MATHEMATICS 1
ADDITIONAL EXERCISES N. 6

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Notation: log stands for the natural logarithm (i.e. the logarithm with the basis e)

1. Derivatives and Applications

(1) Use the definition of the derivative to compute the derivative of the following functions at the given point x_0

$$\begin{aligned} f(x) &= 3x^2 + 1, & x_0 &= 0 \\ f(x) &= 5x + 7, & x_0 &= 1 \\ f(x) &= \sqrt{x}, & x_0 &= 2 \\ f(x) &= \frac{x+1}{x}, & x_0 &= -3 \\ f(x) &= \log(x+3), & x_0 &= 0 \\ f(x) &= \sqrt{x^2+2}, & x_0 &= 3 \\ f(x) &= 3x^2 + 1, & x_0 &= 0 \\ f(x) &= |x^2 - 4|, & x_0 &= -1 \\ f(x) &= \frac{5}{x+1}, & x_0 &= -2 \\ f(x) &= e^{2x}, & x_0 &= 0 \end{aligned}$$

(2) Compute the derivatives of the following functions

1. $f(x) = x + \sin(x) + 3\cos(x)$
2. $f(x) = e^x + \log(x) + 3x^2$
3. $f(x) = \sqrt{x} + 3x\sqrt{x} + 2\sqrt[3]{x^2}$
4. $f(x) = \dfrac{7}{x^2} + \dfrac{2}{e^x}$

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5. $f(x) = x^3 - 3\sin(x)$
6. $f(x) = x^2 e^x$
7. $f(x) = 2\sqrt{x}\log(x)$
8. $f(x) = x\log(x)$
9. $f(x) = \sqrt{x}\cos(x) + \sin(x)$
10. $f(x) = x^2\sin(x)$
11. $f(x) = e^x(2 - e^x) + \sin(x) + 3\cos(x)$
12. $f(x) = (x^2 - x)e^x$
13. $f(x) = e^x(\cos(x) + \sin(x))$
14. $f(x) = \tan(x)$
15. $f(x) = \tan(x)(1 - \tan(x))$
16. $f(x) = (1 - 2\cos(x))\tan(x)$
17. $f(x) = x^2\arctan(x)$
18. $f(x) = \sqrt{1 - x^2}\arcsin(x)$
19. $f(x) = x^3\sin(x)\log(x)$
20. $f(x) = (3x + 1)^4$
21. $f(x) = 2\sin(3x) - 4$
22. $f(x) = e^{2x+1} + \log(x^2 + x)$
23. $f(x) = (x^2 + 2x)^4\left(\dfrac{3}{x} - 2x\right)$
24. $f(x) = \sqrt{3x^2 + 2x + 5}$
25. $f(x) = \sqrt{\log(x^4 + x^2 + 1)}$
26. $f(x) = \log^2(x) + 3x + 5$
27. $f(x) = (x + 1)\log^3(x)$
28. $f(x) = \dfrac{4x^2 - 5}{x + 1}$
29. $f(x) = \dfrac{2x - 1}{\sqrt{x^2 + 1}}$
30. $f(x) = \dfrac{1}{x + \sqrt{x^2 - 1}}$
31. $f(x) = \dfrac{\sqrt{x}}{x + 2}$
32. $f(x) = \dfrac{e^x + e^{-x}}{1 - e^x}$
33. $f(x) = \dfrac{\log(x)}{1 + \log(x)}$
34. $f(x) = \dfrac{1 - 2\cos(x)}{\sin(x)}$
35. $f(x) = \dfrac{1 - \tan(x)}{\sin(x) - \cos(x)}$
36. $f(x) = \cos(x^4 + 4x)$
37. $f(x) = e^{5-x^2}$
38. $f(x) = \log\left(\dfrac{x}{x - 1}\right)$
39. $f(x) = \log(\sqrt{x^2 + 4})$
40. $f(x) = \log(\sin^2(3x + 1))$
41. $f(x) = \log(e^x + e^{-x})$
42. $f(x) = \log(\cos(x)) + x\tan(x)$
43. $f(x) = xe^{\frac{x-1}{x}}$
44. $f(x) = \log\left(\dfrac{x}{x + 1}\right) + \dfrac{1}{x} - \dfrac{1}{2x^2}$
45. $f(x) = \arctan 2x + 3$
46. $f(x) = \arcsin e^x$
47. $f(x) = (e^{3x^2+2} - 2x)^5$
48. $f(x) = \sqrt{x}\log(\sqrt{x})$
49. $f(x) = xe^{\sqrt{x}}$
50. $f(x) = x^2\log(x + 1)$

(3) For each of the following functions, compute the equation of the tangent line at the given point x_0

$$\begin{aligned} f(x) &= 3x^2 - 9x + 4, & x_0 &= 1 \\ f(x) &= x^2 + 1, & x_0 &= 1 \\ f(x) &= \frac{x+1}{x}, & x_0 &= -1 \\ f(x) &= 5 + \log(x), & x_0 &= 1 \\ f(x) &= \sqrt{1-x^3}, & x_0 &= 1 \\ f(x) &= e^{4x-1}, & x_0 &= -\frac{1}{2} \\ f(x) &= \frac{x^2}{x+3}, & x_0 &= -1 \\ f(x) &= \log(3x+2), & x_0 &= 0 \end{aligned}$$

(4) For each of the following functions, say if they continuous and differentiable, and if not, identify the nature of discontinuity and non-differentiability points

1. $f(x) = |9 - x^2|$
2. $f(x) = |x^2 + 3|$
3. $f(x) = \sqrt{x^2 - 4x + 3}$
4. $f(x) = \sqrt[3]{2x^2 - 8}$
5. $f(x) = \sqrt[3]{3x - 1}$
6. $f(x) = x\sqrt{x}$
7. $f(x) = |x| + |x + 1|$
8. $f(x) = \dfrac{1}{\log(x)}$
9. $f(x) = x\sqrt[3]{x}$
10. $f(x) = \dfrac{|x^2 - x|}{x}$

(5) For each of the following functions, compute the first order Taylor approximation $P(x)$ at the point x_0. Evaluate also the error (absolute and in percentage) that is made by

approximating the function $f(x_1)$ with $P(x_1)$.

$$
\begin{aligned}
f(x) = \sqrt{x+1}, &\qquad x_0 = 0, &\qquad x_1 = 0.1 \\
f(x) = x^3, &\qquad x_0 = 2, &\qquad x_1 = 2.01 \\
f(x) = \frac{1}{x+1}, &\qquad x_0 = 0, &\qquad x_1 = 0.01 \\
f(x) = \frac{x+1}{x}, &\qquad x_0 = -1, &\qquad x_1 = -0.9 \\
f(x) = \log(x+1), &\qquad x_0 = 0, &\qquad x_1 = 0.2 \\
f(x) = \sqrt{1-x^3}, &\qquad x_0 = 1, &\qquad x_1 = 1.01 \\
f(x) = e^x, &\qquad x_0 = 0, &\qquad x_1 = -0.2 \\
f(x) = \sin(x), &\qquad x_0 = 0, &\qquad x_1 = \frac{\pi}{6}
\end{aligned}
$$

(6) Determine the intervals in which the functions are increasing and decreasing

1. $f(x) = x^2 + 2x$
2. $f(x) = x^2 - 2x + 3$
3. $f(x) = 2x^4 - 27x$
4. $f(x) = \dfrac{1}{x+1}$
5. $f(x) = \dfrac{x^2-1}{x}$
6. $f(x) = \sqrt{x+1}$
7. $f(x) = x\sqrt{2x+1}$
8. $f(x) = x + \sin(x)$
9. $f(x) = x + \cos(x)$
10. $f(x) = \log(x) - \dfrac{e}{x}$
11. $f(x) = x^2 - \log(x^2-1)$
12. $f(x) = \dfrac{1+\log(x)}{x}$
13. $f(x) = \log^3(x) - \log^2(x)$
14. $f(x) = \log\left(\dfrac{x^2+4}{x^2-4}\right)$
15. $f(x) = e^x - e^{-x}$
16. $f(x) = (x+1)e^x$
17. $f(x) = \dfrac{x-1}{e^x}$
18. $f(x) = \dfrac{x+1}{x-2}$
19. $f(x) = 1 + 2\sqrt{x^2-1}$
20. $f(x) = \dfrac{1}{1-\sqrt{x}}$

(7) Determine the intervals in which the functions are concave and convex

1. $f(x) = 4x^3 - 1$
2. $f(x) = x^3 - 3x^2 + 3$
3. $f(x) = 2x^4 - 2x^3$
4. $f(x) = \dfrac{1}{x-1}$
5. $f(x) = \dfrac{x}{x+2}$
6. $f(x) = x\sqrt{x^2+1}$
7. $f(x) = e^{-x^2+2x}$
8. $f(x) = x^2 - x\log(x)$
9. $f(x) = \dfrac{x^3-1}{x^3+1}$
10. $f(x) = \log^3(x) - 9\log(x)$
11. $f(x) = x(x+2)^3$
12. $f(x) = \dfrac{1}{4}x^4 - 2x^2$
13. $f(x) = \dfrac{x^3+1}{x}$
14. $f(x) = x^2 - \sin(x)$
15. $f(x) = e^{x^2-1}$
16. $f(x) = \log(x^3-1)$
17. $f(x) = \dfrac{x}{e^x}$
18. $f(x) = \log^2(x+1)$
19. $f(x) = \sqrt{2x-1}$
20. $f(x) = x^2 - \sqrt{x}$

(8) Compute the stationary points of the following functions and determine if these are local maxima, local minima or inflection points with horizontal tangents

1. $f(x) = x^3 - 9x$
2. $f(x) = x^4 - 2x^2$
3. $f(x) = x^4 - 4x^3 + 5$
4. $f(x) = \dfrac{1}{x-1}$
5. $f(x) = \dfrac{x}{x+1}$
6. $f(x) = \dfrac{3-x}{x+2}$
7. $f(x) = \dfrac{x^2}{x^2+1}$
8. $f(x) = x\sqrt[3]{x}$
9. $f(x) = e^{\frac{1-x}{x^2}}$
10. $f(x) = \log(x^2+1)$
11. $f(x) = \dfrac{\log(x)}{x}$
12. $f(x) = \log(2x - x^2)$
13. $f(x) = xe^{-x^2}$
14. $f(x) = e^{x^3-6x^2}$
15. $f(x) = \dfrac{x^2-3}{x-5}$
16. $f(x) = x^2 + \dfrac{1}{x}$
17. $f(x) = \dfrac{x^2}{x^2+3x-3}$
18. $f(x) = \dfrac{x^2+4x}{x^2+6x+5}$
19. $f(x) = \log^4(x) - \log^2(x)$
20. $f(x) = x - \sqrt{x}$

(9) Compute the maxima and minima of the following functions in the given intervals

1. $f(x) = \sqrt{x^2+1}$ in $[-1,1]$
2. $f(x) = \dfrac{x^2}{x^2+1}$ in $[-1,1]$
3. $f(x) = x^3 - 3x + 7$ in $[0,4]$
4. $f(x) = x\sqrt[3]{x}$ in $[-2,-1]$
5. $f(x) = \dfrac{x^2+x+1}{(x-1)^2}$ in $[-2,2]$
6. $f(x) = e^{|x-1|}$ in $[-3,3]$
7. $f(x) = \dfrac{\log(x)}{x}$ in $[1,e^2]$
8. $f(x) = \dfrac{1}{x^2-3}$ in $[-1,2]$

(10) Use De L'Hopital rule to compute the following limits

1. $\lim_{x\to 0} \dfrac{x\cos(x) - \sin(x)}{x^3}$
2. $\lim_{x\to 1} \dfrac{\sqrt[3]{x} - 1}{x-1}$
3. $\lim_{x\to 0} \dfrac{e^x - e^{\sin(x)}}{x + \sin(x)}$
4. $\lim_{x\to 1} \dfrac{e^x - e}{x-1}$
5. $\lim_{x\to 0} \dfrac{x - \log(1+x)}{x}$
6. $\lim_{x\to -\infty} xe^x$
7. $\lim_{x\to 0} \dfrac{1}{\sin^2(x)} - \dfrac{1}{x^2}$
8. $\lim_{x\to +\infty} \dfrac{\log(x^2+3x+5)}{x^2+1}$
9. $\lim_{x\to 0} \dfrac{2\cos(x) - 2 + x^2}{x^4}$
10. $\lim_{x\to 0^+} \dfrac{1}{x} + \log(x)$

2. Sketch of the plot of a function

(1) for each of the following functions determine, if possible
   (a) the domain
   (b) the sign
   (c) the asymptotes
   (d) the intervals in which functions are increasing and decreasing
   (e) the intervals where the functions are concave and convex
   (f) local maxima, local minima and inflection points

   Finally use the information collected above to sketch the graph of the functions.

1. $f(x) = x^3 - x^2$
2. $f(x) = 3x^4 - 4x^3$
3. $f(x) = \dfrac{3x^2}{x^2+1}$
4. $f(x) = \dfrac{x^2+2}{x^2-x}$
5. $f(x) = \left(1 + \dfrac{1}{x}\right)^3$
6. $f(x) = \sqrt{x + \dfrac{1}{x}}$
7. $f(x) = \dfrac{\sqrt{1-x}}{x}$
8. $f(x) = \dfrac{(x+2)|x|}{1-x^2}$
9. $f(x) = 2 + e^{-x+3}$
10. $f(x) = e^{\frac{x-1}{2-x}}$
11. $f(x) = \dfrac{\log(x)}{2 - \log(x)}$
12. $f(x) = \dfrac{\log(x+1)}{x}$
13. $f(x) = \dfrac{\log^2(x)}{x}$
14. $f(x) = 1 - e^{-x^2}$
15. $f(x) = \sqrt{1 - e^x}$
16. $f(x) = x - \sqrt{x^2 - 2x}$
17. $f(x) = \dfrac{x-1}{e^x}$
18. $f(x) = \dfrac{x+1}{x-2}$
19. $f(x) = 1 + 2\sqrt{x^2 - 1}$
20. $f(x) = \dfrac{1}{1 - \sqrt{x}}$