Mathematics I - Practice V

Sequences

1. Compute the following limits

a)
$$\lim_{n\to+\infty} \frac{e^{\frac{n+1}{n}}-1}{\sin\left(\frac{1}{n}\right)} = \lim_{n\to+\infty} \frac{e^{\frac{n+1}{n}}-1}{1/n}\cdot \overbrace{\frac{1/n}{\sin(1/n)}}^{\to 1}$$

$$= \lim_{n\to+\infty} \frac{e^{\frac{n+1}{n}}-1}{1/n}\cdot 1 = \lim_{n\to+\infty} \frac{e^{1+\frac{1}{n}}-1}{1/n} =$$

$$= \lim_{n\to+\infty} \underbrace{n}_{\to+\infty} \left(\underbrace{e^{1+1/n}}_{\to 1.6\ldots}-1\right) = +\infty$$

$$\lim_{n\to+\infty} \frac{\sin(1/n)}{1/n} = 1$$

$$\lim_{n\to+\infty} \frac{e^{\frac{1}{n}}-1}{1/n} = 1$$

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e \simeq 2.6\ldots$$

b)
$$\lim_{n\to+\infty} \sqrt{\log(n)}\,\sin\left((\log n)^{-1/2}\right) = \lim_{n\to+\infty} \frac{\sin\left(\frac{1}{\sqrt{\log(n)}}\right)}{\frac{1}{\sqrt{\log(n)}}} =$$

$$= 1$$

If f is an increasing monotonic function

$$\lim_{n\to+\infty} \frac{\sin\left(\frac{1}{f(n)}\right)}{1/f(n)} = 1$$

$$\lim_{n\to+\infty} \frac{\sin(1/n)}{1/n} = 1$$

c)
$$\lim_{n\to+\infty} n\left(e^{\frac{2n+2}{n-1}}-1\right) =$$

$$\frac{2n+2}{n-1} = \frac{2n-2+2+2}{n-1} = \frac{2(n-1)+4}{n-1} = 2+\frac{4}{n-1} \to 2$$

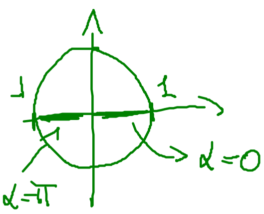
$$= \lim_{n\to+\infty} \underbrace{n}_{\to+\infty} \underbrace{\left(e^{2+\frac{4}{n-1}}-1\right)}_{\to e^2-1} = +\infty$$

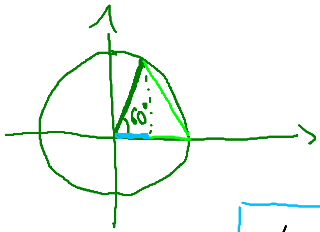
Series

2. Compute the following series, if they exist

$$\boxed{\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}} \qquad |a| < 1$$

a) $$\sum_{n=0}^{+\infty} \left[\cos\left(\pi/3\right)\right]^n = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1 \cdot 2}{\frac{1}{2} \cdot 2} = \frac{2}{1} = 2$$

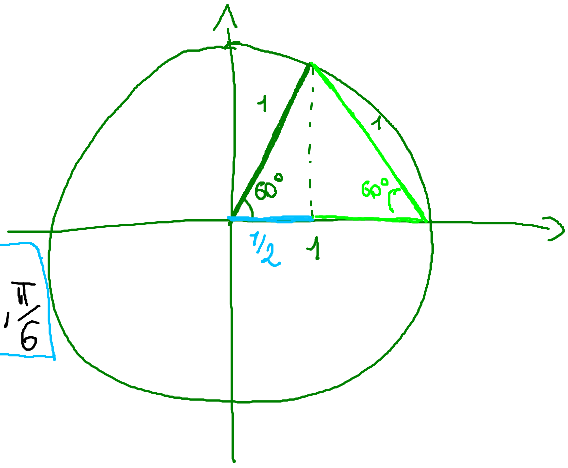




$\cos(\pi/3) = 1/2$

$$\boxed{\alpha = 0, \pi, \pi/2, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}}$$

$\sin(\alpha)$ $\qquad$ $\cos(\alpha)$



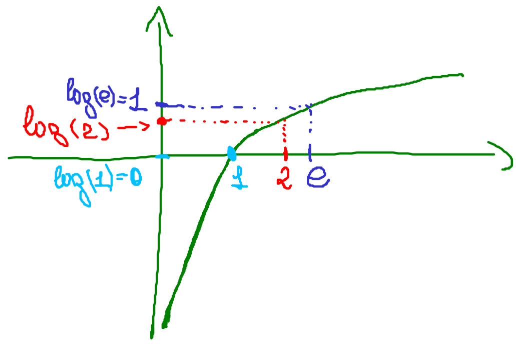
b) $$\sum_{n=0}^{+\infty} \Big(\underbrace{\log(2)}_{}\Big)^n = \frac{1}{1-\log(2)} \simeq 3.26$$

$a = \log(2)$

$\cancel{\log(0)} = \lim_{x \to 0} \log(x) = -\infty$

$\log(1) = 0$ * $\qquad$ $\log(e)$ *

$\log(e) = 1$ * $\qquad$ $\log(2) < 1$



$$\sum_{n=0}^{M} a^n = 1 + a + a^2 + a^3 + \dots + a^M = \frac{a^{M+1}-1}{a-1}$$

$$\boxed{\left(a^M - 1\right) = (a-1)\left(a^{M-1} + a^{M-2} + \dots + 1\right)}$$

$M \to M+1$

$$\left(a^{M+1} - 1\right) = (a-1)\left(a^M + a^{M-1} + \dots + 1\right)$$

$$\rightarrow \left(a^M + a^{M-1} + \dots + a^3 + a^2 + 1\right) = \frac{\left(a^{M+1} - 1\right)}{(a-1)}$$

3

$$\sum_{n=0}^{M} a^n = 1 + a + a^2 + \dots + a^M = \frac{a^{M+1} - 1}{(a-1)}$$

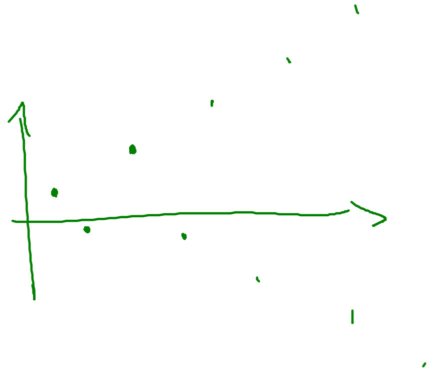
$M \to +\infty$

$|a| < 1 \qquad a^{M+1} \xrightarrow[M\to+\infty]{} 0$

$$\boxed{\sum_{n=0}^{+\infty} a^n} = \frac{-1}{a-1} = \boxed{\frac{1}{1-a}}$$

c) $$\sum_{n=0}^{+\infty} (-\sqrt{3})^n = \times \text{ DOES NOT CONVERGE} \qquad \cancel{= \pm\infty}$$

$$1 - \sqrt{3} \,|\, + 3 \,|\, - 3\sqrt{3} \,|\, + 9 \,|$$



$$|-\sqrt{3}| > 1$$

d) $$\sum_{n=1}^{+\infty} \frac{2^n + 8^n}{9^n} = \sum_{n=1}^{+\infty} \underbrace{\frac{2^n}{9^n}}_{\text{FINITE}} + \underbrace{\frac{8^n}{9^n}}_{\text{FINITE}} = \sum_{n=1}^{+\infty} \left(\frac{2}{9}\right)^n + \left(\frac{8}{9}\right)^n =$$

$$= \sum_{n=1}^{+\infty} \left(\frac{2}{9}\right)^n + \sum_{n=1}^{+\infty} \left(\frac{8}{9}\right)^n = \underbrace{\frac{1}{1-\frac{2}{9}}}_{\sum_{n=0}^{+\infty} a^n} \underbrace{- 1}_{- a^0} + \frac{1}{1-\frac{8}{9}} - 1 = *$$

$$\sum_{n=0}^{+\infty} a^n = 1 + a + a^2 + a^3 + \dots$$

$$\sum_{n=0}^{+\infty} (a^n + b^n) \overset{\text{if they converge}}{=} \sum_{n=0}^{+\infty} a^n + \sum_{n=0}^{+\infty} b^n$$

$$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=0}^{+\infty} a^n = \sum_{n=1}^{+\infty} a^n + a^0$$

$$\sum_{n=0}^{+\infty} a^n = \sum_{n=1}^{+\infty} a^n + 1$$

$$\sum_{n=1}^{+\infty} a^n = \boxed{\sum_{n=0}^{+\infty} a^n} - 1 = \frac{1}{1-a} - 1$$

$$* = \frac{1}{\frac{7}{9}} - 1 + \frac{1}{\frac{1}{9}} - 1 =$$

$$= \frac{9}{7} - 1 + 9 - 1 = \frac{9}{7} + 7 = \frac{9+49}{7} =$$

$$= \frac{58}{7}$$

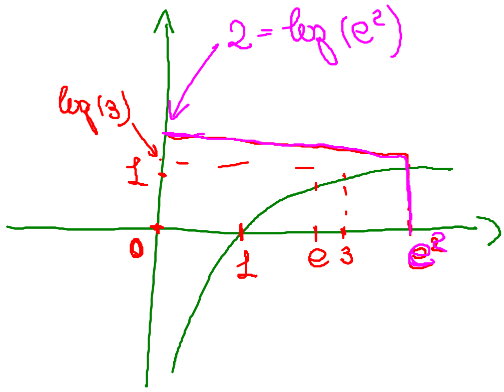
4

$$\sum_{n=4}^{+\infty} a^n = \sum_{n=0}^{+\infty} a^n - a^3 - a^2 - a^1 - a^0 \qquad \sum_{n=6}^{+\infty} a^n = ?$$

e) $$\sum_{n=0}^{+\infty} \left(\frac{\log(3)}{3}\right)^{n+1} \to ?$$

$$\left|\frac{\log(3)}{3}\right| \overset{?}{<} 1 \quad ✓$$

$$\left|\frac{\log(3)}{3}\right| < 1 \qquad \frac{|\log(3)|}{3} < 1 \qquad \frac{\log(3)}{3} < 1$$



$$\log(3) < 3$$

$1 \div 2$

$$\log(e^2) = 2$$

$\parallel$

$2.6\ldots \cdot 2.6\ldots > 3$

$$a^{n+1} = a^n \cdot a^1$$

$$\sum_{n=0}^{+\infty} \left(\frac{\log(3)}{3}\right)^{n+1} = \sum_{n=0}^{+\infty} \left(\frac{\log(3)}{3}\right)^{n} \cdot \left(\frac{\log(3)}{3}\right)^{1} =$$

$$= \left(\frac{\log(3)}{3}\right) \cdot \sum_{n=0}^{+\infty} \left(\frac{\log(3)}{3}\right)^{n} = \left(\frac{\log(3)}{3}\right) \cdot \frac{1}{1 - \frac{\log(3)}{3}} =$$

$$= \left(\frac{\log(3)}{3}\right) \cdot \frac{1}{\frac{3-\log(3)}{3}} = \left(\frac{\log(3)}{\cancel{3}}\right) \cdot \frac{\cancel{3}}{3 - \log(3)} = \frac{\log(3)}{3 - \log(3)}$$

$$\sum_{n=0}^{+\infty} \left(\frac{\log(3)}{3}\right)^{n+1} = \sum_{m=1}^{+\infty} \left(\frac{\log(3)}{3}\right)^{m} = \underbrace{\frac{1}{1 - \frac{\log(3)}{3}}}_{\sum_{m=0}^{+\infty}\left(\frac{\log(3)}{3}\right)^m} - \underbrace{1}_{\left(\frac{\log(3)}{3}\right)^0} =$$

$m = n+1$

$n=0 \to m=1$
$n \to \infty \to m = +\infty$

$$= \frac{1}{\frac{3-\log(3)}{3}} - 1 = \frac{3}{3-\log(3)} - 1 = \frac{\cancel{3} - \cancel{3} + \log(3)}{3-\log(3)} =$$

$$= \frac{\log(3)}{3-\log(3)}$$

3. For which value of the parameter a do the following series converge?

a) $$\sum_{n=0}^{+\infty} e^{an} = \sum_{n=0}^{+\infty} (e^a)^n$$

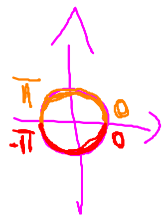
$$\mathbb{R} \xrightarrow{e} \mathbb{R}^+$$

$$|e^a| < 1 \qquad e^a < 1 \qquad \log e^a < \log(1)$$

$$\boxed{a < 0} \Rightarrow \sum_{n=0}^{+\infty} (e^a)^n = \frac{1}{1-e^a}$$

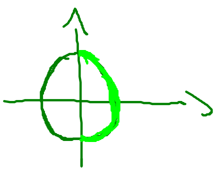
b) $$\sum_{n=0}^{+\infty} (\cos(a))^n$$

$$\boxed{a \in [-\pi, \pi]}$$



$$|\cos(a)| < 1$$

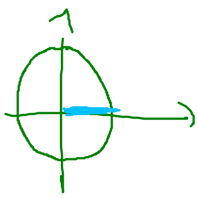
$\cos(a) \geq 0$



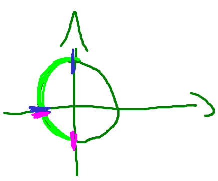
$$\boxed{-\frac{\pi}{2} + 2k\pi \leq a \leq \frac{\pi}{2} + 2k\pi} \quad k \in \mathbb{Z}$$

$$\cos(a) < 1$$

$$\boxed{a \neq 0 + 2k\pi}$$



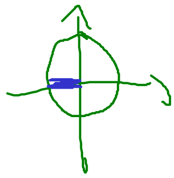
$\cos(a) < 0$



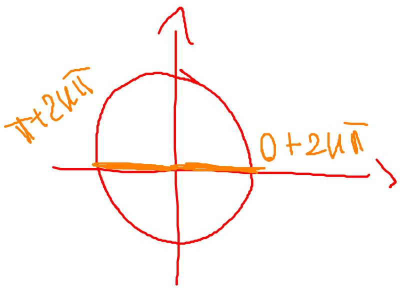
$$\left[\frac{\pi}{2} + 2k\pi < a \leq \pi + 2k\pi\right.$$

$$\left[-\pi + 2k\pi < a < -\frac{\pi}{2} + 2k\pi\right.$$

$-\cos(a) < 1$ $\qquad \cos(a) > -1$



$$\boxed{a \neq \pi + 2k\pi}$$



$$\boxed{a \neq k\pi} \qquad k \in \mathbb{Z}$$

c) $\displaystyle\sum_{n=0}^{+\infty} \left(\frac{a+3}{a^2+1}\right)^n \qquad \left|\frac{a+3}{a^2+1}\right| < 1$

$$\frac{a+3}{a^2+1} \geq 0 \implies a \geq -3$$

$$\frac{a+3}{a^2+1} < 1 \qquad a+3 < a^2+1 \qquad a^2-a-2 > 0$$

$$a_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases} \qquad (a-2)(a+1) > 0$$

	-1		2	
$(a-2)$	$-$		$-$	$+$
$(a+1)$	$-$		$+$	$+$
	$+$		$-$	$+$

$$\boxed{a < -1 \lor a > 2}$$

It must be consistent with the initial assumption

$$\Downarrow$$

$$A \ \boxed{-3 \leq a < -1 \lor a > 2}$$

$$\frac{a+3}{a^2+1} < 0 \implies a < -3$$

$$-\frac{a+3}{a^2+1} < 1 \qquad -a-3 < a^2+1 \qquad a^2+a+4 > 0 \qquad a_{1,2} = \frac{-1 \pm \sqrt{1-16}}{2}$$

No real solution

$\Downarrow$

Always positive

$\boxed{\text{Any } a}$

$$B \ \boxed{a < -3}$$

Overall solution

$A \cup B$

$$\boxed{a < -1 \lor a > 2}$$

Limits of functions

4. Compute the following limits of functions

a) $$\lim_{x\to 1}\frac{\sqrt{x}-1}{x^2+x-2}=\lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x^2+x-2)}=\lim_{x\to 1}\frac{(x-1)}{(\sqrt{x}+1)(x^2+x-2)}=$$

$$x^2+x-2=0$$

$$x_{1,2}=\frac{-1\pm\sqrt{1+8}}{2}=\frac{-1\pm 3}{2}=\begin{cases}1\\-2\end{cases}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}}{(\sqrt{x}+1)\cancel{(x-1)}(x+2)}=\lim_{x\to 1}\frac{1}{\underbrace{(\sqrt{x}+1)}_{2}\underbrace{(x+2)}_{3}}=\frac{1}{6}$$

b) $$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)=\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{\underbrace{(1-x)(x^2+x+1)}_{\cancel{x^2}+\cancel{x}+1-x^3-\cancel{x^2}-\cancel{x}}}\right)=$$

$\rightarrow +\infty - +\infty \neq 0$

$\rightarrow$ ~~$\lim_{x\to 1}\frac{1}{1-x}-\lim_{x\to 1}\frac{3}{1-x^3}$~~

$$=\lim_{x\to 1}\left(\frac{x^2+x+1-3}{(1-x)(x^2+x+1)}\right)=\lim_{x\to 1}\left(\frac{x^2+x-2}{(1-x)(x^2+x+1)}\right)=$$

$$=\lim_{x\to 1}\left(\frac{(x-1)(x+2)}{(1-x)(x^2+x+1)}\right)=\lim_{x\to 1}\left(\frac{-\cancel{(1-x)}(x+2)}{\cancel{(1-x)}(x^2+x+1)}\right)=$$

$$=\lim_{x\to 1}\left[-\frac{(x+2)}{(x^2+x+1)}\right]=-\frac{3}{3}=-1$$