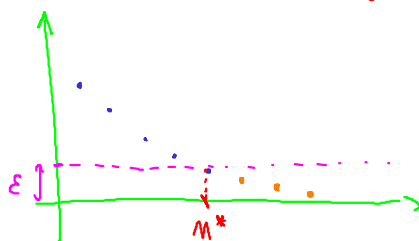
TASK \rightarrow Find n^* given the ε

$$\left| \frac{3}{n} - 0 \right| < \varepsilon$$

$$\left| \frac{3}{n} \right| < \varepsilon$$

$$\frac{3}{n} < \varepsilon \Rightarrow n > \frac{3}{\varepsilon}$$

$$n^* = \left\lceil \frac{3}{\varepsilon} \right\rceil$$



$$\frac{3}{\varepsilon} = 7.235$$
$$\left\lceil \frac{3}{\varepsilon} \right\rceil = 7$$

If I choose $n > n^* \Rightarrow |s_n - l| < \varepsilon$

$$3 < \varepsilon n$$
$$\frac{3}{\varepsilon} < n$$

b) $\lim_{n \rightarrow +\infty} (1 - n^3) = -\infty$

TASK Find n^* given M

$$1 - n^3 < -M$$

$$-n^3 < -M - 1$$

$$n^3 > M + 1$$

$$n > \sqrt[3]{M+1} \Rightarrow n^* = \left\lceil \sqrt[3]{M+1} \right\rceil \Rightarrow n > n^* \quad s_n < -M \quad |s_n| > M$$

DEF. s_n diverges

$\lim_{n \rightarrow +\infty} s_n = \pm \infty$ if.

$$\forall M > 0 \quad \exists n^* \in \mathbb{N} : \forall n > n^*$$

$$\Rightarrow |s_n| > M$$

$+\infty$

$$s_n > M$$

$-\infty$

$$s_n < -M$$

Quick example $M = 10^8$ $n^* \approx 461$

$n = 462$
($n > n^*$)

$S_{n=462} = 1 - (462)^3 = -9.8 \cdot 10^8$

$S_n < -M$

2. Prove that the following limits do not exist

> Theorem of sub-sequences

$n^2 \rightarrow +\infty$	even $n = 2k$	odd $n = 2k+1$
	$(2k)^2 \rightarrow +\infty$	$(2k+1)^2 \rightarrow +\infty$

I am walking in the park \Rightarrow it is not raining
A \Rightarrow B

" S_n has the limit l " \Rightarrow "Any subsequence extracted from S_n has the same limit l "

It is raining \Rightarrow I am not walking in the park
~~B~~ \Rightarrow ~~A~~

I find two subsequences that have two different limits
" S_n does not have a limit.
The limit of S_n does not exist"

a) $\lim_{n \rightarrow \infty} \frac{n}{\cos(\pi n)}$

EVEN n $n = 2k$ $k \in \mathbb{N}$

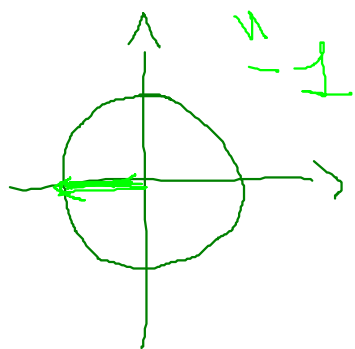
$\lim_{k \rightarrow +\infty} \frac{2k}{\cos(2k\pi)} = \lim_{k \rightarrow +\infty} 2k = \underline{\underline{+\infty}}$

$$|2u| > M \quad 2u > M \quad u > M/2 \quad u^* = M/2$$

$$2u \rightarrow +\infty$$

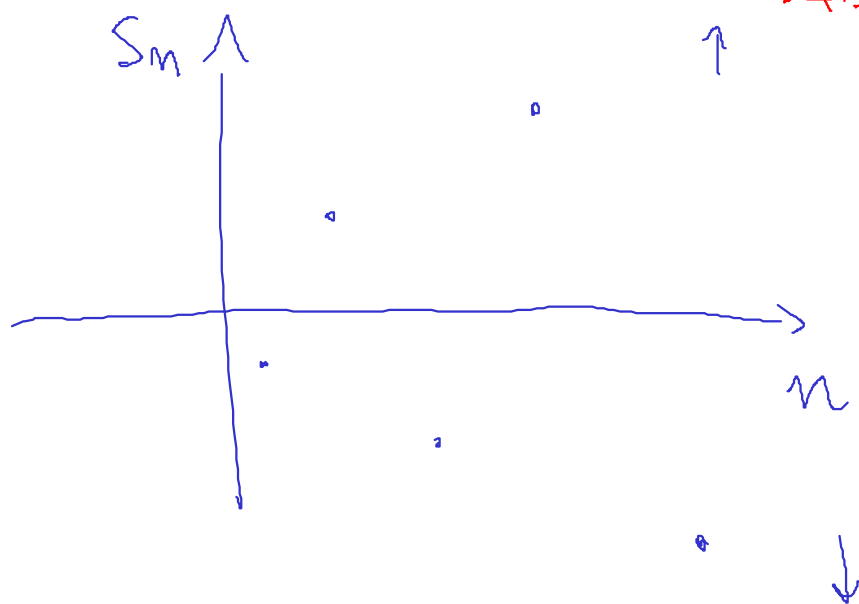
ODD $n = 2u + 1$

$$\lim_{u \rightarrow +\infty} \frac{2u+1}{\cos(2u\pi + \pi)} = \lim_{u \rightarrow +\infty} - (2u+1) = -\infty$$



$$S_n = \frac{n}{\cos(\pi n)}$$

$\lim_{n \rightarrow +\infty} S_n$ does not exist



b) $\lim_{n \rightarrow +\infty} \sin(n)$

PROP. $S_m \rightarrow l \quad Q_m \rightarrow l'$

4

$$S_m \cdot Q_m \rightarrow l \cdot l' \quad S_m \pm Q_m \rightarrow l \pm l'$$

Assume $\lim_{n \rightarrow +\infty} \sin(n) = 0 \quad 0 \in \mathbb{R}$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$$

$$\sin(n+1) - \sin(n-1) = 2 \cos(n) \sin(1)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 0 & - & 0 \end{array}$$

$$\lim_{n \rightarrow +\infty} \underbrace{2 \sin(1)}_{\neq 0} \cos(n) = 0 \Rightarrow \lim_{n \rightarrow +\infty} \cos(n) = 0$$

$$\sin(n+1) = \sin(n) \cos(1) + \cos(n) \sin(1)$$

$$\begin{array}{ccc} \downarrow & & \downarrow & & \downarrow \\ 0 & & \cos(1) \cdot 0 & & 0 \end{array}$$

$$0 = 0 \cos(1) \Rightarrow 0 \neq 0$$

$$\boxed{\cos(1) = 1 \text{ FALSE}}$$

$3=1$

$$\begin{pmatrix} \cos(0) = 1 \\ \cos(1) < 1 \end{pmatrix}$$

$$\cos^2(m) + \sin^2(m) = 1$$

I assume
 $\neq 0 = 0$

$$\lim_{m \rightarrow +\infty} \sin(m) = 0$$

$$\lim_{m \rightarrow +\infty} \cos(m) = 0$$

$$\lim_{m \rightarrow +\infty} \sin^2(m) = 0$$

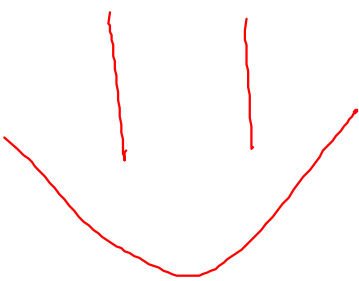
$$\lim_{m \rightarrow +\infty} \cos^2(m) = 0$$

$$\lim_{m \rightarrow +\infty} \sin^2(m) + \cos^2(m) = 0$$

$$0 = 0$$

FALSE

$$\lim_{m \rightarrow +\infty} (\sin^2(m) + \cos^2(m)) = \lim_{m \rightarrow +\infty} 1 = 1$$


 $\lim_{m \rightarrow +\infty} \sin(m)$ does not exist

3. Use the absolute value theorem or the comparison theorem to compute the following limits.

> Abs. value theorem

$$|S_n| \rightarrow \underline{0} \Rightarrow S_n \rightarrow \underline{0}$$

$$|S_n| \rightarrow 3 \Rightarrow \text{?}$$

> Comparison theorem

$$S_n \quad P_n \quad Q_n$$

$$P_n \leq S_n \leq Q_n$$

$$P_n \rightarrow l$$

$$Q_n \rightarrow l$$

$$\Rightarrow S_n \rightarrow l$$

a)

$$\lim_{n \rightarrow +\infty} \frac{3}{n (\cos(n)) + 2n}$$

$$-1 \leq \cos(n) \leq 1$$

||

$$-1 \leq a \leq 1$$

\Downarrow

$$\Rightarrow \frac{1}{1+2} \leq \frac{1}{a+2} \leq \frac{1}{-1+2}$$

$\frac{1}{3} \qquad \qquad \qquad 1$



$$\frac{3}{n(-1)+2n} \ll \frac{3}{n(\cos(n))+2n} \ll \frac{3}{n(-1)+2n}$$

$$\frac{3^1}{1 \cdot 3n} \ll \frac{3}{n(\cos(n))+2n} \ll \frac{3}{n}$$

p_n
 s_n
 q_n

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

$$\lim_{n \rightarrow \infty} \frac{3}{n} = 0$$



$$\lim_{n \rightarrow +\infty} \frac{3}{n(\cos(n))+2n} = 0$$

b)

$$\lim_{n \rightarrow +\infty} \sqrt[n]{3^n + 5^n} = \lim_{n \rightarrow +\infty} (3^n + 5^n)^{1/n} =$$



$$\lim_{n \rightarrow +\infty} \sqrt[n]{3^n + 5^n} = 5$$

$$|2^{1/n} - 1| < \varepsilon \quad a = 2^{1/n}$$

$$\underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}} = 2$$

$$\cancel{a = 1} \quad a > 1$$

$$2^{1/n} - 1 < \varepsilon \quad 2^{1/n} < \varepsilon + 1$$

$$\log_2 2^{1/n} < \log_2 (\varepsilon + 1)$$

$$\frac{1}{n} < \log_2 (\varepsilon + 1)$$

$$n > \frac{1}{\log_2 (\varepsilon + 1)}$$

$$n^* = \left\lceil \frac{1}{\log_2 (\varepsilon + 1)} \right\rceil$$



4. Compute the following limits 10

$$\begin{aligned}
 \text{a) } \lim_{n \rightarrow \infty} \frac{6n + \cos(n) - 3}{1 + \sin(n) - 3n} &= \\
 &= \lim_{n \rightarrow \infty} \frac{6n \left(1 + \frac{\cos(n)}{6n} - \frac{3}{6n} \right)}{-3n \left(\underline{1} - \frac{1}{3n} - \frac{\sin(n)}{3n} \right)} = \\
 &= \lim_{n \rightarrow +\infty} -2 \frac{\left(1 + \frac{\cos(n)}{6n} - \frac{1}{2n} \right)}{\left(1 - \frac{1}{3n} - \frac{\sin(n)}{3n} \right)} = -2
 \end{aligned}$$

\downarrow
0
 \downarrow
0

$$\begin{aligned}
 \text{b) } \lim_{n \rightarrow +\infty} \left(n - \sqrt{2n^2 + 5n} \right) &= \\
 &= \lim_{n \rightarrow +\infty} \frac{\left(n - \sqrt{2n^2 + 5n} \right) \left(n + \sqrt{2n^2 + 5n} \right)}{n + \sqrt{2n^2 + 5n}} \\
 &= \lim_{n \rightarrow +\infty} \frac{n^2 - (2n^2 + 5n)}{n + \sqrt{2n^2 + 5n}} =
 \end{aligned}$$

$$= \lim_{n \rightarrow +\infty} \frac{-n^2 - 5n}{n \left[1 + \frac{1}{n} (2n^2 + 5n)^{1/2} \right]} =$$

$$= \lim_{n \rightarrow +\infty} \frac{-n - 5}{1 + \left(2 + \frac{5}{n} \right)^{1/2}} = -\infty$$

\downarrow
 0

c) $\lim_{n \rightarrow +\infty} \frac{\underline{7^n} + \log(n^3) + \underline{e^n}}{n\sqrt{n} + 5\sin(n^4) + \underline{8^n} + 1} =$

$$\log(n^3) = 3 \log(n)$$

$\sin(n^4)$ oscillating

$$= \lim_{n \rightarrow +\infty} \frac{7^n \left(1 + \frac{\log(n^3)}{7^n} + \left(\frac{e}{7} \right)^n \right)}{8^n \left(1 + \frac{n\sqrt{n}}{8^n} + \frac{5\sin(n^4)}{8^n} + \frac{1}{8^n} \right)}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $0 \quad 0 \quad 0 \quad 0$

$$= 0$$

12

$$d) \lim_{n \rightarrow +\infty} \frac{n! + 2^n}{(n+1)!} = \lim_{n \rightarrow +\infty} \frac{n! \left(1 + \frac{2^n}{n!}\right)}{(n+1)!} =$$

$$\frac{n!}{(n+1)!} = \frac{\cancel{n}(\cancel{n-1})(\cancel{n-2}) \dots \cancel{1}}{(n+1)\cancel{n}(\cancel{n-1})(\cancel{n-2}) \dots \cancel{1}} =$$

$$= \frac{1}{n+1}$$

$$= 0$$