MATHEMATICS 1
ADDITIONAL EXERCISES N. 5

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Notation: log stands for the natural logarithm (i.e. the logarithm with the basis e)

1. Limits

(1) Compute the following limits

$$(a)\ \lim_{x\to 1}\frac{x^3-1}{x-1} \qquad (b)\ \lim_{x\to -2}\frac{x^3+8}{x^2-x-2}$$

$$(c)\ \lim_{x\to 0}\frac{\sqrt{x^2+x+2}-1}{x} \qquad (d)\ \lim_{x\to -2}\frac{\sqrt{2+x}-\sqrt{2-x}}{x}$$

$$(e)\ \lim_{x\to 1}\frac{\sqrt{1+x^2}-\sqrt{2}}{x-1} \qquad (f)\ \lim_{x\to -2}\frac{x^3+4x^2+4x}{x^2-x-6}$$

$$(g)\ \lim_{x\to +\infty}\frac{x^2-4x}{\sqrt{5+x^4}} \qquad (h)\ \lim_{x\to -\infty}\frac{5x+7}{\sqrt{x^2+1}}$$

$$(i)\ \lim_{x\to +\infty}\sqrt{\frac{2x^2+1}{x^2-1}} \qquad (j)\ \lim_{x\to -\infty}\frac{\sqrt{x^2+1}}{x+1}$$

$$(k)\ \lim_{x\to +\infty}\sqrt{x+3}-\sqrt{x} \qquad (l)\ \lim_{x\to -\infty}\sqrt{3-x}-\sqrt{1-x}$$

$$(m)\ \lim_{x\to +\infty}\frac{3x}{\sqrt{3+x^2}-\sqrt{x+x^2}} \qquad (n)\ \lim_{x\to 1}\frac{1}{1-x}-\frac{3}{1-x^3}$$

$$(o)\ \lim_{x\to +\infty}\frac{\sqrt{x^2+x-2}}{x+1+\sqrt{x^2+x-2}} \qquad (p)\ \lim_{x\to -\infty}\frac{\sqrt{x^2+x-2}}{x+1+\sqrt{x^2+x-2}}$$

$$(q)\ \lim_{x\to -\infty}\frac{3x-4}{x-\sqrt{2-x}} \qquad (r)\ \lim_{x\to -\infty}\frac{2x}{3x-\sqrt{5x^2+3}}$$

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(2) Compute the following limits

$$(a) \lim_{x\to 0} \frac{\sin(3x)}{x} \qquad (b) \lim_{x\to 0} \frac{1}{\sqrt{1-\cos(2x)}}$$

$$(c) \lim_{x\to 0} \frac{\sin^2(x)}{1-\cos(x)} \qquad (d) \lim_{x\to \frac{\pi}{4}} \frac{1+\cos(4x)}{\cos^2(x)}$$

$$(e) \lim_{x\to 0} \frac{\sin(x^2+x)}{x} \qquad (f) \lim_{x\to 0} \frac{\sqrt{1-\cos(x)}}{2x}$$

$$(g) \lim_{x\to -\infty} \frac{\sin(e^x)}{3e^x} \qquad (h) \lim_{x\to 0} \frac{\sin(6x)}{3x} - x^2$$

$$(i) \lim_{x\to 0} \frac{\sin^2\left(\frac{x}{3}\right)}{x^2} \qquad (j) \lim_{x\to 1} \frac{\sin(\log(x))}{x-1}$$

$$(k) \lim_{x\to 0} \frac{\sin(3x)-5\sin(2x)+x}{x-2\tan(x)} \qquad (l) \lim_{x\to 1} \frac{1-cos(x)+x^2}{2\sin^2(x)-5x^2}$$

$$(m) \lim_{x\to +\infty} \left(\frac{x+3}{x+2}\right)^x \qquad (n) \lim_{x\to +\infty} \left(\frac{2x+3}{2x-1}\right)^{3x+4}$$

$$(o) \lim_{x\to +\infty} \left(\frac{x-1}{x+6}\right)^{2x} \qquad (p) \lim_{x\to 0} \frac{\log(1+3x)}{2x^2+x}$$

$$(q) \lim_{x\to 0} \frac{e^{2x^2+x}-1}{2x} \qquad (r) \lim_{x\to 0} \frac{e^{\sin(x)}-1}{x}$$

$$(s) \lim_{x\to +\infty} \left(\frac{1+x^3}{x^3+3x}\right)^{x^2} \qquad (t) \lim_{x\to +\infty} \left(1+\frac{1}{x^2+1}\right)^{2x^2+1}$$

$$(u) \lim_{x\to +\infty} \log(x+3)-\log(x-1) \qquad (v) \lim_{x\to +\infty} x(\log(x+3)-\log(x+1))$$

$$(w) \lim_{x\to 0} \frac{3\sin(x)-2x-4x^2}{5\sin(x)-4x} \qquad (x) \lim_{x\to 0} \frac{\sin^2(2x)+3x^2}{x^2+x\log(1+5x)-4x^2}$$

$$(y) \lim_{x\to 0} \frac{1-\cos^2(x)}{x\sin(x)\cos(x)} \qquad (z) \lim_{x\to 0} \frac{1-\cos^3(x)}{x\sin(x)\cos(x)}$$

2. Continuous Functions

(1) For each function determine the set where the function is continuous and identify the nature of discontinuity points, if any.

$$(a) f(x) = \frac{|4-x^2|}{x+2} \qquad (b) f(x) = \frac{\log(x+1)}{x}$$

$$(c) f(x) = 5+\log(|x|) \qquad (d) f(x) = \frac{e^x-1}{x}$$

$$(e) f(x) = \frac{1}{e^x 2 - x + 1} \qquad (f) f(x) = e^{-\frac{1}{x}}$$

$$(g) f(x) = \frac{x^2 + 2x}{x^3 + 8} \qquad (h) f(x) = \frac{e^{2x} - 1}{e^x - 1}$$

$$(h) f(x) = \frac{e^x + e^{-x}}{x} \qquad (i) f(x) = \log\left|\frac{x-1}{x}\right|$$

$$(j)\ f(x) = \begin{cases} \log(-x) & \text{if } \ x \le -1 \\ x + 1 & \text{if } \ x > -1 \end{cases}$$

$$(k)\ f(x) = \begin{cases} e^{-x-2} & \text{if } \ x \le -2 \\ \sqrt{x+2} & \text{if } \ x > -2 \end{cases}$$

$$(l)\ f(x) = \begin{cases} \dfrac{1 + x^3}{x + 1} & \text{if } \ x \ne -1 \\ 4 & \text{if } \ x = -1 \end{cases}$$

$$(m)\ f(x) = \begin{cases} \dfrac{e^{2x} - 1}{x} & \text{if } \ x \le 0 \\ 2 & \text{if } \ x = 0 \end{cases}$$

$$(n) f(x) = \frac{x + 1}{|x + 2| - 1} \qquad (o) f(x) = \frac{x^2 - 1}{\sqrt{x - 1}}$$

$$(p) f(x) = \frac{e^x - 1}{\log(x + 1)} \qquad (q) f(x) = \frac{e^{\frac{1}{x}} + e^{\frac{2}{x}}}{e^{\frac{1}{x}} + 5}$$

(2) For each function, say if Weierstrass theorem applies in the indicated interval

$$(a)\ f(x) = \frac{x^3}{4 - x^2} \qquad I = [-1, 0]$$

$$(b)\ f(x) = \frac{1}{\log(x)} \qquad I = \left[\frac{1}{3}, 2\right]$$

$$(c)\ f(x) = \frac{1}{|x|} \qquad I = [-1, 1]$$

$$(d)\ f(x) = \frac{\sin(x)}{1 - \cos(x)} \qquad I = [0, \pi]$$

$$(e)\ f(x) = \sqrt{|x - 1|} \qquad I = [0, 2]$$

$$(f)\ f(x) = \log(|x - 1|) \qquad I = [0, 2]$$

(3) For each equation, prove that a solution exists in the indicated interval and identify the solution graphically (as intersection point of suitable functions)

(a) $\sin(x) - x = 0$ $\qquad I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $\frac{x-1}{x} + x^3 = 0$ $\qquad I = \left[\frac{1}{2}, 2\right]$

(c) $e^x + 3x = 0$ $\qquad I = [-1, 0]$

(d) $x^3 - 3x^2 + 1 = 0$ $\qquad I = [0, 2]$

(e) $3 - x - \log(x)$ $\qquad I = [0, 3]$

(4) Compute all horizontal and vertical asymptotes of the functions below

(a) $f(x) = \frac{1}{x^3} - x$ $\qquad$ (b) $f(x) = \frac{1}{x^2+2}$

(c) $f(x) = \frac{x^2-4x-1}{x^2-4}$ $\qquad$ (d) $f(x) = \frac{x^2}{x^2-x}$

(e) $f(x) = x + \sqrt{x}$ $\qquad$ (f) $f(x) = \frac{\sqrt{x}}{x+1}$

(g) $f(x) = \sqrt{3-x^2} - 4x$ $\qquad$ (h) $f(x) = \frac{x^2-x-3}{x+1}$

(i) $f(x) = e^{-\frac{x+3}{x}}$ $\qquad$ (j) $f(x) = x\left(e^{\frac{1}{x}} - 1\right)$

(k) $f(x) = \frac{3}{x^3(x-1)}$ $\qquad$ (l) $f(x) = \frac{1}{\sqrt{x} - \sqrt{x-1}}$

(m) $f(x) = \frac{\log(x+1)}{x^2}$ $\qquad$ (n) $f(x) = \frac{1}{\log(x)}$