MATHEMATICS 1
ADDITIONAL EXERCISES N. 4

KATIA COLANERI

Notation: log stands for the natural logarithm (i.e. the logarithm with the basis e)

1. Sequences

(1) Compute the following limits

(a) $\lim_{n\to\infty} \frac{(n+1)^2}{n^2+1} = 1$

(b) $\lim_{n\to\infty} \frac{n^2+5n+1}{3n+7} = +\infty$

(c) $\lim_{n\to\infty} \frac{2n^2-3n-4}{\sqrt{n^4+4}} = 2$

(d) $\lim_{n\to\infty} n - \sqrt{2n^2+5n} = -\infty$

(e) $\lim_{n\to\infty} \sqrt{5n+6} - \sqrt{2n+1} = +\infty$

(f) $\lim_{n\to\infty} n(\sqrt{n^2+1} - n) = \frac{1}{2}$

(g) $\lim_{n\to\infty} n + \sqrt[3]{1-n^3} = +\infty$

(h) $\lim_{n\to\infty} \frac{n+1}{n^2+1} = 0$

(i) $\lim_{n\to\infty} \frac{n^4+5}{n^5+7n-1} = 0$

(j) $\lim_{n\to\infty} \frac{1-n}{\sqrt{n}+1} = -\infty$

(k) $\lim_{n\to\infty} \frac{n+(-1)^n}{n-(-1)^n} = 1$

(l) $\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n-1} = 0$

(m) $\lim_{n\to\infty} \sqrt{n^2+1} - \sqrt{n} = +\infty$

(n) $\lim_{n\to\infty} n\sqrt{\frac{1}{n+1}} = +\infty$

(o) $\lim_{n\to\infty} e^n - 2^n = +\infty$

(p) $\lim_{n\to\infty} 3^n + 4^n - 5^n = -\infty$

(q) $\lim_{n\to\infty} \frac{2^{n+1}-4^{n-1}}{3^n} = -\infty$

(r) $\lim_{n\to\infty} \frac{2^{n+1}+1}{3^n+1} = 0$

(s) $\lim_{n\to\infty} n - \log(n) = +\infty$

(t) $\lim_{n\to\infty} \frac{(n3^n+n^5+\sin(n))n}{(3^n+2^n)n^2} = 0$

(u) $\lim_{n\to\infty} \frac{\log^2(n)}{n} = 0$

(v) $\lim_{n\to\infty} \frac{n^4+5}{n^5+7n-1} = 0$

Katia Colaneri, Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, 00133 Rome, Italy.

E-mail address: katia.colaneri@uniroma2.it.

$$(w)\quad \lim_{n\to\infty} 3^{n+1} - 3^{\sqrt{n^2-1}} = \lim_{n\to\infty} 3^{n+1}(1 - 3^{\sqrt{n^2-1}-(n+1)} =$$

$$\lim_{n\to\infty} 3^{n+1}(1 - 3^{\frac{-2n}{\sqrt{n^2-1}-(n+1)}} = +\infty \cdot (1 - 3^{-1}) = +\infty$$

$$(x)\quad \lim_{n\to\infty} \frac{5^n - n^5}{4^n + n^6} = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \frac{1 - \frac{n^5}{5^n}}{1 + \frac{n^6}{4^n}} = +\infty$$

$$(y)\quad \lim_{n\to\infty} \frac{(n^2+1)\log(n)}{n^3} = \lim_{n\to\infty} \frac{(n^2+1)}{n^2}\frac{\log(n)}{n} = 1 \cdot 0 = 0$$

$$(z)\quad \lim_{n\to\infty} \frac{n! + 2^n}{(n+1)!} = \lim_{n\to\infty} \frac{n!\left(1 + \frac{2^n}{n!}\right)}{n!(n+1)} = \lim_{n\to\infty} \frac{\left(1 + \frac{2^n}{n!}\right)}{(n+1)} = 0$$

(2) Compute the following limits

$$(a)\quad \lim_{n\to\infty} \left(\frac{n+1}{n+3}\right)^n = e^{-2}$$

$$(b)\quad \lim_{n\to\infty} \left(\frac{3n+1}{3n-3}\right)^{n-1} = e^4$$

$$(c)\quad \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n^2} = +\infty$$

$$(d)\quad \lim_{n\to\infty} \left(\frac{n^2+1}{n^2-1}\right)^n = 1$$

$$(e)\quad \lim_{n\to\infty} \left(1 + \frac{1}{2^n}\right)^{n^2} = 1$$

$$(f)\quad \lim_{n\to\infty} \left(1 + 3^{-n}\right)^n = 1$$

$$(g)\quad \lim_{n\to\infty} \left(1 + \frac{1}{\log(n)}\right)^n = +\infty$$

$$(h)\quad \lim_{n\to\infty} (2n+1)\log\left(\frac{2n+1}{2n-3}\right) = 4$$

$$(i)\quad \lim_{n\to\infty} \left(1 + e^{-n}\right)^{n!} = +\infty$$

$$(j)\quad \lim_{n\to\infty} n^3 \log\left(1 + 3^{-n}\right) = 0$$

$$(k)\quad \lim_{n\to\infty} n\log\left(\frac{n+1}{n+3}\right) = -2$$

$$(l)\quad \lim_{n\to\infty} n\left(e^{\frac{1}{n^2}-1}\right) = -+\infty$$

$$(m)\quad \lim_{n\to\infty} n^2\left(e^{\frac{1}{n}-1}\right) = +\infty$$

$$(n)\quad \lim_{n\to\infty} n\left(e^{\frac{n+1}{n-1}-1}\right) = +\infty$$

$$(o)\quad \lim_{n\to\infty} 2n\left(e^{\frac{3}{n}-1}\right) = +\infty$$

$$(p)\quad \lim_{n\to\infty} (n^2+1)\left(e^{\frac{2}{n^2}-1}\right) = +\infty$$

$$(q)\quad \lim_{n\to\infty} n^2 \sin\left(\frac{2}{n^2}\right) = 2$$

$$(r)\quad \lim_{n\to\infty} n\sin\left(\frac{1}{n+1}\right) = 1$$

$$(s)\quad \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{1 - \cos\left(\frac{1}{n}\right)} = 2$$

$$(t)\quad \lim_{n\to\infty} \frac{e^{2+\frac{2}{n}} - 1}{\sin\left(\frac{1}{n}\right)} = +\infty$$

$$(u)\quad \lim_{n\to\infty} \frac{\log\left(\frac{n+2}{n-1}\right)}{\left(1 - \cos\left(\frac{1}{2n}\right)\right)} = +\infty$$

$$(v)\quad \lim_{n\to\infty} \frac{\log\left(\frac{n^2-1}{n^2+1}\right)}{\left(1 - \cos\left(\frac{2}{n}\right)\right)} = -1$$

$(w)\quad \lim_{n\to\infty} \frac{n^3+3n+1}{\sin\left(\frac{2}{3n^3}\right)} = +\infty$ $\qquad (x)\quad \lim_{n\to\infty} \frac{5^n-n^5}{\log(1+5^{-n})} = +\infty$

$(y)\quad \lim_{n\to\infty} \frac{(n^2+1)}{n\sin\left(\frac{1}{2n}\right)} = +\infty$ $\qquad (z)\quad \lim_{n\to\infty} \frac{n!+2^n}{\log\left(1+\frac{1}{(n+1)!}\right)} = +\infty$

2. Series

(1) Compute the following series (if they exist)

$$(a)\quad \sum_{n=0}^{+\infty}\left(\cos\left(\frac{\pi}{3}\right)\right)^n = \sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$(b)\quad \sum_{n=0}^{+\infty}\left(\cos\left(\frac{\pi}{4}\right)\right)^n = \sum_{n=0}^{+\infty}\left(\frac{\sqrt{2}}{2}\right)^n = \frac{1}{1-\frac{\sqrt{2}}{2}} = 2+\sqrt{2}$$

$$(c)\quad \sum_{n=0}^{+\infty}\left(\cos\left(\frac{\pi}{6}\right)\right)^n = \sum_{n=0}^{+\infty}\left(\frac{\sqrt{3}}{2}\right)^n = 2\left(2+\sqrt{3}\right)$$

$$(d)\quad \sum_{n=0}^{+\infty}\left(\cos\left(\frac{4\pi}{3}\right)\right)^n = \frac{2}{3}$$

$$(e)\quad \sum_{n=0}^{+\infty}\frac{\left(\sin\left(\frac{\pi}{4}\right)\right)^n+1}{(\sqrt{2})^n} = 2+\frac{\sqrt{2}}{\sqrt{2}-1}$$

$$(f)\quad \sum_{n=0}^{+\infty}\frac{\left(\sin\left(\frac{\pi}{3}\right)\right)^n+2}{(-3)^n} = \frac{6}{6+\sqrt{3}}+\frac{3}{2}$$

$$(g)\quad \sum_{n=0}^{+\infty}\frac{2^n+4^n}{5^n} = \sum_{n=0}^{+\infty}\frac{2^n}{5^n}+\sum_{n=0}^{+\infty}\frac{4^n}{5^n} = \sum_{n=0}^{+\infty}\left(\frac{2}{5}\right)^n+\sum_{n=0}^{+\infty}\left(\frac{4}{5}\right)^n = \frac{1}{1-\frac{2}{5}}+\frac{1}{1-\frac{4}{5}} = \frac{5}{3}+5$$

$$(h)\quad \sum_{n=0}^{+\infty}\frac{3^n+5^n}{4^n} = \sum_{n=0}^{+\infty}\left(\frac{3}{4}\right)^n+\sum_{n=0}^{+\infty}\left(\frac{5}{4}\right)^n = +\infty$$

$$(i)\quad \sum_{n=0}^{+\infty}\frac{2^n-5^n}{7^n} = -\frac{21}{10}$$

$$(j)\quad \sum_{n=0}^{+\infty}\frac{e^n+1}{e^{2n}} = \frac{2e^2+e}{e^2-1}$$

$$(k)\quad \sum_{n=0}^{+\infty}\frac{3^{n+2}}{5^n} = \frac{45}{2}$$

$$(l)\quad \sum_{n=0}^{+\infty}\left(-\frac{2}{3}\right)^{n-3} = -\frac{81}{40}$$

$$(m)\quad \sum_{n=1}^{+\infty}\left(-\frac{1}{3}\right)^{n} = \frac{3}{4} - 1 = -\frac{1}{4}$$

$$(n)\quad \sum_{n=2}^{+\infty}\left(\sin\left(\frac{\pi}{3}\right)\right)^{n} = \frac{1}{1-\frac{\sqrt{3}}{2}} - 1 - \frac{\sqrt{3}}{2}$$

$$(o)\quad \sum_{n=2}^{+\infty}(1-\log(2))^{n} = \frac{1}{\log(2)} - 1 - 1 + \log(2)$$

$$(p)\quad \sum_{n=1}^{+\infty}\frac{e^n}{3^n} = \frac{e}{3-e} - 1$$

$$(q)\quad \sum_{n=1}^{+\infty}\frac{e^n}{2^n} = +\infty$$

$$(r)\quad \sum_{n=1}^{+\infty}(-\sqrt{2})^{n} \quad \text{it does not exist}$$

(2) Say for which values of the parameter a the following series exist.

$(a)\quad \sum_{n=0}^{+\infty}(2a+1)^n$ It exists for $|2a+1| < 1$, that is: $-1 < a < 0$

$(b)\quad \sum_{n=0}^{+\infty}\left(a^2+3a+2\right)^n$ It exists for $|a^2+3a+2| < 1$, that is: $\frac{-\sqrt{5}-3}{2} < a < \frac{\sqrt{5}-3}{2}$

$(c)\quad \sum_{n=0}^{+\infty}(1+\log(a))^n$ It exists for $|1+\log(a)| < 1$, that is: $\frac{1}{e^2} < a < 1$

$(d)\quad \sum_{n=0}^{+\infty}\left(\frac{a+1}{a}\right)^n$ It exists for $\left|\frac{a+1}{a}\right| < 1$, that is: $a < -\frac{1}{2}$

$(e)\quad \sum_{n=0}^{+\infty}(a^2-1)^n$ It exists for $\left|a^2-1\right| < 1$, that is: $-\sqrt{2} < a < 0$ or $0 < a < \sqrt{2}$

$(f)\quad \sum_{n=0}^{+\infty}(\cos(a))^n$ It exists for $|\cos(a)| < 1$, that is: $a \neq k\pi,\ \forall k \in \mathbb{Z}$

$(g)\quad \sum_{n=0}^{+\infty} \left(\frac{2a+1}{a}\right)^n \qquad$ It exists for $\left|\frac{2a+1}{a}\right| < 1$, that is: $-1 < a < -\frac{1}{3}$

$(h)\quad \sum_{n=0}^{+\infty} \left(\frac{3^a}{2}\right)^n \qquad$ It exists for $\left|\frac{3^a}{2}\right| < 1$, that is: $a < \log_3(2)$