Mathematics I - practice 7

1) Determine for the following function
   a. the intervals in which it is increasing and decreasing
   b. the intervals in which it is concave and convex
   c. maxima, minima and inflection points (with horizontal tangent)

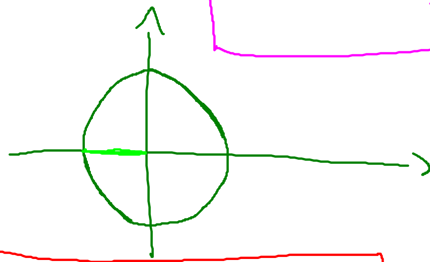
$f(x) = x + \sin(x)$ C.E. $x \in \mathbb{R}$

$f'(x) > 0$ INCREASING 

$f'(x) < 0$ DECREASING 

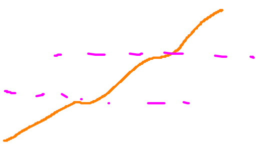
$f'(x) = 1 + \cos(x)$ $f'(x) > 0$

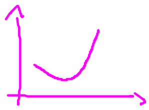
$1 + \cos(x) > 0$ $\cos(x) > -1$

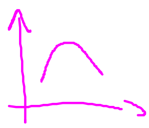


a. $\boxed{x \neq \pi + 2k\pi \quad k \in \mathbb{Z} \Rightarrow f'(x) > 0 \longrightarrow \text{INCREASING}}$

$x = \pi + 2k\pi \quad k \in \mathbb{Z} \Rightarrow f'(x) = 0 \longrightarrow$ STATIONARY POINT (MAX, MIN, INFL. POINT)



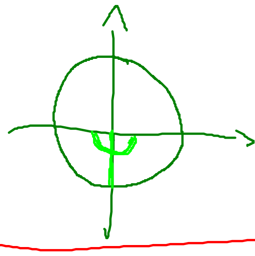
$f''(x) > 0$ CONVEX 

$f''(x) < 0$ CONCAVE 

$f''(x) = -\sin(x)$ $f''(x) > 0$

$-\sin(x) > 0 \longrightarrow \sin(x) < 0$

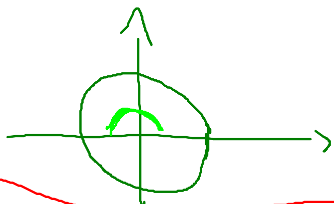
$\cdot(-1)$ $\cdot(-1)$



$\boxed{\pi + 2k\pi < x < 2\pi + 2k\pi \longrightarrow f''(x) > 0 \longrightarrow f(x) \text{ is CONVEX}}$

b. $f''(x) < 0$

$\sin(x) > 0$



$\boxed{0 + 2k\pi < x < \pi + 2k\pi \longrightarrow f''(x) < 0 \longrightarrow f(x) \text{ is CONCAVE}}$

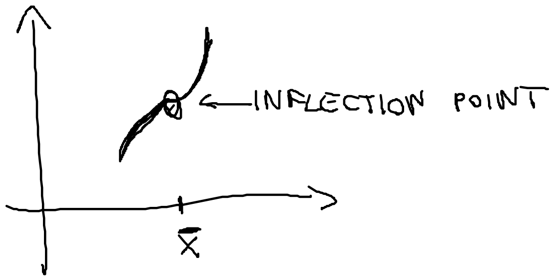
c. $\bar{x} = \pi + 2k\pi \quad k \in \mathbb{Z}$

$f'(x) = 1 + \cos(x)$

$f''(x) = -\sin(x)$

$f'(\bar{x}) = 0 \quad f''(\bar{x}) = 0$

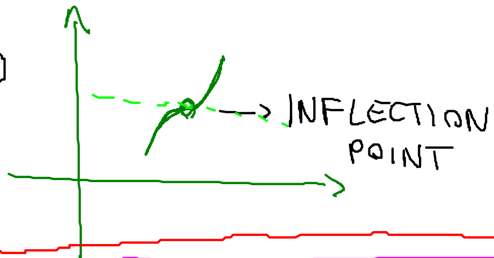
1. Look at $f'(x)$ before and after the point $\bar{x}$



2. Compute the $f'''(\bar{x})$

$f'''(x) = -\cos(x) \qquad f'''(\bar{x}) = 1$

$f'''(\bar{x}) > 0$



$\bar{x} \quad f'(\bar{x}) = f''(\bar{x}) = 0$

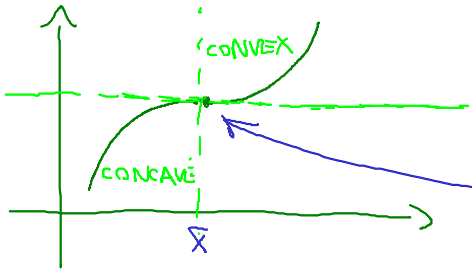
$f'''(\bar{x}) > 0$ INFLECTION

$f'''(\bar{x}) < 0$ INFLECTION

$f'''(\bar{x}) = 0$ CANNOT SAY

compute $f''''(\bar{x})$

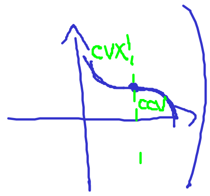
Inflection point with horizontal tangent



$f'(\bar{x}) = 0$

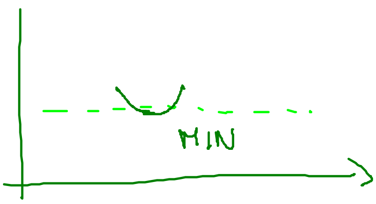
$f''(\bar{x}) = 0$

$f'''(\bar{x}) > 0$ $\quad \Big(f'''(\bar{x}) < 0$



$\Big)$

Example, what happens when $f'(\bar{x}) = 0$

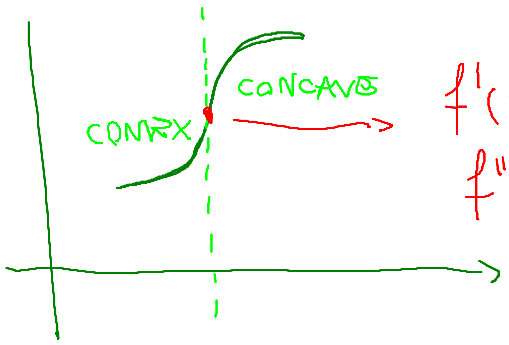


$f''(\bar{x}) > 0$

$f''(\bar{x}) < 0$

$f''(\bar{x}) = 0$

Inflection point with non-zero tangent

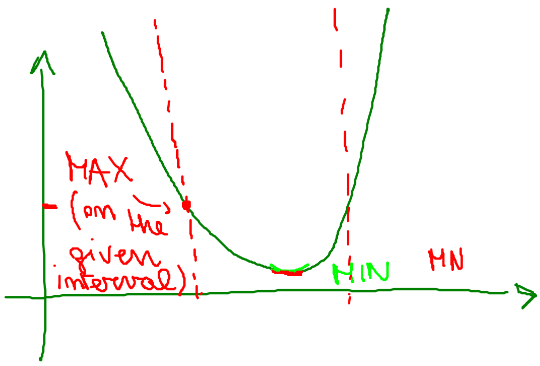


$f'(x) \neq 0$

$f''(x) = 0$

2) Compute maxima and minima of the following function in the given interval.

$f(x) = x^3 - 3x + 7$ in $[0,4]$

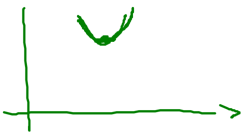


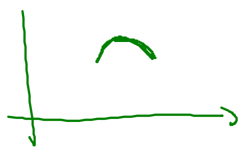
$f'(x) = 3x^2 - 3$

$f'(\bar{x}) = 0 \qquad 3\bar{x}^2 - 3 = 0 \qquad 3\bar{x}^2 = 3 \qquad \bar{x}^2 = 1 \qquad \boxed{\bar{x}_\pm = \pm 1}$

$f''(x) = 6x$

$\boxed{f''(\underset{+1}{\bar{x}_+}) = 6 \quad \text{MINIMUM}}$ $\qquad f''(\underset{-1}{\bar{x}_-}) = -6$ MAXIMUM $\to$ $\bar{x}_-$ is outside the interval $[0,4]$





- Check stationary p. ($f'(x) = 0$)
- Check boundary p.

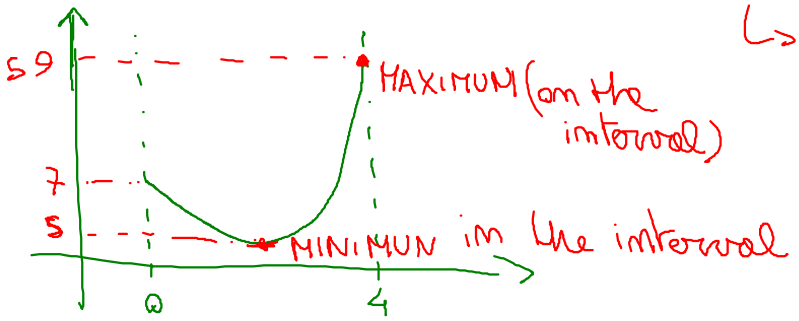
$f(\bar{x}_+) = 1 - 3 + 7 = 5$ MINIMUM (stationary point)

$f(0) = 7$

$f(4) = 4^3 - 12 + 7 = 64 - 12 + 7 = 71 - 12 = 59$

} Boundary points

$\to$ MAXIMUM in the interval



$\boxed{\begin{array}{l} \text{MAX at } x = 4 \\ \text{MIN at } x = 1 \end{array}}$

Sketch of the plot of a function

3) For each function determine (if possible)

a. the domain

b. the sign

c. the asymptotes (compute the limits)

d. the intervals in which the function is increasing and decreasing

e. the intervals " " " " " concave and convex

f. local maxima, local minima and inflection points

Finally, use the information collected above to sketch the graph of the function.

1. $f(x) = x^3 - x^2$

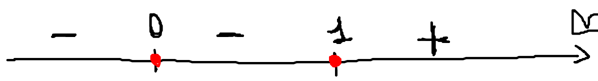
a. Domain

$$\boxed{\begin{array}{l} \text{C.E.} \\ x \in \mathbb{R} \end{array}}$$

b. Sign

$f(x) > 0$ $\qquad$ $f(x) = 0$ $\qquad$ $x^2(x-1) = 0 \longrightarrow \begin{array}{l} x = 0 \\ x = 1 \end{array}$ $\qquad$ • = zero of the function

$x^3 - x^2 > 0$ $\qquad$ $\underbrace{x^2}_{\geq 0}(x-1) > 0 \Rightarrow (x-1) > 0$ $\qquad$ $x > 1$



$$\boxed{f(x) > 0 \quad x > 1} \Rightarrow \boxed{f(x) < 0 \quad x < 1}$$

$$\boxed{f(x) = 0 \quad \begin{array}{l} x = 0 \\ x = 1 \end{array}}$$

c. asymptotes

$f(x) = x^3 - x^2$

A vertical

• $\lim_{x \to a} f(x) = \pm\infty$

a is finite

B horizontal

• $\lim_{x \to \pm\infty} f(x) = a$

A We do not have vertical asymptotes

B $\lim\limits_{x \to +\infty} x^3 - x^2 = \lim\limits_{x \to +\infty} x^3 \left(1 - \underbrace{\frac{1}{x}}_{0}\right) = +\infty$

$\lim\limits_{x \to -\infty} x^3 - x^2 = \lim\limits_{x \to -\infty} x^3 \left(1 - \underbrace{\frac{1}{x}}_{0}\right) = -\infty$

No horizontal asymptotes

$\boxed{\lim\limits_{x \to +\infty} f(x) = +\infty \qquad \lim\limits_{x \to -\infty} f(x) = -\infty}$

$\boxed{\text{No asymptotes}}$

d. Increasing and decreasing

$f(x) = x^3 - x^2$

$f'(x) = 3x^2 - 2x \qquad f'(x) > 0$

$3x^2 - 2x > 0 \qquad \underset{A}{x}\,\underset{B}{(3x-2)} > 0$

		$x<0$	$0<x<\frac{2}{3}$	$x>\frac{2}{3}$
A	$x > 0$	−	+	+
B	$x > \frac{2}{3}$	−	−	+
		+	−	+
$f(x)$		↗	↘	↗
		$f'(x)>0$	$f'(x)<0$	$f'(x)>0$

$f'(x) = 0$

$x = 0 \qquad x = \frac{2}{3}$

• = stationary point

$\boxed{\begin{array}{l} f(x) \text{ is increasing for } x<0 \vee x>\frac{2}{3} \\ f(x) \text{ is decreasing for } 0<x<\frac{2}{3} \\ f(x) \text{ is stationary for } x=0 \vee x=\frac{2}{3} \end{array}}$

e. Concave or convex

$f(x) = x^3 - x^2$

$f'(x) = 3x^2 - 2x$

$f''(x) = 6x - 2$ $\qquad f''(x) > 0$

$6x - 2 > 0 \qquad x > \frac{2}{6}$

$\frac{2}{6}$

$-$ $\qquad +$

$f''(x) < 0 \qquad f''(x) > 0$

$f(x)$

Inflection point with nonzero tangent

$f(x)$ is concave for $x < \frac{2}{6}$

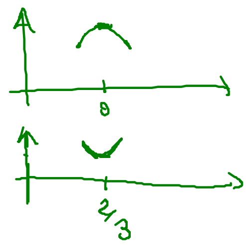
$f(x)$ is convex for $x > \frac{2}{6}$

f. Max, min and inflection points

$\bar{x}_1 = 0 \qquad \bar{x}_2 = \frac{2}{3} \qquad\qquad f''(x) = 6x - 2$

$f''(\bar{x}_1) = f''(0) = -2 \longrightarrow$ MAX

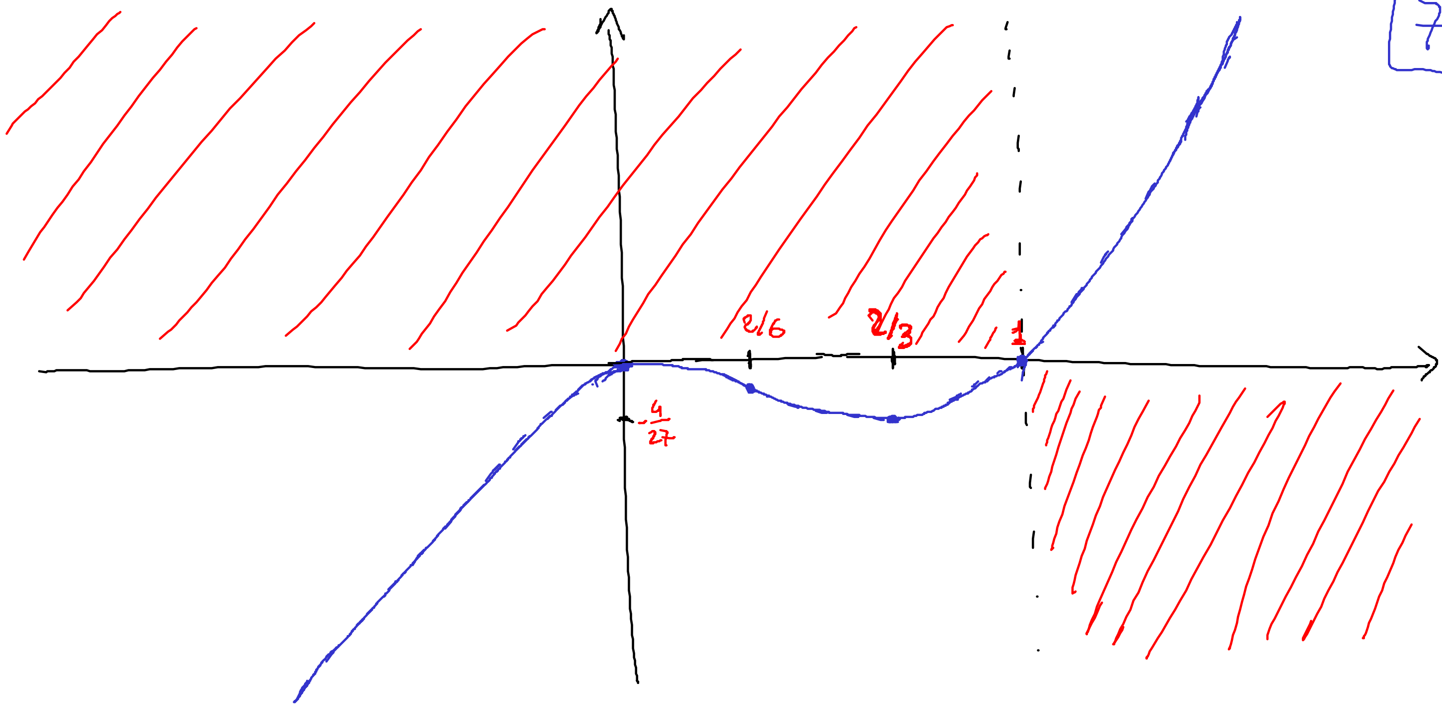
$f''(\bar{x}_2) = f''\left(\frac{2}{3}\right) = \cancel{6}^{2} \cdot \frac{2}{\cancel{3}_{1}} - 2 = 2 \longrightarrow$ MIN



$\bar{x}_1 = 0$ is a maximum

$\bar{x}_2 = \frac{2}{3}$ is a minimum

$\bar{x}_3 = \frac{2}{6}$ is an inflection point with nonzero tangent



$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2 = \frac{8}{27} - \frac{4}{9} = \frac{8-12}{27} = -\frac{4}{27}$$

$$\sim -\frac{4}{30}$$

$$\sim -\frac{0.4}{3}$$

$$\sim -0.13$$