Week 6

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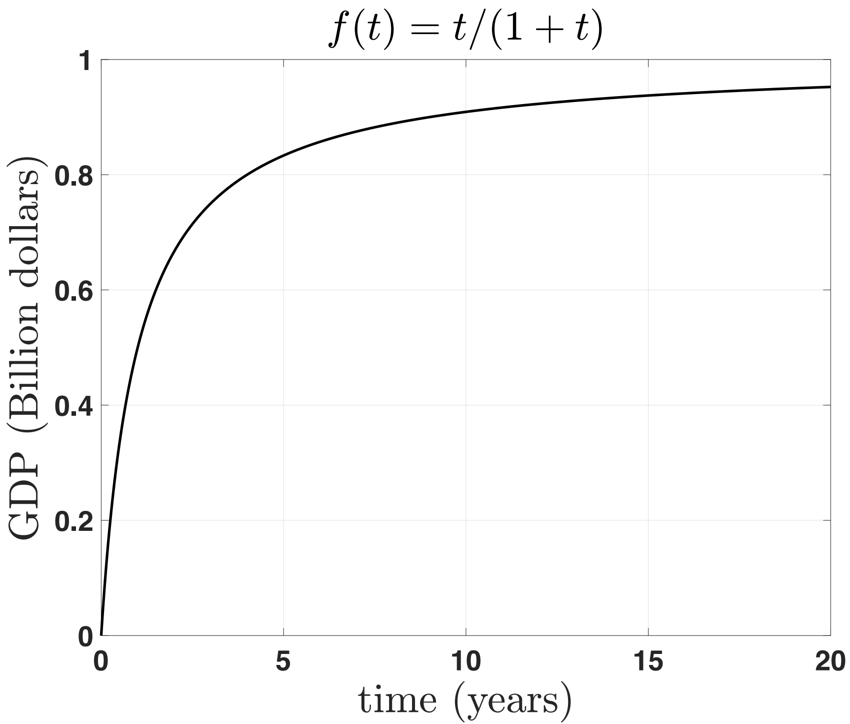
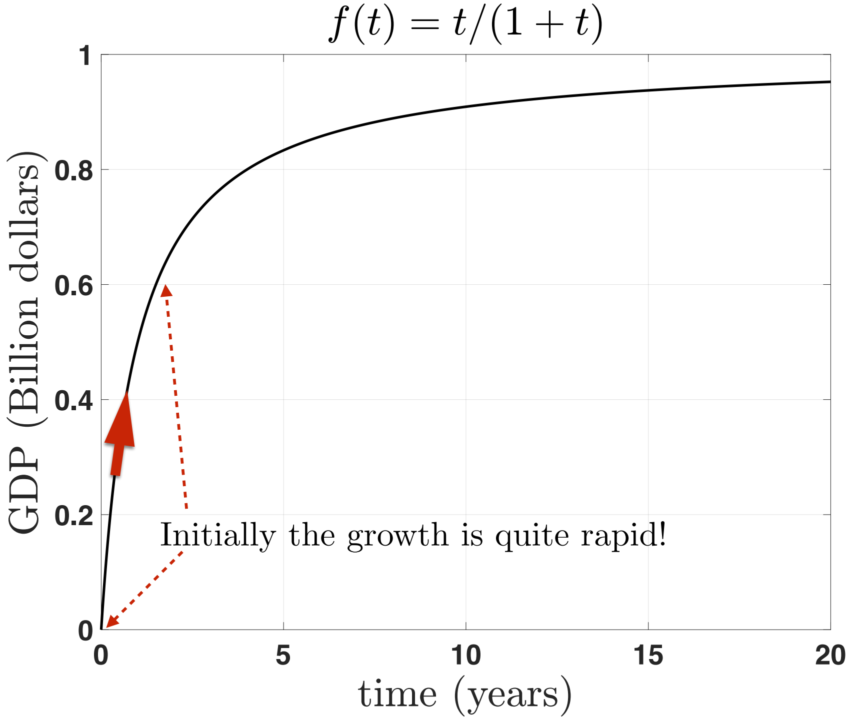
Mathematics I

University of Rome Tor Vergata

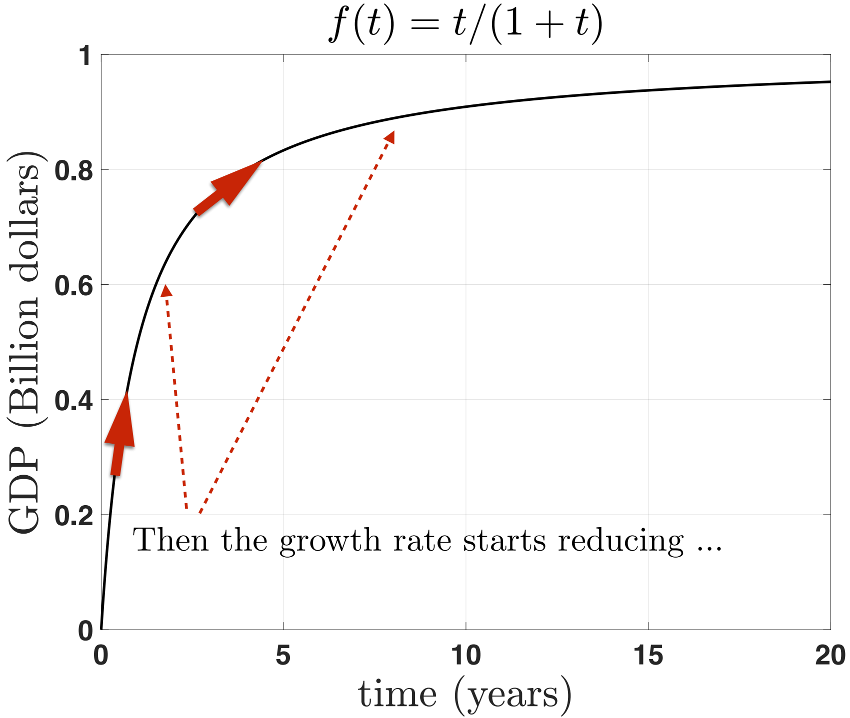
18-21 October, 2021

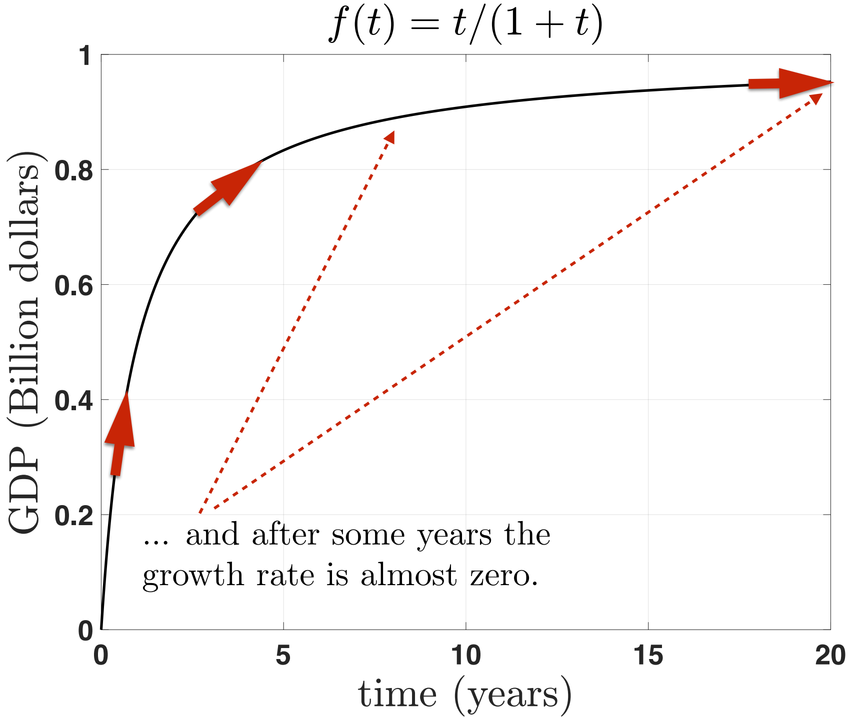
Outline

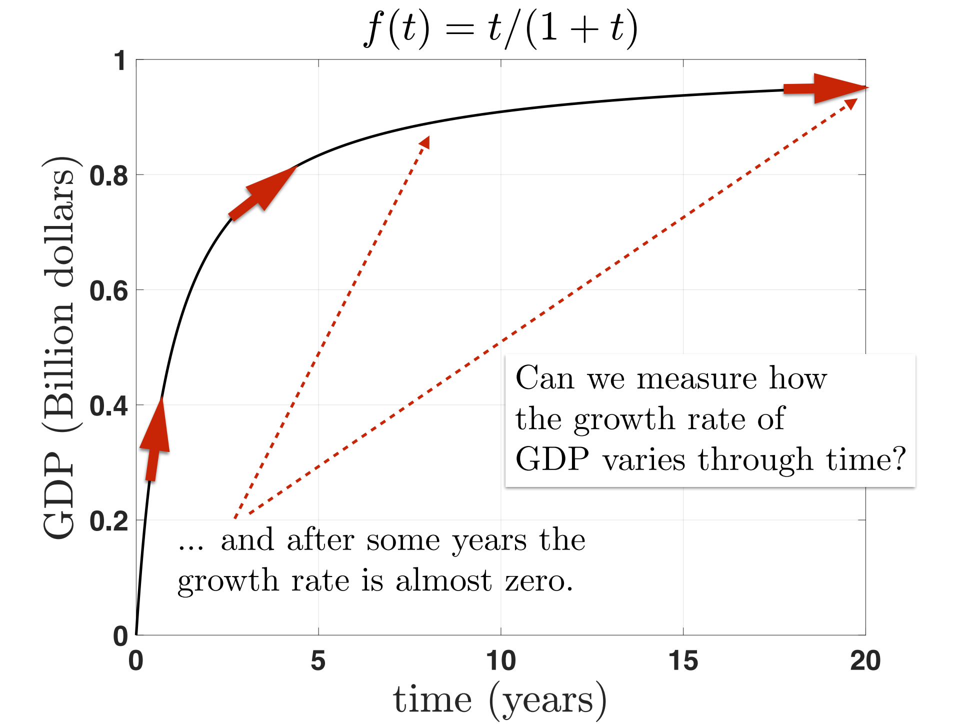

$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

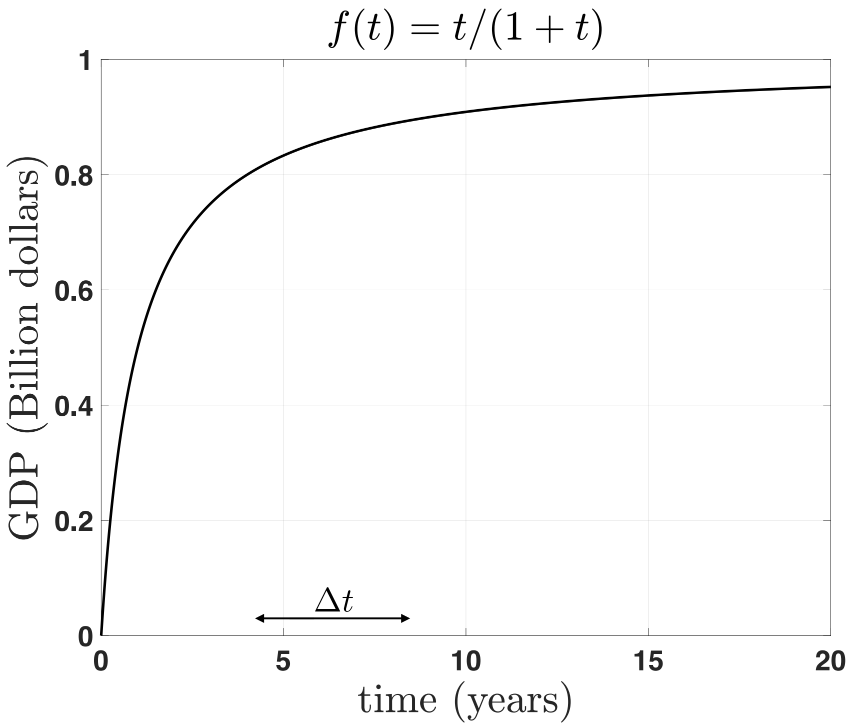


$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Initially the growth is quite rapid!
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

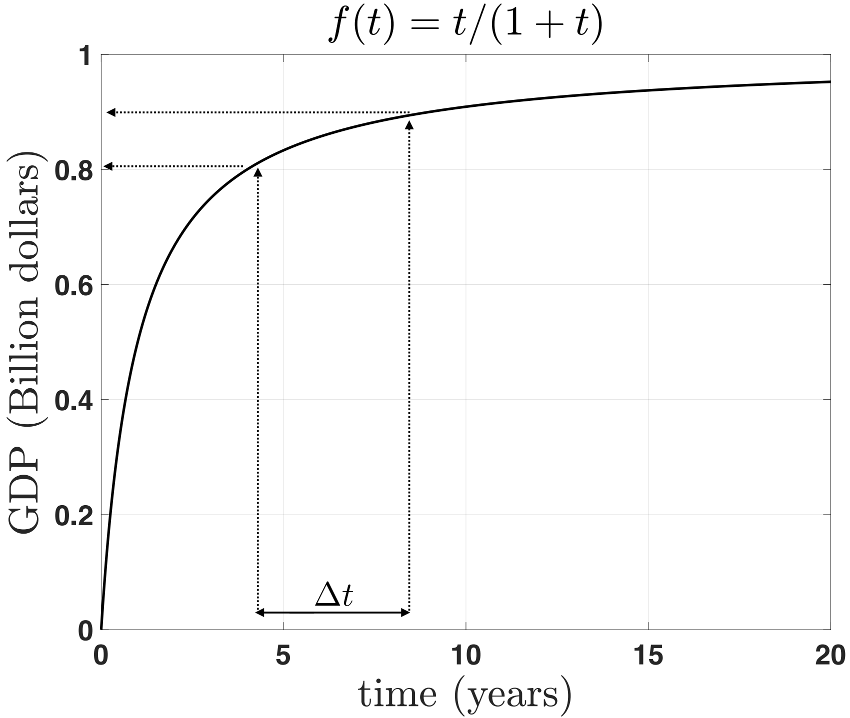

$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Then the growth rate starts reducing ...
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20


$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
... and after some years the growth rate is almost zero.

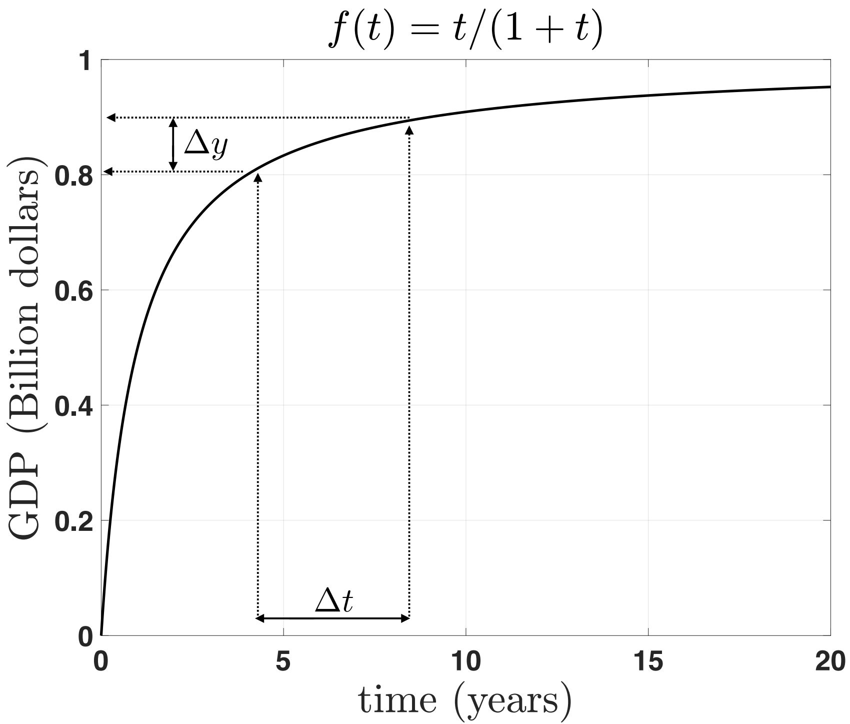

$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
Can we measure how the growth rate of GDP varies through time?
... and after some years the growth rate is almost zero.



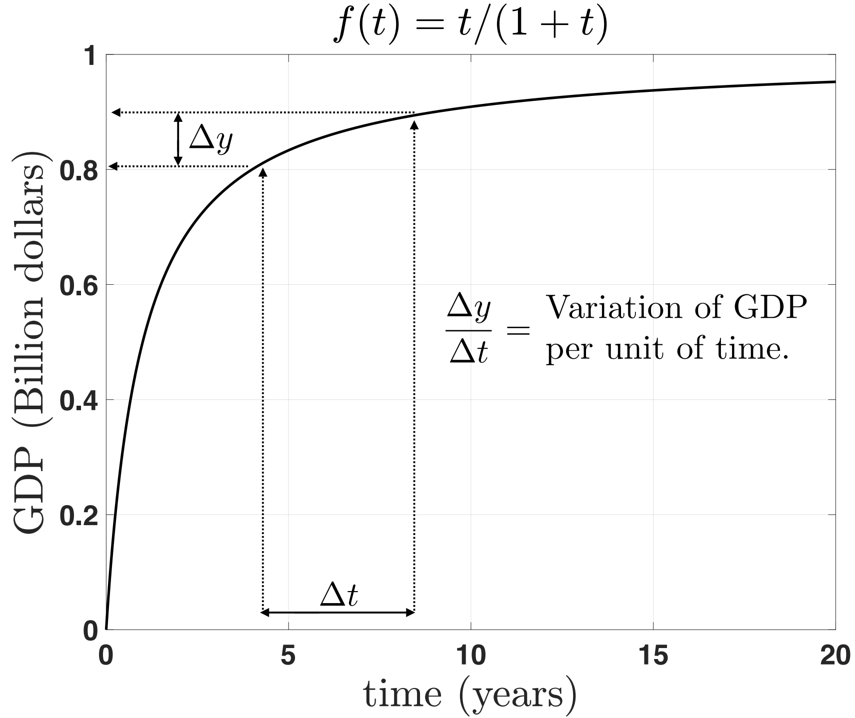
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δt
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

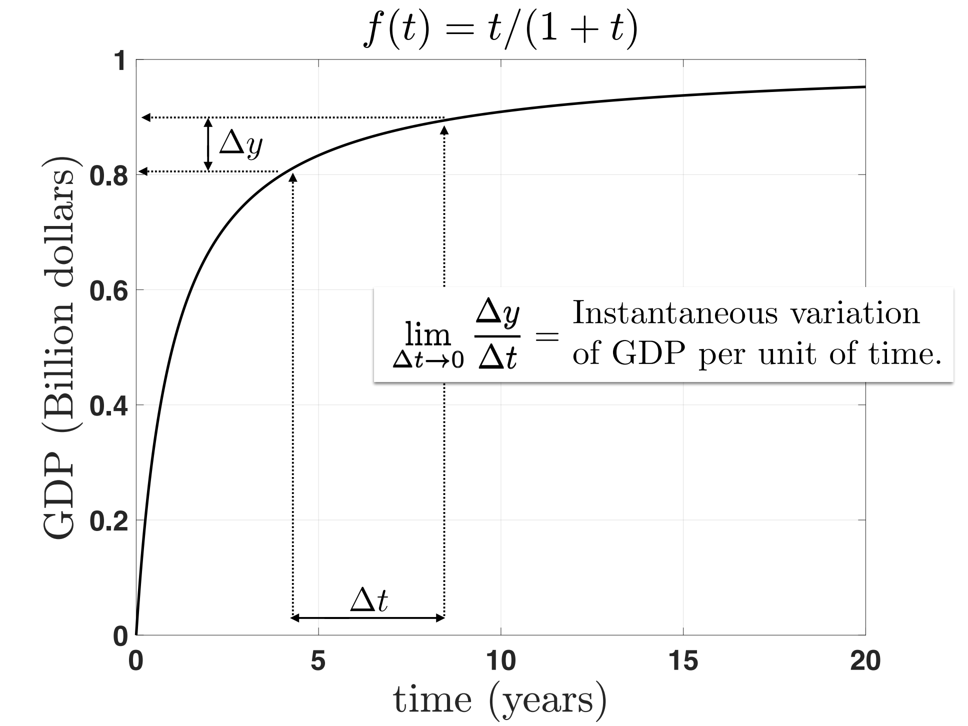


f(t) = t/(1 + t)
GDP (Billion dollars)
time (years)
Δt
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

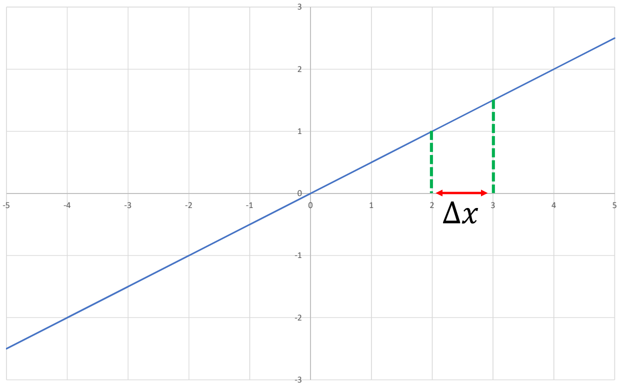


$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δy
Δt
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

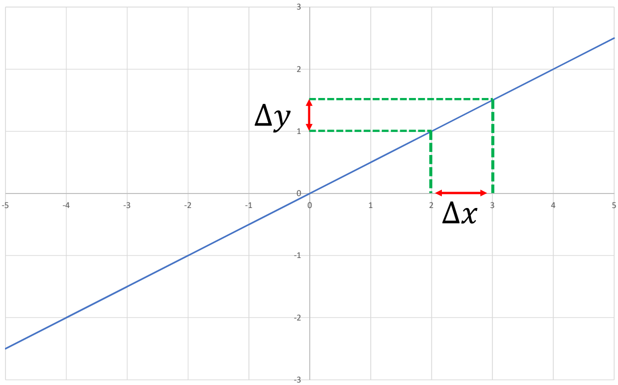

$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δy
Δt
$\frac{\Delta y}{\Delta t} =$ Variation of GDP per unit of time.
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20


$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
Δy
Δt
$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} =$ Instantaneous variation of GDP per unit of time.

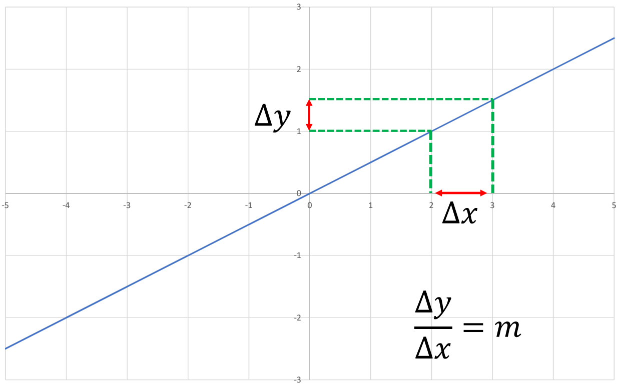
3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5



3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Δx



3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Δy
Δx



Slope of the line indicates the rate of change

Incremental Ratio

Definition

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0, x_1 \in (a, b)$. We call incremental ratio the ratio

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_1; f(x_1))$

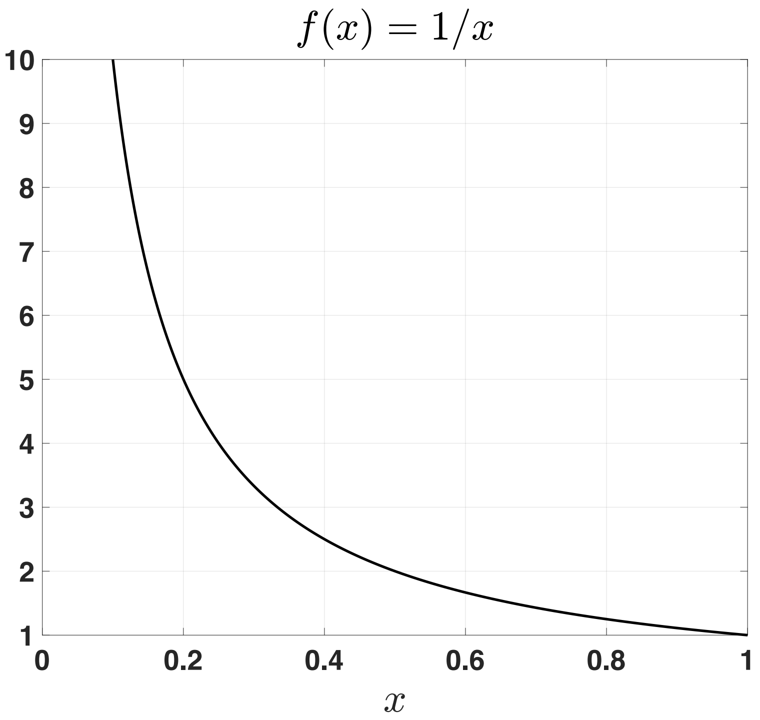
Incremental Ratio

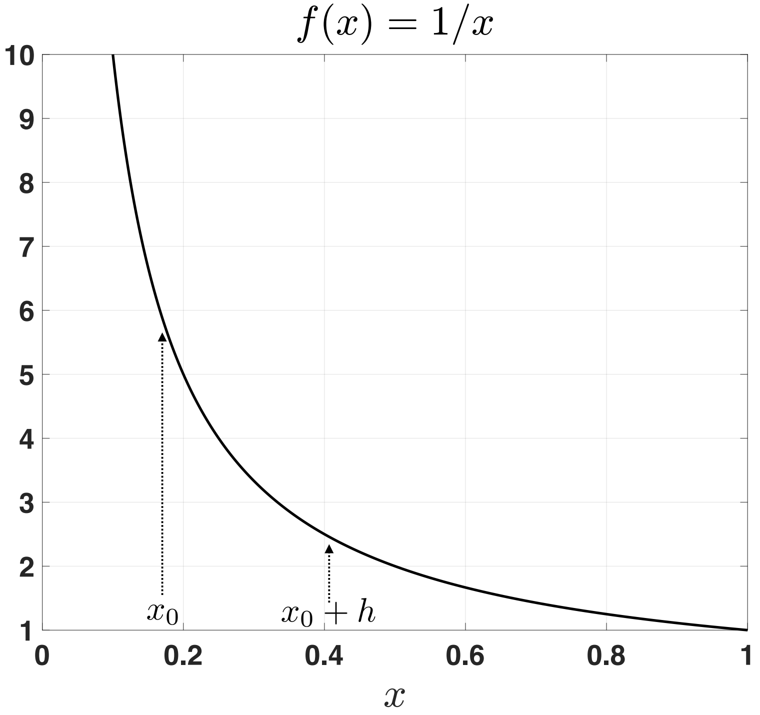
Definition

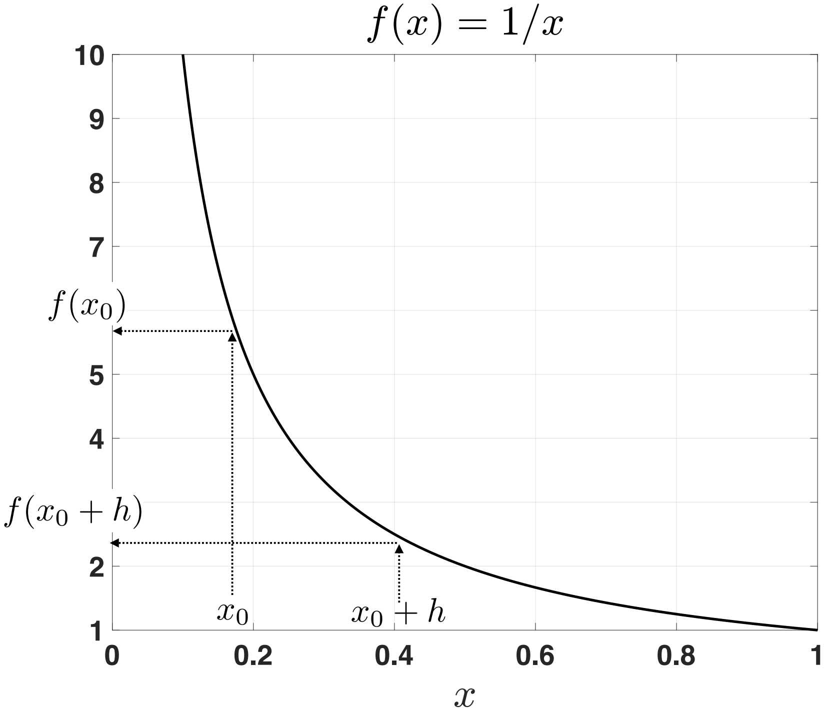
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0 \in (a, b)$ and let $h \in \mathbb{R}$ such that $x_0 + h \in (a, b)$. We call incremental ratio the ratio

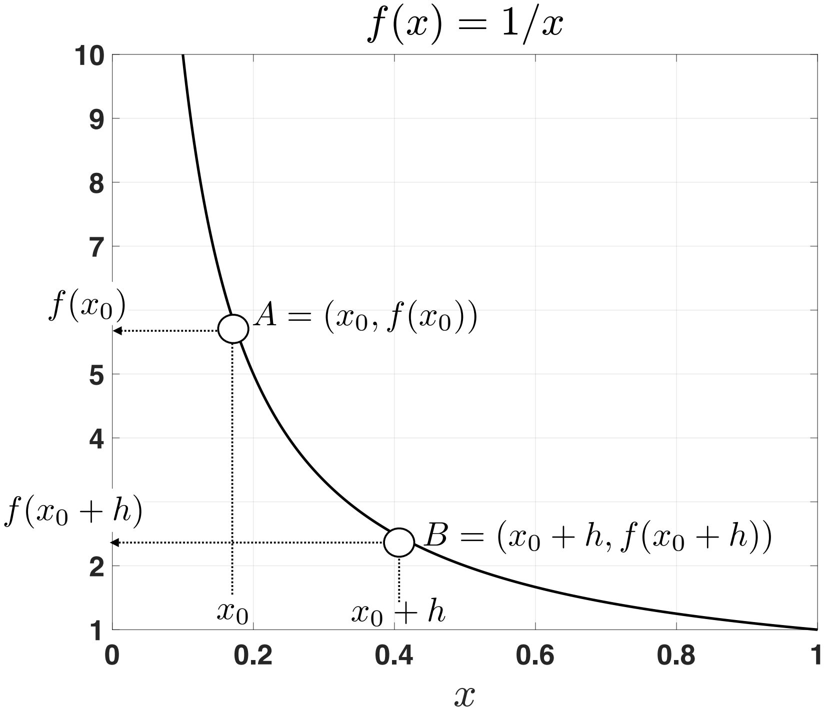
$$\frac{f(x_0 + h) - f(x_0)}{h}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_0 + h; f(x_0 + h))$

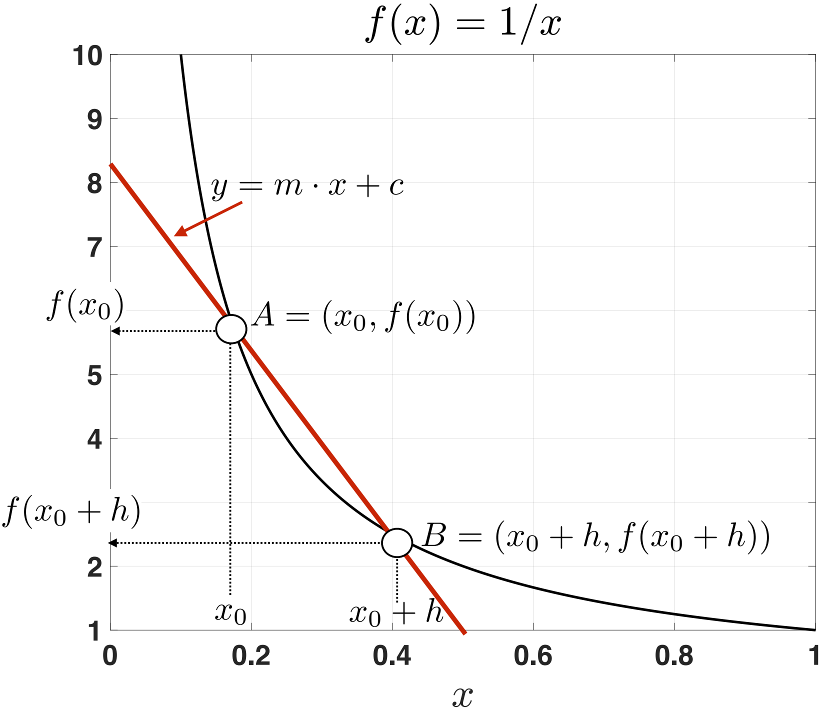

$f(x) = 1/x$
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x


$f(x) = 1/x$
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x_0
$x_0 + h$
x


$f(x) = 1/x$
10
9
8
7
$f(x_0)$
5
4
$f(x_0 + h)$
2
1
x_0
$x_0 + h$
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
10
9
8
7
$f(x_0)$
$A = (x_0, f(x_0))$
5
4
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
2
1
x_0
$x_0 + h$
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
$y = m \cdot x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
x_0
$x_0 + h$
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1

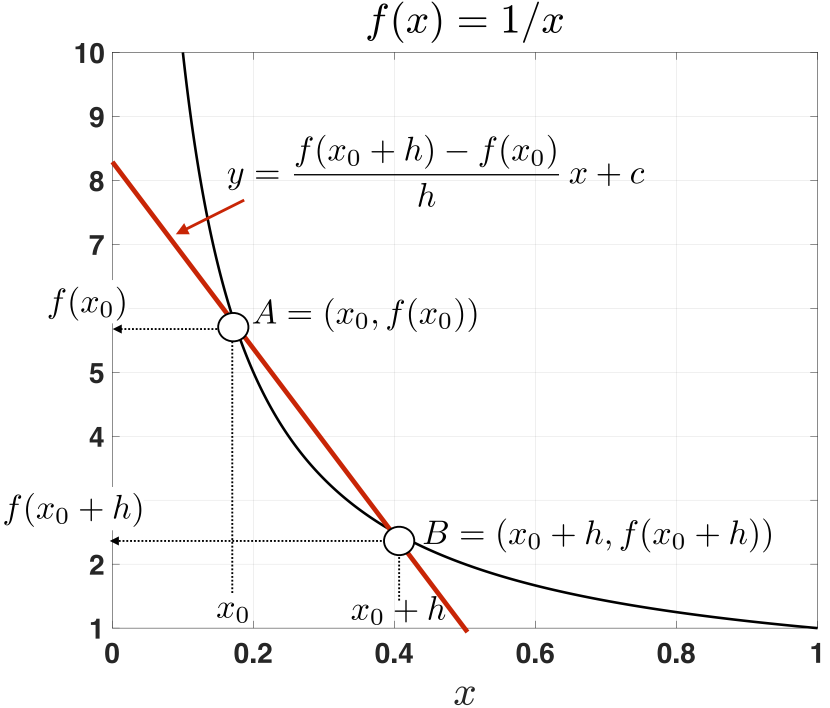
Derivatives

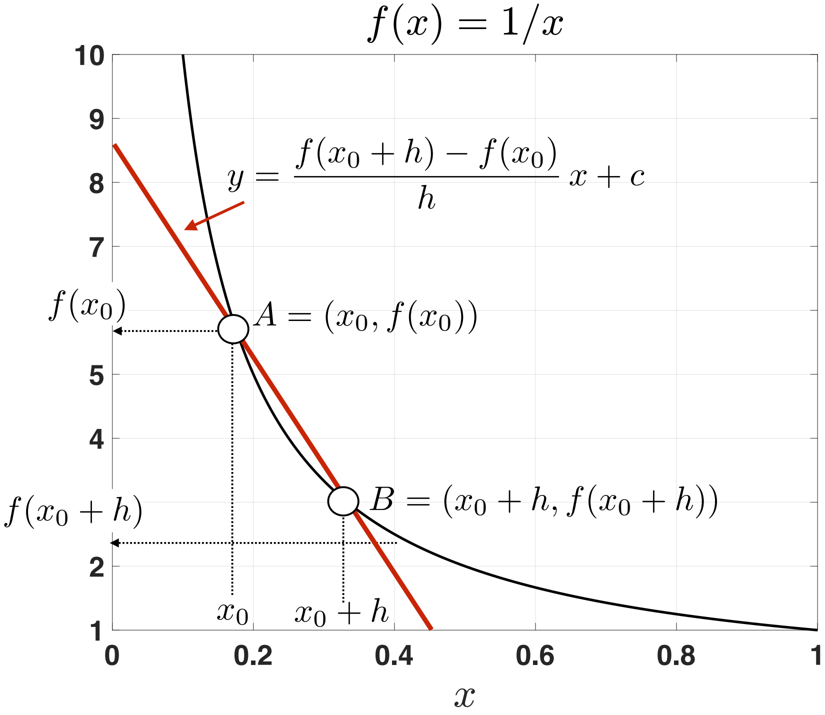
Definition

Let $f : D \to \mathbb{R}$ be a function and let $[a, b] \subset D$. We say that f is differentiable at x_0 if:

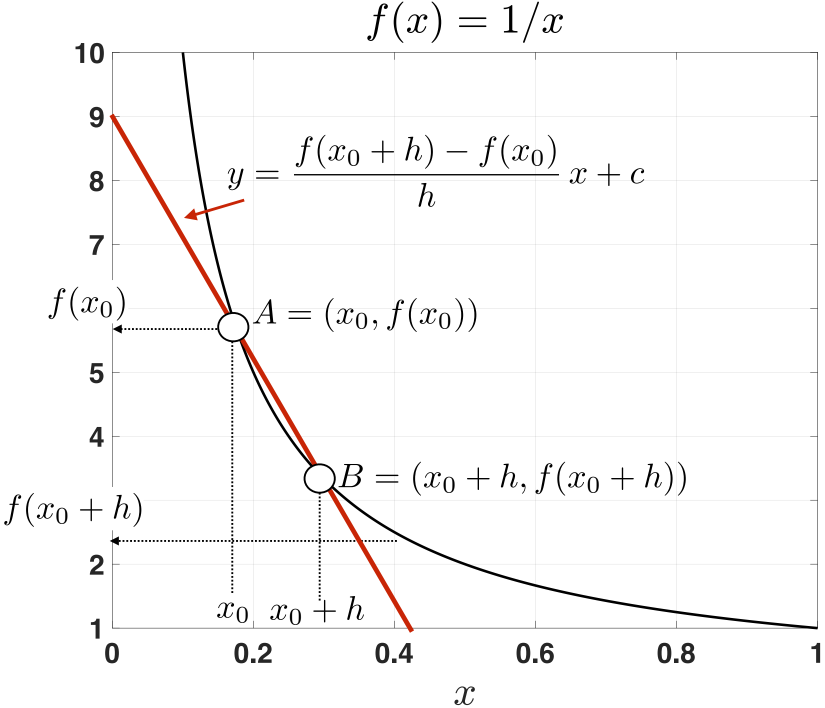
- $x_0 \in (a, b)$
- $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h} = d$ exists and it is finite.

If this is the case we call $d = f'(x_0)$ and we say that d is the derivative of f at the point x_0.

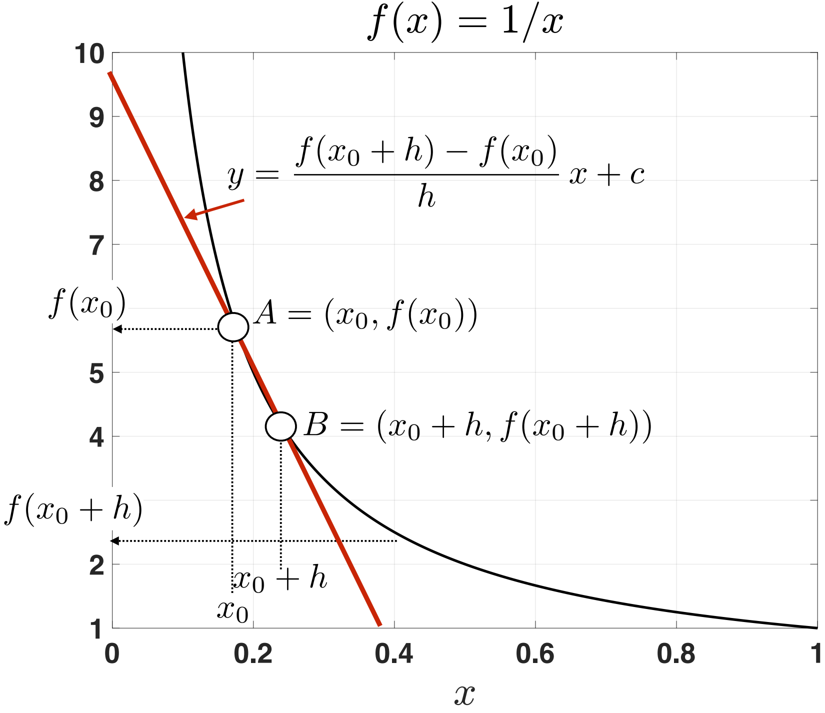

$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0+h)$
$B = (x_0+h, f(x_0+h))$
x_0
$x_0 + h$
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0+h)$
$B = (x_0+h, f(x_0+h))$
x_0
$x_0 + h$
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$B = (x_0+h, f(x_0+h))$
$f(x_0+h)$
x_0
$x_0 + h$
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$A = (x_0, f(x_0))$
$B = (x_0 + h, f(x_0 + h))$
$f(x_0)$
$f(x_0 + h)$
x_0
$x_0 + h$
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1

Derivatives of elementary functions

- Let $f(x) = k$, with $k \in \mathbb{R}$ (i.e. the constant function). $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k-k}{h} = 0$$

  The last equality is not an undetermined form since the numerator is equal to 0 no matter the value of h.

- Let $f(x) = x$. $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Derivatives of elementary functions

- Let $f(x) = x^2$. $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2hx + h^2}{h} = 2x$$

- Let $f(x) = e^x$. $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} e^x \underbrace{\frac{e^h - 1}{h}}_{1} = e^x$$

Derivatives of elementary functions

- Let $f(x) = \log(x)$. $D = (0, +\infty)$
  Using the definition of derivative we get that, for any $x \in (0, +\infty)$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{h} = \lim_{h \to 0} \frac{1}{x} \underbrace{\frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}}_{1} = \frac{1}{x}$$

- Let $f(x) = \sin(x)$. $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\sin(x)(\cos(h) - 1)}{h}}_{0} + \lim_{h \to 0} \underbrace{\frac{\sin(h)\cos(x)}{h}}_{\cos(x)} = \cos(x)$$

Derivatives of elementary functions

- Let $f(x) = \cos(x)$. $D = \mathbb{R}$
  Using the definition of derivative we get that, for any $x \in \mathbb{R}$:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(h)\sin(x)-\cos(x)}{h}$$
$$= \lim_{h\to 0} \underbrace{\frac{\cos(x)(\cos(h)-1)}{h}}_{0} - \lim_{h\to 0} \underbrace{\frac{\sin(h)\sin(x)}{h}}_{\sin(x)} = -\sin(x)$$

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Exponential	e^x	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Exponential	e^x	e^x

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Exponential	e^x	e^x
Logarithm	$\log(x)$	

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Exponential	e^x	e^x
Logarithm	$\log(x)$	$\frac{1}{x}$

Derivatives of elementary functions: Summary

Function	$f(x)$	$f'(x)$
Constant	k	0
Powers	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
Sine	$\sin x$	$\cos x$
Cosine	$\cos x$	$-\sin x$
Exponential	e^x	e^x
Logarithm	$\log(x)$	$\frac{1}{x}$

Operations with derivatives

Theorem

Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ be two differentiable functions. Then:

$$\begin{aligned}
D[f(x) + g(x)] &= f'(x) + g'(x) \\
D[f(x) - g(x)] &= f'(x) - g'(x) \\
D[kf(x)] &= kf'(x) \\
D[f(x) \cdot g(x)] &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\
D\left[\frac{f(x)}{g(x)}\right] &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \\
D[(f \circ g)(x)] &= D[f(g(x))] = f'(g(x)) \cdot g'(x)
\end{aligned}$$

Differentiation rules: exercises

Compute the derivative of the following functions:

1. $f(x) = 4x^3 - 3x^2 + x + 3$
2. $f(x) = \sin x + \cos x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \tan x$
5. $f(x) = x^{\sqrt{2}}$
6. $f(x) = \log x + \log_5 x$
7. $f(x) = 2^x + 3e^x$
8. $f(x) = (x+1)^2$
9. $f(x) = (2x+3)^4 + e^{2x}$
10. $f(x) = \cos x^2$
11. $f(x) = \cos^2 x$
12. $f(x) = \sin x \log x$
13. $f(x) = x^2 \sin 3x$
14. $f(x) = \frac{x \log x}{\sin x}$

Checklist for differentiability

Let $f : D \to \mathbb{R}$ be a function. Then f is differentiable at x_0 if

- f is continuous in x_0
- left and right limits of the incremental ratio at x_0 must coincide
- the limit must be finite

Non-differentiability points

- if f is not defined at x_0, that is $x_0 \notin D$, then clearly it cannot be differentiable (because in this case the function does not exists at the point x_0)
- if

$$\lim_{h \to 0^+} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h} = d_1, \qquad d_1 \neq \pm\infty$$
$$\lim_{h \to 0^-} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h} = d_2, \qquad d_2 \neq \pm\infty$$

but $d_1 \neq d_2$, then the function is NOT differentiable at x_0, and $\left(x_0, f(x_0)\right)$ is called an **angle point**

Non-differentiability points

- if

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty, \qquad (-\infty)$$
$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = +\infty, \qquad (-\infty)$$

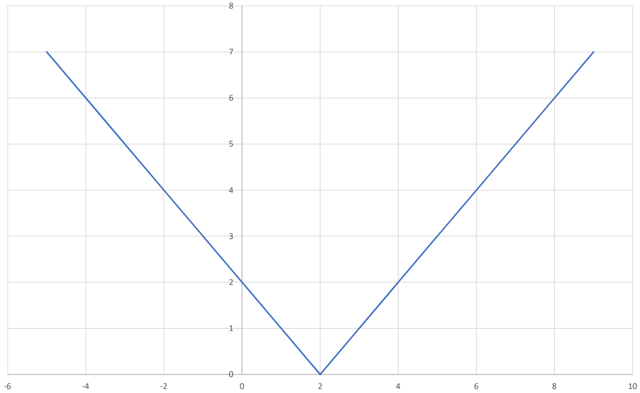
(i.e. left and right limits are infinite of the same sign!) then the function is NOT differentiable at x_0, and $(x_0, f(x_0))$ is called an **inflection point with vertical tangent**

- if

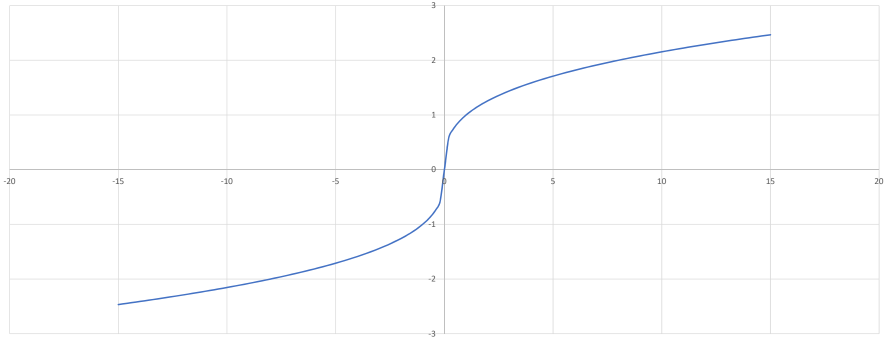
$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty \qquad (-\infty)$$
$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = -\infty, \qquad (+\infty)$$

(i.e. left and right limits are infinite of the different sign!) then the function is NOT differentiable at x_0, and $(x_0, f(x_0))$ is called a **cusp point**

$$f(x) = |x - 2|$$



$$f(x) = \sqrt[3]{x}$$



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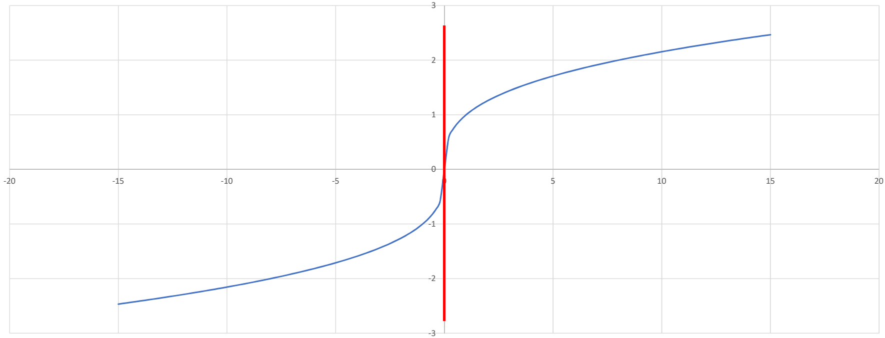
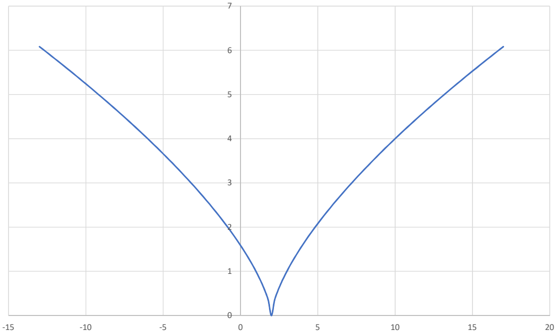


$$f(x) = \sqrt[3]{(x-2)^2}$$



First order Taylor approximation

Suppose that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} = f'(x_0) \tag{1}$$

Then if x is close to x_0, we can say that

$$f'(x_0) \sim \frac{f(x) - f(x_0)}{(x - x_0)}$$

Equivalently

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0)$$

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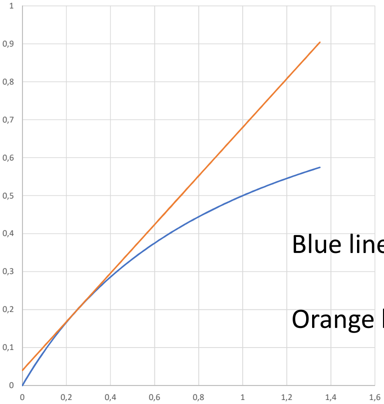

Blue line: $f(x)$

Orange line: tangent at $x_0 = \frac{1}{4}$

First order Taylor approximation

- The function

$$P(x) = f(x_0) + f'(x_0)(x - x_0)$$

  is called the first order Taylor approximation of f or the linearization of f or the first order Taylor polynomial of f.

- The quantity

$$R(x_1) = f(x_1) - P(x_1)$$

  is the reminder or the (absolute) error, for every $x_1 \neq x_0$

- The relative error (or error in percentage) is $\epsilon(x_1) = \frac{R(x_1)}{f(x_1)}$

The approximation $P(x)$ is good (i.e. if we use $P(x)$ in place of $f(x)$ we are not making a large error) if

- if x is close to x_0, i.e. $|x - x_0| < \delta$
- f is almost flat

Differentiability and continuity

Theorem

Let $f : D \to \mathbb{R}$ and let $x_0 \in D$. If f is differentiable in x_0 then it is continuous in x_0.

Proof: Notice that we already know that $x_0 \in D$. Therefore we need to show that $\lim_{x\to x_0} f(x)$ exists and it is equal to $f(x_0)$.
For all $x \neq x_0$ we can write

$$f(x) = f(x) - f(x_0) + f(x_0)$$

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0)$$

Now, we take the limit as $x \to x_0$ on both sides:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} \underbrace{(x - x_0)}_{0} + f(x_0) = f(x_0)$$

This concludes the proof.

Differentiability and continuity

Problem: If a function is continuous, is it differentiable ? **No!**

Differentiability and continuity

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Example: The function absolute value

$$f(x) = |x| = \begin{cases} -x & \text{if } \ x < 0 \\ x & \text{if } \ x \geq 0 \end{cases},$$

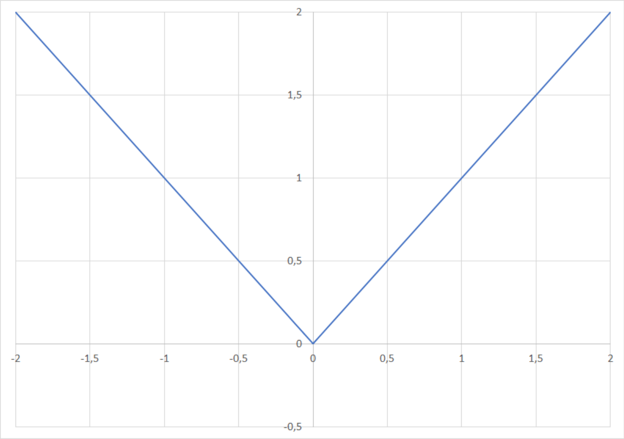
This function is continuous in for all $x \in \mathbb{R}$.
However, if we compute the left and right limit of incremental ratio at $x_0 = 0$ we get

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$
$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

Hence f is not differentiable at $x_0 = 0$ and $x_0 = 0$ is an angle point.

Example

$$f(x) = |x|$$



Increasing and decreasing functions

Theorem (Necessary and sufficient conditions for monotonicity of differentiable functions)

Let $f : D \to \mathbb{R}$ be a function and assume that f is differentiable in an **open** *interval $I \subseteq D$. Then:*

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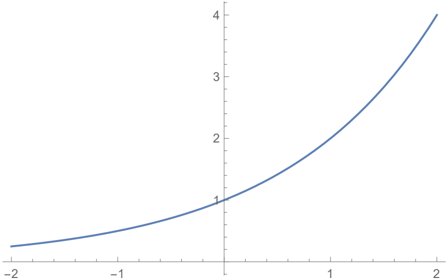
Pay attention: the open interval I **can** be unbounded.

Increasing and decreasing functions: examples

$$f(x) = e^x$$



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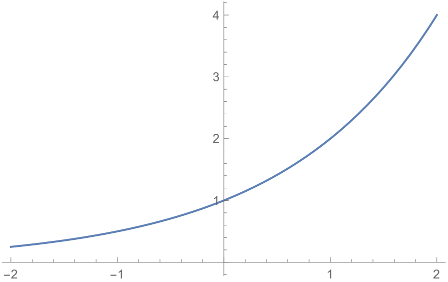

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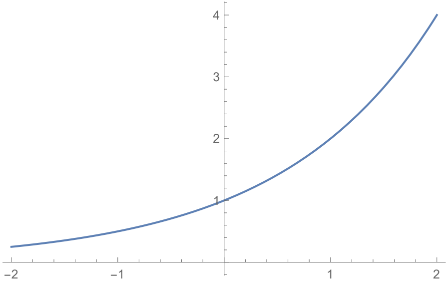

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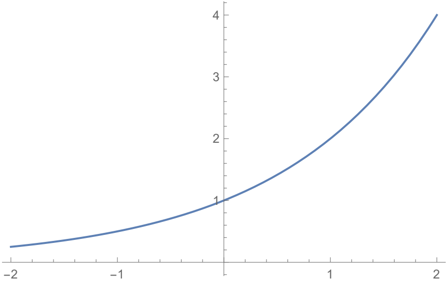

The function $f(x) = e^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = e^x > 0$, $\forall x \in \mathbb{R}$.

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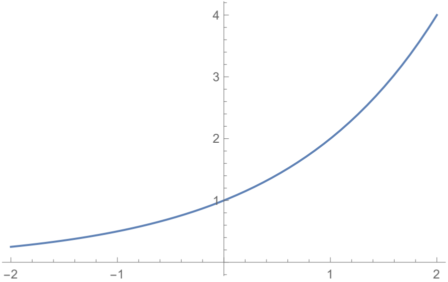

The function $f(x) = e^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = e^x > 0$, $\forall x \in \mathbb{R}$. Thus, the function is strictly increasing in $\mathbb{R}$.

Increasing and decreasing functions: examples, cont'd

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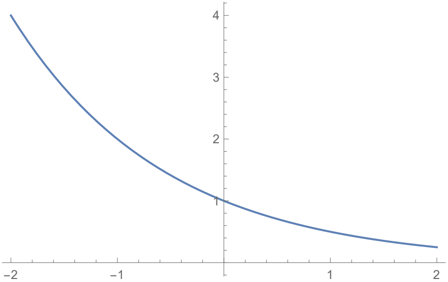

Increasing and decreasing functions: examples, cont'd

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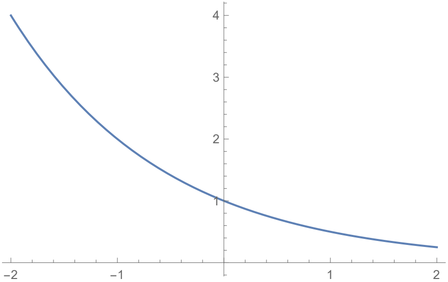

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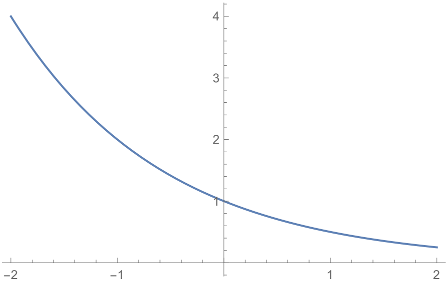

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Increasing and decreasing functions: examples, cont'd

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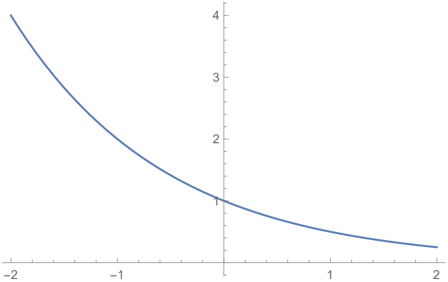
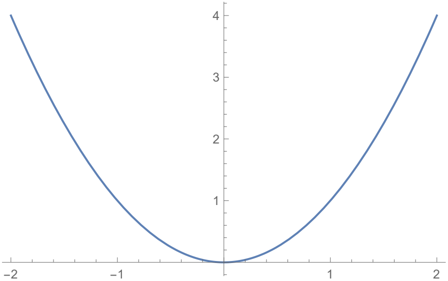


The function $f(x) = \left(\frac{1}{e}\right)^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = -\left(\frac{1}{e}\right)^x < 0$, $\forall x \in \mathbb{R}$. Thus, the function is strictly decreasing in $\mathbb{R}$.

Increasing and decreasing functions: examples

$$f(x) = x^2$$



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Increasing and decreasing functions: examples

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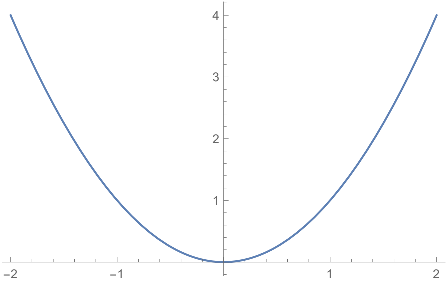

The function $f(x) = x^2$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2x$ which is positive for $x > 0$ and negative for $x < 0$.

Increasing and decreasing functions: examples

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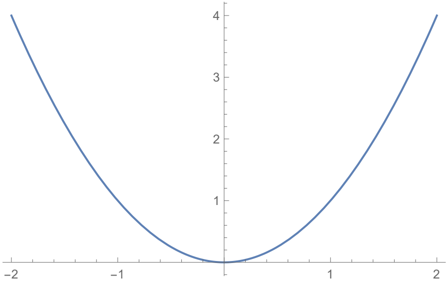

The function $f(x) = x^2$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2x$ which is positive for $x > 0$ and negative for $x < 0$. Thus, the function is strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$.

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$
2. $f(x) = \log x - x$
3. $f(x) = \sin(x)$
4. $f(x) = x^3 - 6x^2 + 4x + 12$
5. $f(x) = \log x - x^2$

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Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

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Pay attention: Note the difference between local maxima/minima and maxima/minima in an interval that we saw in the Weierstrass theorem.

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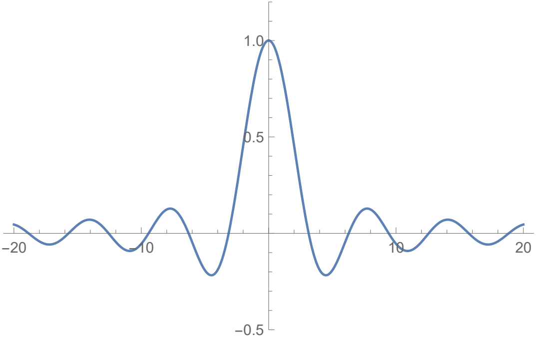
Pay attention: Note the difference between local maxima/minima and maxima/minima in an interval that we saw in the Weierstrass theorem.

In a closed and bounded interval $[a, b]$ a function may have multiple local maxima/minima but ONLY one maximum/minimum.

Local maxima and local minima: examples

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



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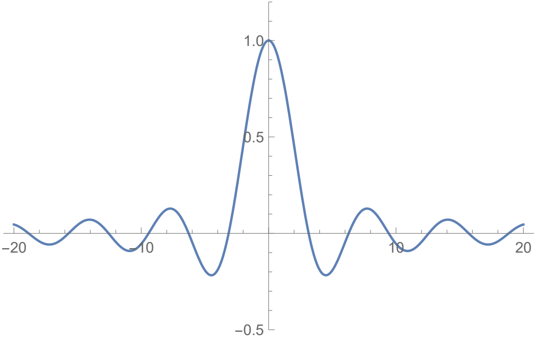
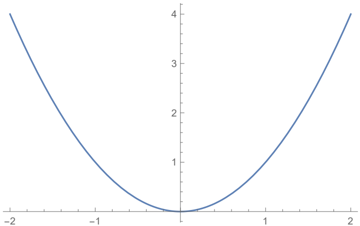


This function is continuous in $\mathbb{R}$.
By the Weierstrass theorem, it admits a maximum and a minimum in every closed bounded interval $[a, b]$.
However, the function may have multiple local maxima and minima in $[a, b]$.

Fermat's theorem: the intuition



$$f(x) = x^2$$

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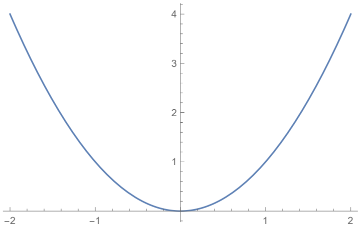

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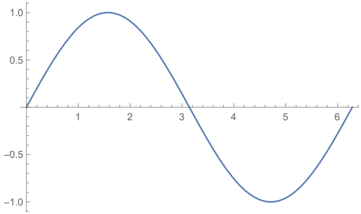

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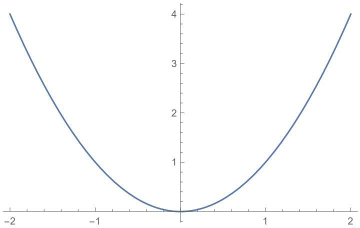

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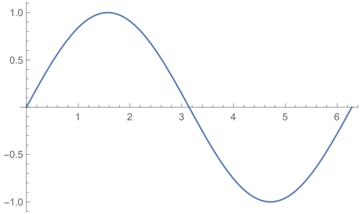
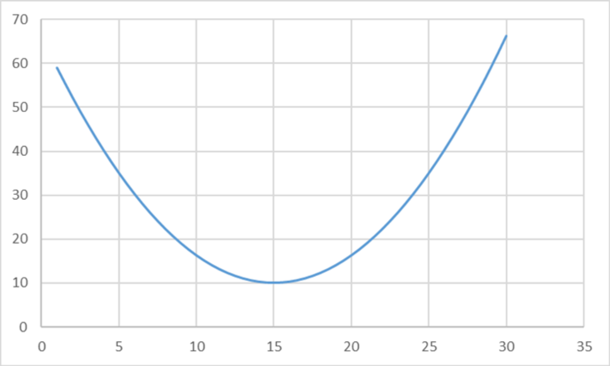


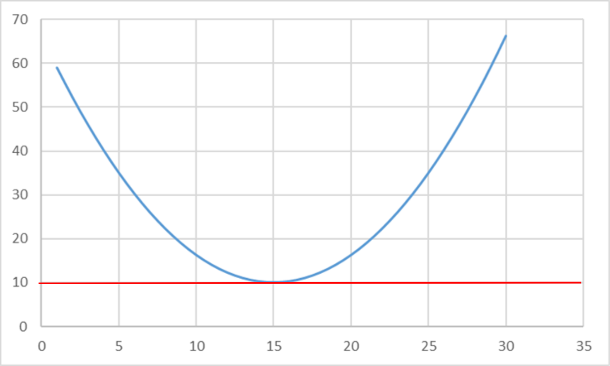
$g(x) = \sin x$

If a function is differentiable, the tangent line at local maxima and local minima is **horizontal**: that means, the derivative at local maxima and local minima is **zero**

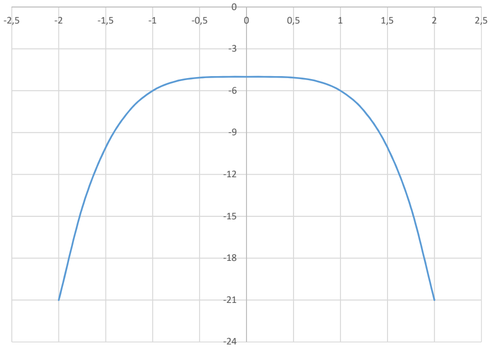
$$f(x) = x^2 - 2x + 35$$



$$f(x) = x^2 - 2x + 35$$



$$f(x) = -5 - x^4$$



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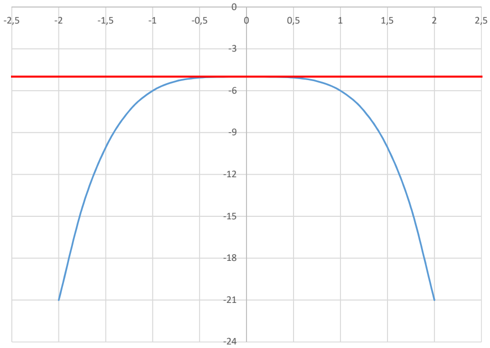

Fermat's theorem

Theorem (First order necessary condition for local maxima/minima)

Let $f : D \to \mathbb{R}$ and let f be differentiable in $x_0 \in D$.

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Let $f : D \to \mathbb{R}$ and let f be differentiable in $x_0 \in D$. If x_0 is a local maximum or a local minimum then $f'(x_0) = 0$.

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Does the converse hold?

Fermat's theorem

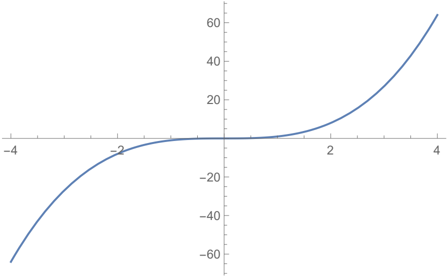
Theorem (First order necessary condition for local maxima/minima)

Let $f : D \to \mathbb{R}$ and let f be differentiable in $x_0 \in D$. If x_0 is a local maximum or a local minimum then $f'(x_0) = 0$.

Does the converse hold? Namely, if a function $f : D \to \mathbb{R}$ is differentiable in $x_0 \in D$ and if $f'(x_0) = 0$, can we conclude that x_0 is a local maximum or a local minimum? NO! (see next slide)

Fermat's theorem, cont'd

Indeed, consider the function $f(x) = x^3$:



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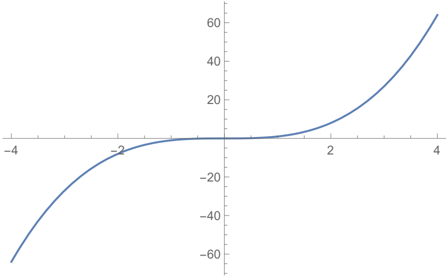

This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \Rightarrow$$

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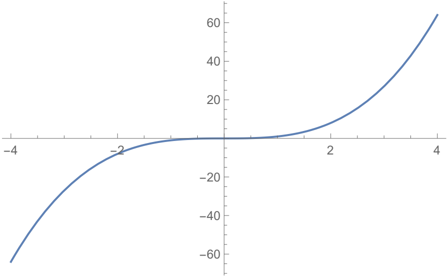

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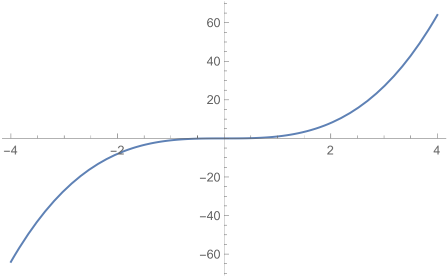

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However, $x = 0$ is NOT a local maximum/minimum.

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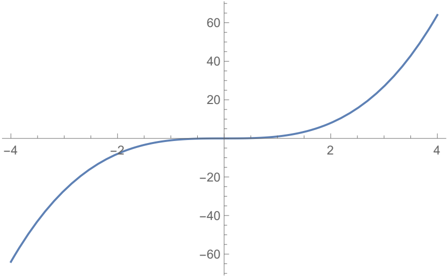


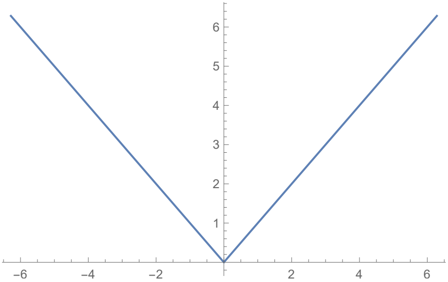
This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \Rightarrow \quad f'(0) = 0$$

However, $x = 0$ is NOT a local maximum/minimum. It is an **inflection point with an horizontal tangent**.

Fermat's theorem, cont'd

What if the function is not differentiable?



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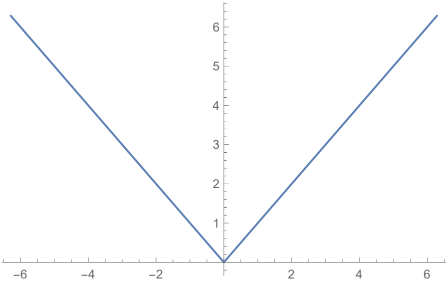

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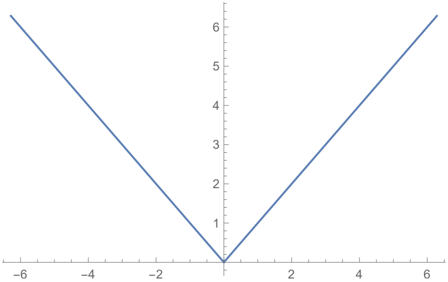

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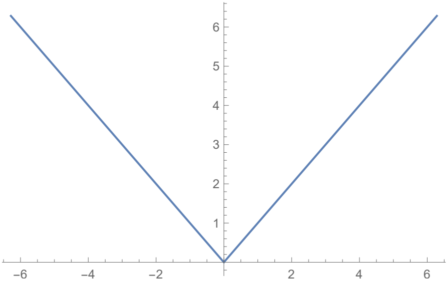

The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist. However note that $x = 0$ is a local minimum. **We CANNOT use derivatives to find the local maxima/minima if a function is not differentiable. In that case, we need to use the definition!!**

Stationary points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$.

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In order to understand whether x_0 is a local minimum, a local maximum or an inflection point, we need additional conditions that involve the **second derivative**.

Concavity and Convexity: the definition

Definition

A function $f : D \to \mathbb{R}$ is said to be concave in $(a, b) \subseteq D$ if for all $x_1, x_2 \in (a, b)$ the segment that joins the point $(x_1, f(x_1))$ and the point $(x_2, f(x_2))$ lies below the graph of $f(x)$ in the interval (x_1, x_2).

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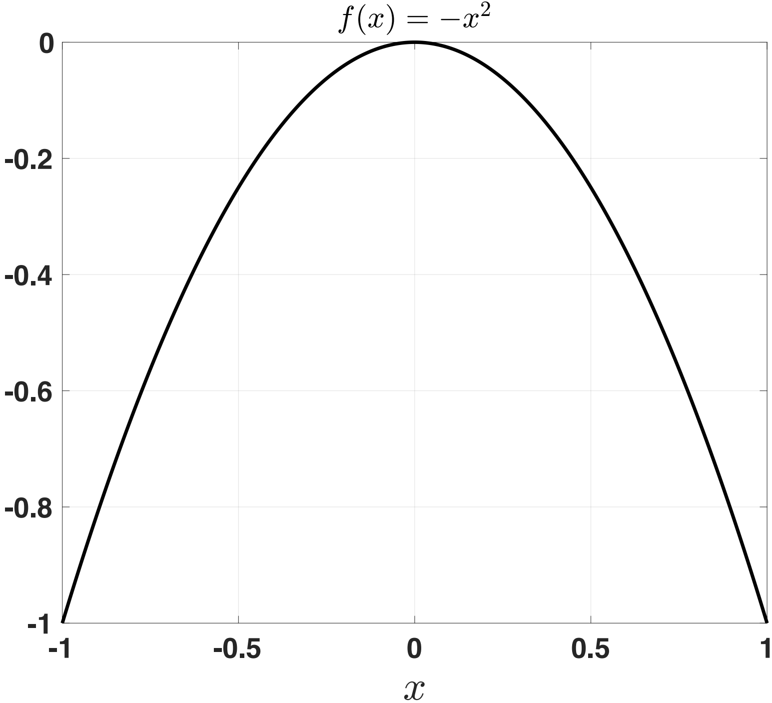
$f(x) = -x^2$
0
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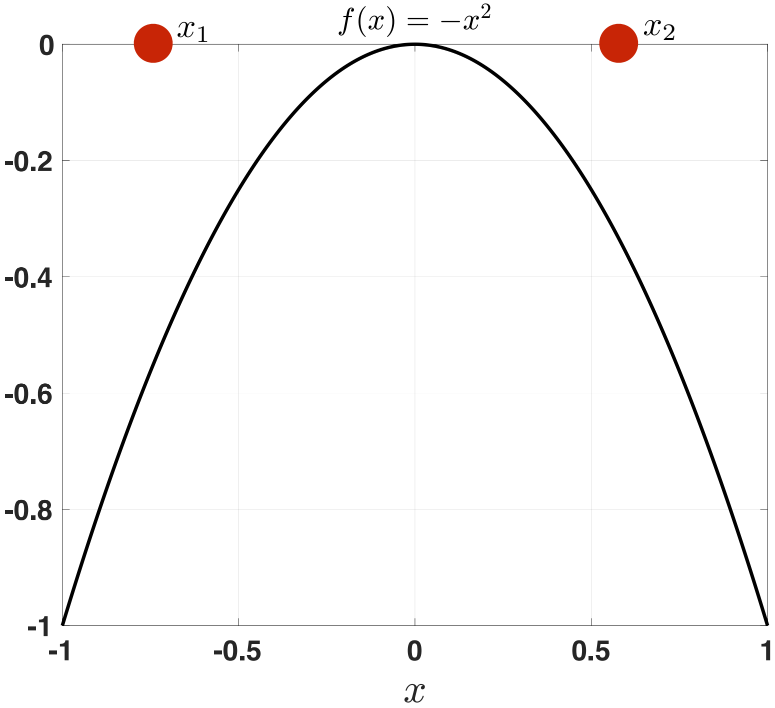
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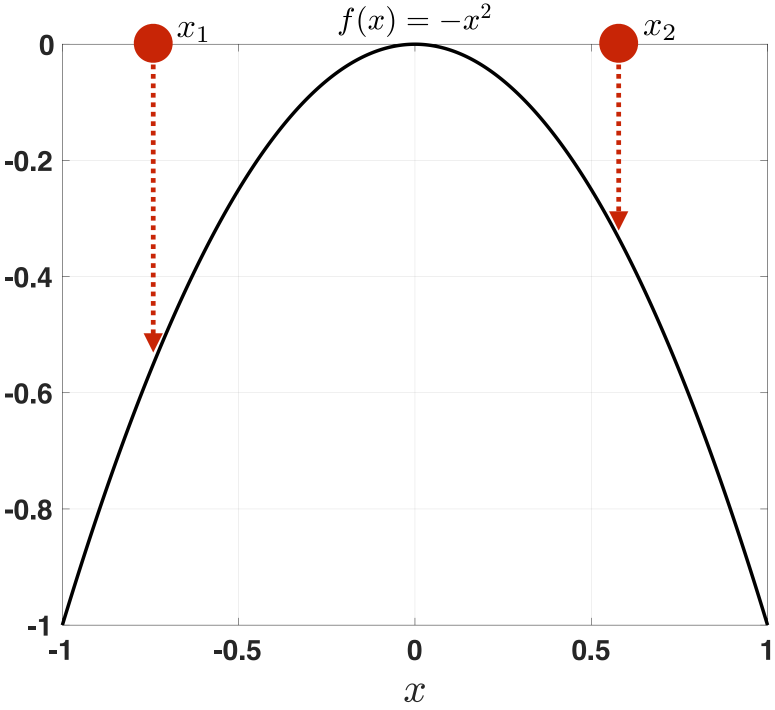
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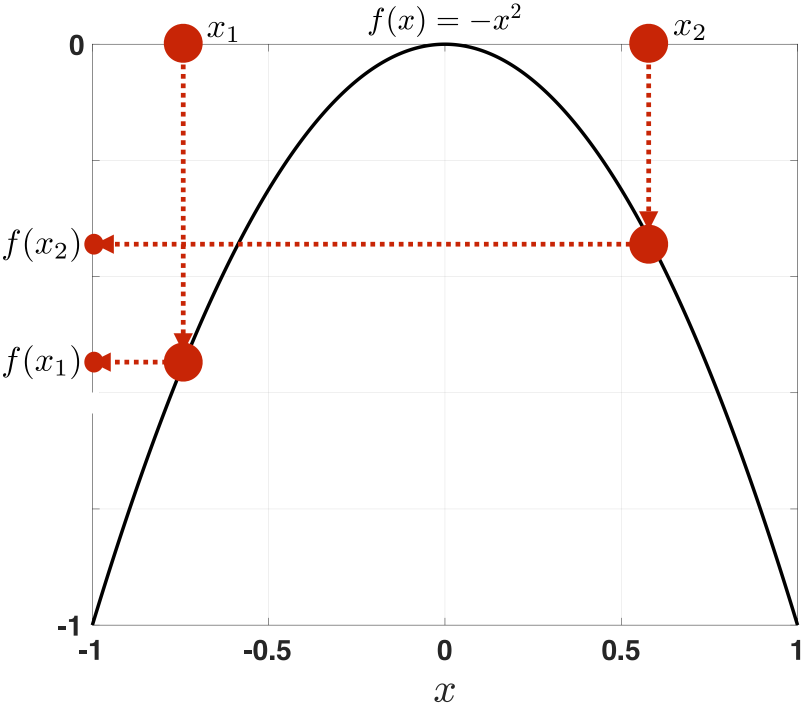

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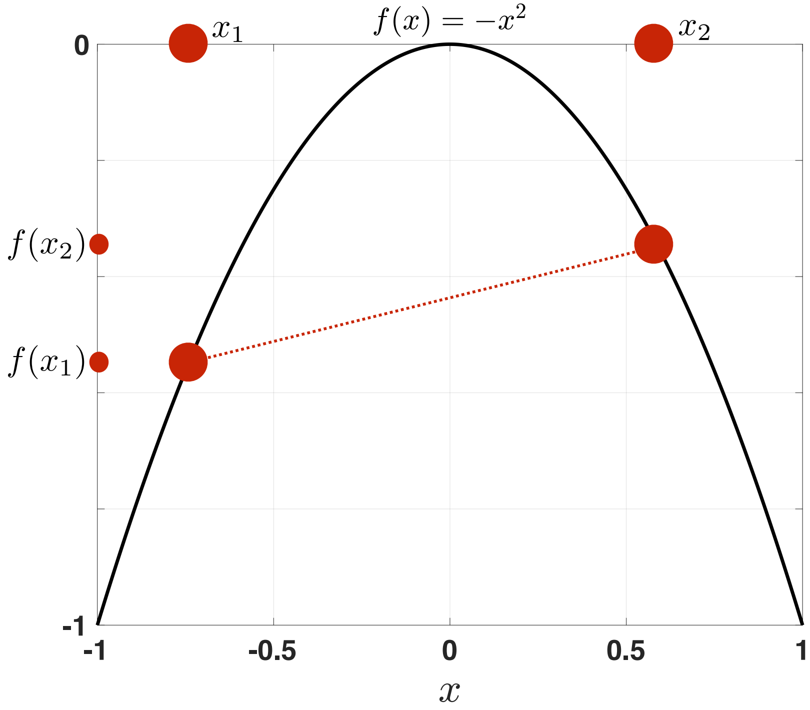
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The derivative of concave and convex functions

Theorem (Necessary and sufficient conditions for concavity/convexity of differentiable functions)

Let $f : D \to \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- *f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*

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Intuition: observe that, if a function is concave, the slope of the line tangent to a point decreases. Instead, if a function is convex, the slope of the line tangent to a point increases.

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. Suppose that also f' is differentiable.

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Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$
- $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$

The derivative of concave and convex functions, cont'd

We have seen that:

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

Then we have the following result

Theorem (Necessary and sufficient conditions for concavity/convexity of twice differentiable functions)

Let $f : D \to \mathbb{R}$ be a twice differentiable function in an interval $(a, b) \subseteq D$. Then

- *f is concave in (a, b) if and only if $f''(x) < 0$ for all $x \in (a, b)$.*
- *f is convex in (a, b) if and only if $f''(x) > 0$ for all $x \in (a, b)$.*

Second order sufficient condition for local maxima and minima

Theorem (Second order sufficient condition for local maxima and minima)

Let $f : D \to \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$ (that is, x_0 is a critical point). Then:

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- *If $f''(x_0) < 0$ then x_0 is a local maximum.*
- *If $f''(x_0) > 0$ then x_0 is a local minimum.*

Intuition: observe that, if x_0 is a local maximum, the function is concave in a neighborhood of x_0 and therefore $f''(x) < 0$.
Similarly, if x_0 is a local minimum, the function is convex in a neighborhood of x_0 and therefore $f''(x_0) > 0$.

Second order sufficient condition for local maxima and minima

What if $f''(x) = 0$?

Definition

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$. Moreover, $f''(0) = 0$. Since $f'''(0) = 6 \neq 0$, $x = 0$ is an inflection point.

Second order sufficient condition for local maxima and minima

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$
7. $f(x) = \frac{\log x}{x}$
8. $f(x) = \frac{\log^2 x}{x}$
9. $f(x) = x + e^{-3x}$
10. $f(x) = \log(1 + \log(x)) - \log(x)$

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Theorem

Let $f : X \to Y$ *be* **differentiable** *in* X.

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Then $f^{(-1)}$ is differentiable in Y and

$$[f^{(-1)}]'(y) = \frac{1}{f'\left(f^{(-1)}(y)\right)}$$

for all $y \in Y$ such that $f'\left(f^{(-1)}(y)\right) \neq 0$.

Derivative of the inverse function: an example

The function $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is injective and surjective and therefore it can be inverted. The inverse is called "arcsin":

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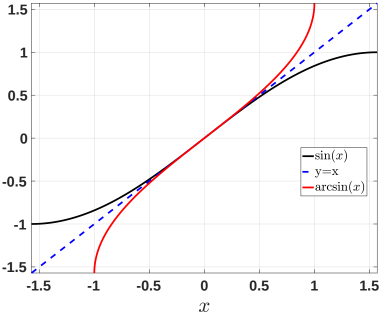

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$$\begin{aligned}
\sin(0) = 0 \quad &\Rightarrow \quad \arcsin(0) = 0 \\
\sin\left(\frac{\pi}{2}\right) = 1 \quad &\Rightarrow \quad \arcsin(1) = \frac{\pi}{2} \\
\sin\left(-\frac{\pi}{2}\right) = -1 \quad &\Rightarrow \quad \arcsin(-1) = -\frac{\pi}{2} \\
\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad &\Rightarrow \quad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\
\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \quad &\Rightarrow \quad \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \\
\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad &\Rightarrow \quad \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \\
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Let's compute the derivative of $\arcsin(x)$ using the previous theorem.

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and thus:

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which is differentiable in $[-1, 1]$ and its derivative is given by:

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which is differentiable in $\mathbb{R}$ and its derivative is given by:

$$(\arctan x)' = \frac{1}{1 + x^2}$$

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D.

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D. Assume that f and g are both differentiable in $D \setminus \{x_0\}$ and $g(x_0) \neq 0$.

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The point x_0 can be either finite or $\pm\infty$.

De L'Hôpital rule: examples

Compute the following limit:

$$\lim_{x \to 0^+} x \log x$$

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$$\begin{aligned}
\lim_{x \to 0^+} x \log(x) &= \lim_{x \to 0^+} \frac{\log(x)}{\frac{1}{x}} \\
&= \lim_{x \to 0^+} \frac{(\log(x))'}{\left(\frac{1}{x}\right)'} \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -\frac{x^2}{x} = \lim_{x \to 0^+} -x = 0
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$$= \lim_{x \to +\infty} \frac{2}{\sqrt{x}} = 0$$

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$$\lim_{x \to +\infty} \arctan(x) = +\frac{\pi}{2}$$

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Remind that the limit for $x \to +\infty$ of $\sin x$ and $\cos x$ do not exist because these functions oscillate between -1 and 1.

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