Mathematics I - Practice VI

1. For each of the following functions determine the set of points at which the function is continuous and describe the type of discontinuities, if any.

a)

$$f(x)=\frac{e^x-1}{\sin(x)}$$

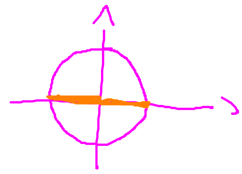
C.E.

$\sin(x)\neq 0$

$x\neq 0 \vee x\neq \pi$

$\Downarrow$

$\boxed{x\neq k\pi \quad k\in\mathbb{Z}}$



$k\in\mathbb{Z}$

$D:\mathbb{R}\setminus\{k\pi\}$

$$\lim_{x\to k\pi^+} f(x)=? \qquad \lim_{x\to k\pi^-} f(x)=?$$

$k=0$ $\qquad$ $k\neq 0$

$$\lim_{x\to 0^+}\frac{e^x-1}{\sin(x)}=\lim_{x\to 0^+}\underbrace{\frac{e^x-1}{x}}_{\to 1}\cdot\overbrace{\frac{x}{\sin(x)}}^{\to 1}=1$$

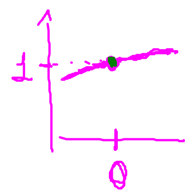
$$\lim_{x\to 0^-}\frac{e^x-1}{\sin(x)}=1$$

$x=0$

Removable discontinuity

$f(x=0)\ \nexists$

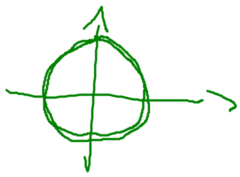
$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^-}f(x)=1$$



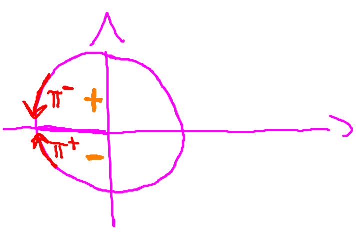
$$\to f(x)=\begin{cases}\dfrac{e^x-1}{\sin(x)} & x\neq 0\\ 1 & x=0\end{cases}$$

$k\neq 0$ $\quad (k=1,\ k=-1,\ k=2,\ k=-2)$

$x=\pi$	$x=-\pi$	$k=2\pi$	$k=-2\pi$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
ODD POSITIVE K	ODD NEGATIVE K	EVEN POSITIVE K	EVEN NEGATIVE K



$$\lim_{x\to\pi^+}\frac{e^x-1}{\sin(x)}=-\infty \qquad e^\pi-1>0,\quad \sin(x)\to 0$$

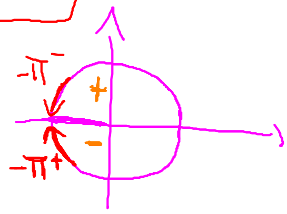


$$\lim_{x\to\pi^-}\frac{e^x-1}{\sin(x)}=+\infty \qquad e^\pi-1>0,\quad \sin(x)\to 0$$

$\boxed{\text{Essential discontinuity } x = \pi + 2k\pi \quad k \in \mathbb{Z}}$

$$\lim_{x \to -\pi^+} \frac{e^x - 1}{\sin(x)} = +\infty$$

$e^x - 1 \to e^{-\pi} - 1 < 0$, $\sin(x) \to 0$



$$\lim_{x \to -\pi^-} \frac{e^x - 1}{\sin(x)} = -\infty$$

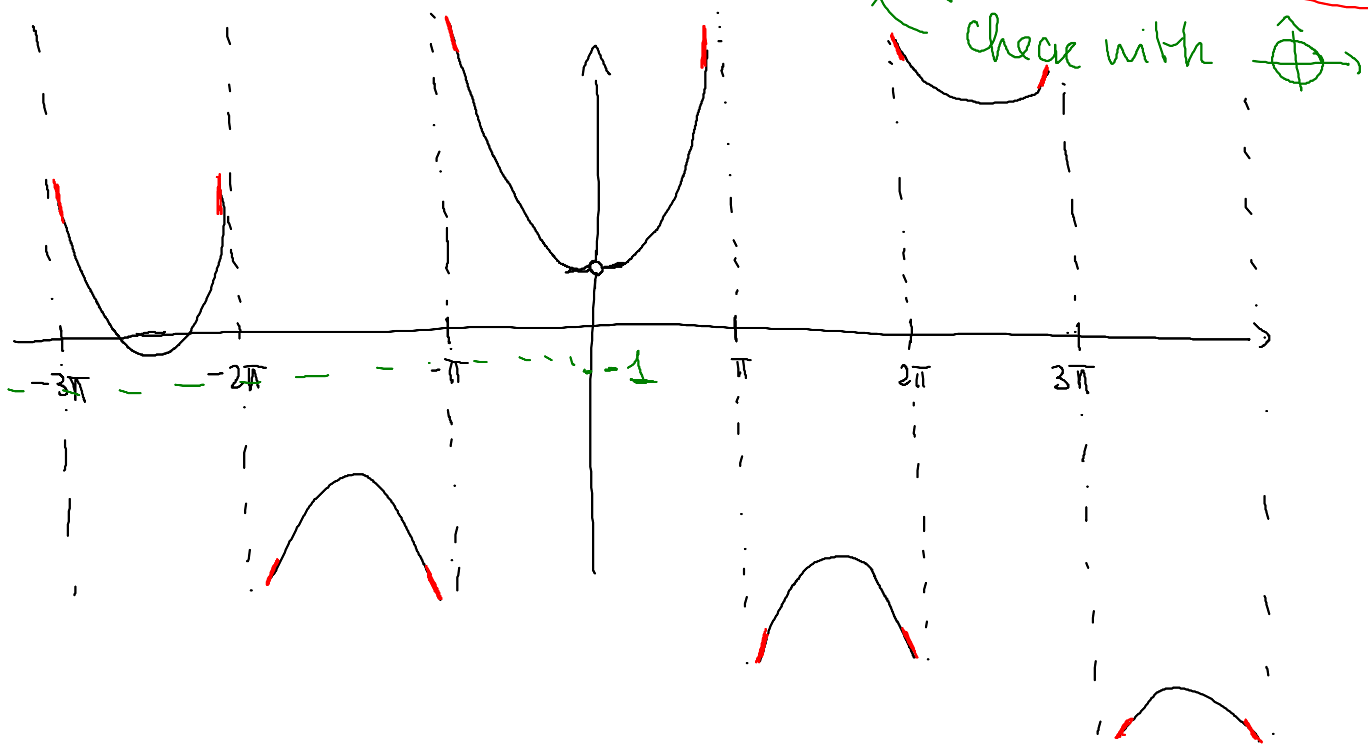
$e^x - 1 \to e^{-\pi} - 1 < 0$, $\sin(x) \to 0$

$\boxed{\text{Essential disc. } x = -\pi + 2k\pi \quad k \in \mathbb{Z}}$

$$\lim_{x \to 2\pi^+} \frac{e^x - 1}{\sin(x)} = +\infty \qquad \lim_{x \to 2\pi^-} \frac{e^x - 1}{\sin(x)} = -\infty \quad \to \quad \boxed{\text{Essential disc. } x = 2\pi + 2k\pi}$$

$$\lim_{x \to -2\pi^+} \frac{e^x - 1}{\sin(x)} = -\infty \qquad \lim_{x \to -2\pi^-} \frac{e^x - 1}{\sin(x)} = +\infty \quad \to \quad \boxed{\text{Essential disc. } x = -2\pi + 2k\pi}$$

Check with



b)

$$f(x) = \begin{cases} x^2 + x + \log(x) & x > 1 \\ 3x - 1 & 0 \leq x \leq 1 \\ \sqrt{4 - x} & x < 0 \end{cases}$$

C.E. $x \in \mathbb{R}$

$x = 0$, $x = 1$

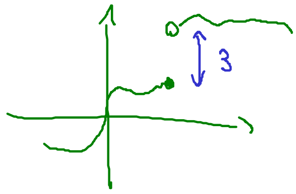
$x = 0$

$f(x=0) = 3\cdot 0 - 1 = -1$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{4-x} = \sqrt{4} = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x-1 = -1$$

$$f(0) = \lim_{x\to 0^+} f(x) = -1 \neq \lim_{x\to 0^-} f(x) = 2$$



$\boxed{x=0 \quad \text{Jump discontinuity}}$

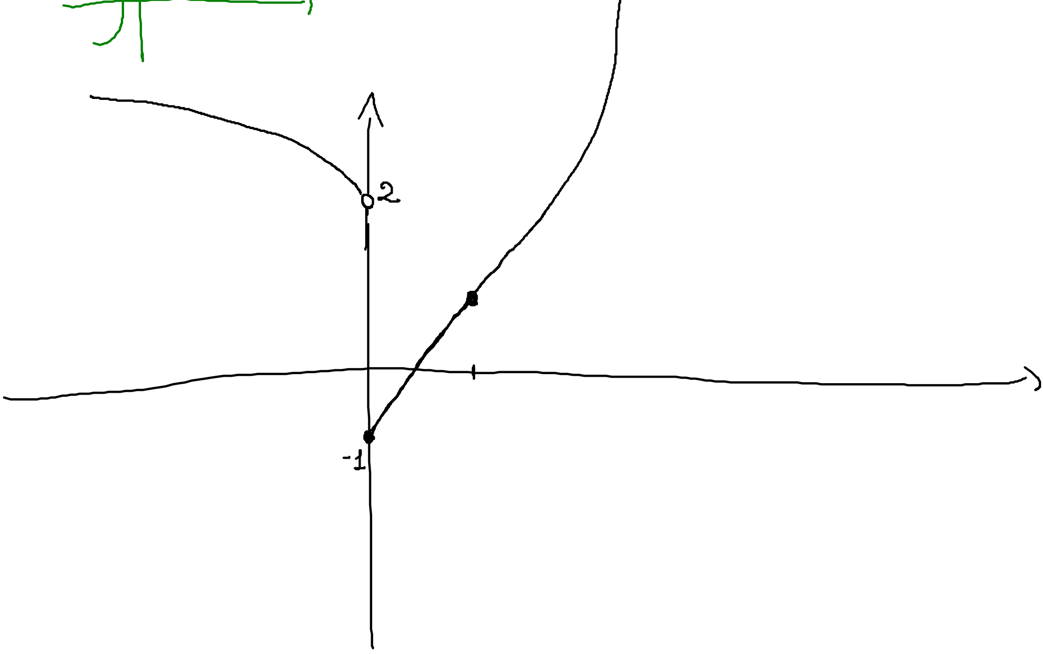
$x=1$

$f(x=1) = 2$ $\qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x-1 = 2$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+x+\log(x) = 2$$

$$2 = f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

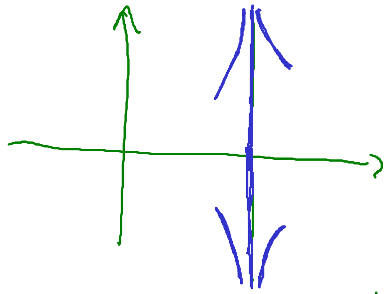
$\boxed{\text{the function is continuous at } x=1}$



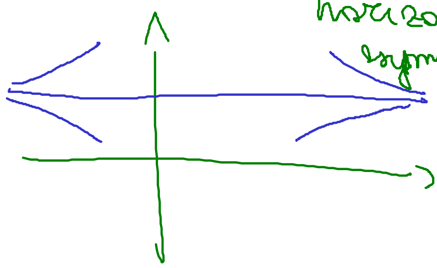
2. Determine horizontal and vertical asymptotes if they exist, for each following function.

a) $f(x)=\dfrac{x^2-4x-1}{x^2-4}$

C.E. $x \neq \pm 2$



Vertical asymptote



horizontal asymptote

Check what happens for these values of x

$x \to 2^{\pm}$ A $x \to -2^{\pm}$ B $x \to +\infty$ C $x \to -\infty$ D

A

$$\lim_{x\to 2^+} \frac{\overbrace{x^2-4x-1}^{4-8-1<0}}{\underbrace{x^2-4}_{0^+}} = -\infty \qquad \lim_{x\to 2^-} \frac{\overbrace{x^2-4x-1}^{<0}}{\underbrace{x^2-4}_{0^-}} = +\infty$$

$\boxed{\text{vertical asymptote } x=2}$

B

$$\lim_{x\to -2^+} \frac{\overbrace{x^2-4x-1}^{4+8-1>0}}{\underbrace{x^2-4}_{0^-}} = -\infty \qquad \lim_{x\to -2^-} \frac{\overbrace{x^2-4x-1}^{>0}}{\underbrace{x^2-4}_{0^+}} = +\infty$$

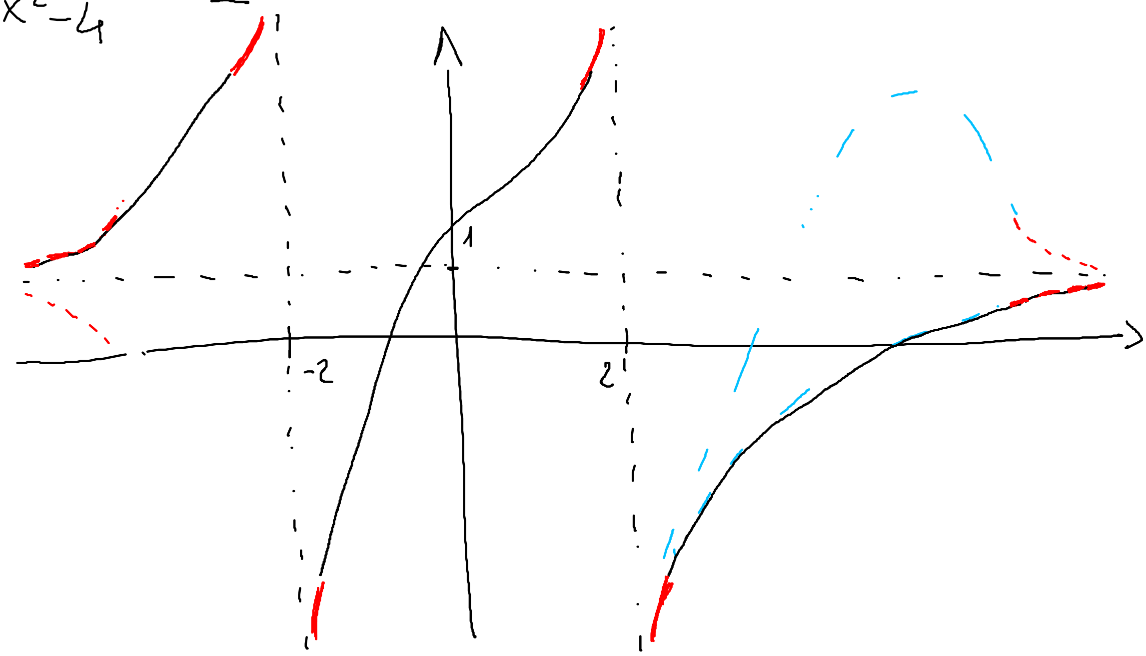
$\boxed{\text{vertical asymptote } x=-2}$

C & D

$$\lim_{x\to +\infty} \frac{x^2-4x-1}{x^2-4} = \lim_{x\to +\infty} \frac{\cancel{x^2}\left(1-\overbrace{\frac{4}{x}}^{\to 0}-\overbrace{\frac{1}{x^2}}^{\to 0}\right)}{\cancel{x^2}\left(1-\underbrace{4/x^2}_{\to 0}\right)} = 1$$

$\boxed{\text{Horizontal asymptote } y=1}$

$$\lim_{x\to -\infty} \frac{x^2-4x-1}{x^2-4} = 1$$



3. Compute the function's derivative. 5

Derivatives box
$f'(x) \to$ deriv. of $f(x)$

$(x^n)' = n x^{n-1}$

$(e^x)' = e^x$

$(a)' = 0$

$(\cos(x))' = -\sin(x)$

$(f(x)+g(x))' = f'(x) + g'(x)$

$(\sin(x))' = \cos(x)$

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$(\log(x))' = \frac{1}{x}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

a) $f(x) = x\cos(x) + xe^x + e^2$

$f'(x) = 1\cdot\cos(x) + x(-\sin(x)) + 1\cdot e^x + xe^x = \cos(x) - x\sin(x) + (x+1)e^x$

$0 \xleftarrow{\text{deriv.}} \sin(\log(\cos(\pi/4)))$

b) $f(x) = \frac{x^2+1}{x-3} = \frac{f(x)}{g(x)}$

$f'(x) = \frac{x^2-6x-1}{(x-3)^2}$

$$f'(x) = \frac{\overset{f'}{2x}\,\overset{g}{(x-3)} - \overset{f}{(x^2+1)}\cdot \overset{g'}{1}}{(x-3)^2} =$$

$$= \frac{2x^2-6x-x^2-1}{(x-3)^2} = \frac{x^2-6x-1}{(x-3)^2}$$

c)

$$f(x) = \log\left(\frac{x^2-x+1}{x^2+3}\right)$$

$$f'(x) = \left(\frac{x^2+3}{x^2-x+1}\right)\cdot\frac{(2x-1)(x^2+3)-(x^2-x+1)\cdot 2x}{(x^2+3)^2} =$$

$$= \frac{(x^2+3)[\cancel{2x^3}-x^2+6x-3-\cancel{2x^3}+2x^2-2x]}{(x^2+3)^2(x^2-x+1)} =$$

$$= \frac{\cancel{(x^2+3)}(x^2+4x-3)}{(x^2+3)^{\cancel{2}}(x^2-x+1)} = \frac{(x^2+4x-3)}{(x^2+3)(x^2-x+1)}$$

CAN HAPPEN:

$$f'(x) = \frac{(x-1)\sim\sim\sim}{\sim\sim\sim(x-1)} \quad x \to 1$$

$$\lim_{x\to 1} \frac{\cancel{(x-1)}\sim\sim}{\sim\sim\cancel{(x-1)}} = \frac{\sim\sim}{\sim\sim}\Big|_{x=1}$$

$x = 1$

$f'(x=1) = \nexists$

$$\lim_{x\to 1^\pm} f'(x) = \frac{\sim\sim}{\sim\sim}\Big|_{x=1}$$

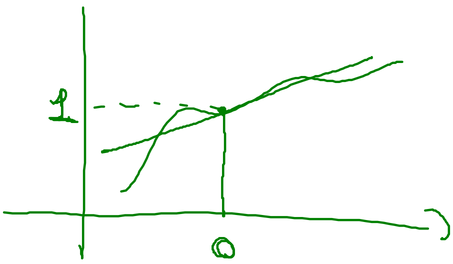
4. Find the equation of the tangent line to the graph of the following function at $x_0 = 0$

$$f(x) = \underset{f}{3x}\,\underset{g}{\cos(2x)} - e^{2x}$$

$$f'(x) = \underset{f'}{3}\,\underset{g}{\cos(2x)} - \underset{fg'}{6x\sin(2x)} - 2e^{2x}$$

$$\boxed{m} = f'(x_0=0) = 3\cdot 1 - 2 \boxed{= 1}$$

$$f(x_0=0) = -1$$



$y = mx + q$

$\downarrow f'(x_0)$

$y = 1\cdot x + q$

$$\underset{f(x_0)}{-1} = 1\cdot \underset{x_0}{0} + q \to \boxed{q = -1}$$

$$\Rightarrow \boxed{y = x - 1}$$