MATHEMATICS 1
ADDITIONAL EXERCISES N. 5

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Notation: log stands for the natural logarithm (i.e. the logarithm with the basis e)

1. Limits

(1) Compute the following limits

$(a)\ \lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

$(b)\ \lim_{x\to -2} \frac{x^3+8}{x^2-x-2} = 0$

$(c)\ \lim_{x\to 0} \frac{\sqrt{x^2+x+2}-1}{x} = \nexists$

$(d)\ \lim_{x\to -2} \frac{\sqrt{2+x}-\sqrt{2-x}}{x} = 1$

$(e)\ \lim_{x\to 1} \frac{\sqrt{1+x^2}-\sqrt{2}}{x-1} = \frac{1}{\sqrt{2}}$

$(f)\ \lim_{x\to -2} \frac{x^3+4x^2+4x}{x^2-x-6} = 0$

$(g)\ \lim_{x\to +\infty} \frac{x^2-4x}{\sqrt{5+x^4}} = 1$

$(h)\ \lim_{x\to -\infty} \frac{5x+7}{\sqrt{x^2+1}} = -5$

$(i)\ \lim_{x\to +\infty} \sqrt{\frac{2x^2+1}{x^2-1}} = \sqrt{2}$

$(j)\ \lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1} = -1$

$(k)\ \lim_{x\to +\infty} \sqrt{x+3}-\sqrt{x} = 0$

$(l)\ \lim_{x\to -\infty} \sqrt{3-x}-\sqrt{1-x} = 0$

$(m)\ \lim_{x\to +\infty} \frac{3x}{\sqrt{3+x^2}-\sqrt{x+x^2}} = -\infty$

$(n)\ \lim_{x\to 1} \frac{1}{1-x} - \frac{3}{1-x^3} = -1$

$(o)\ \lim_{x\to +\infty} \frac{\sqrt{x^2+x-2}}{x+1+\sqrt{x^2+x-2}} = \frac{1}{2}$

$(p)\ \lim_{x\to -\infty} \frac{\sqrt{x^2+x-2}}{x+1+\sqrt{x^2+x-2}} = +\infty$

$(q)\ \lim_{x\to -\infty} \frac{3x-4}{x-\sqrt{2-x}} = 3$

$(r)\ \lim_{x\to -\infty} \frac{2x}{3x-\sqrt{5x^2+3}} = \frac{3-\sqrt{5}}{2}$

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(2) Compute the following limits

$(a)\ \lim_{x\to 0}\dfrac{\sin(3x)}{x}=3$

$(b)\ \lim_{x\to 0}\dfrac{1}{\sqrt{1-\cos(2x)}}=+\infty$

$(c)\ \lim_{x\to 0}\dfrac{\sin^2(x)}{1-\cos(x)}=2$

$(d)\ \lim_{x\to \frac{\pi}{4}}\dfrac{1+\cos(4x)}{\cos^2(x)}=0$

$(e)\ \lim_{x\to 0}\dfrac{\sin(x^2+x)}{x}=1$

$(f)\ \lim_{x\to 0}\dfrac{\sqrt{1-\cos(x)}}{2x}=-\dfrac{1}{2\sqrt{2}}$

$(g)\ \lim_{x\to -\infty}\dfrac{\sin(e^x)}{3e^x}=\dfrac{1}{3}$

$(h)\ \lim_{x\to 0}\dfrac{\sin(6x)}{3x}-x^2=2$

$(i)\ \lim_{x\to 0}\dfrac{\sin^2\left(\frac{x}{3}\right)}{x^2}=\dfrac{1}{9}$

$(j)\ \lim_{x\to 1}\dfrac{\sin(\log(x))}{x-1}=1$

$(k)\ \lim_{x\to 0}\dfrac{\sin(3x)-5\sin(2x)+x}{x-2\tan(x)}=6$

$(l)\ \lim_{x\to 1}\dfrac{1-cos(x)+x^2}{2\sin^2(x)-5x^2}=\dfrac{-\cos(1)+2}{2\sin^2(1)-5}$

$(m)\ \lim_{x\to +\infty}\left(\dfrac{x+3}{x+2}\right)^x=e$

$(n)\ \lim_{x\to +\infty}\left(\dfrac{2x+3}{2x-1}\right)^{3x+4}=e^6$

$(o)\ \lim_{x\to +\infty}\left(\dfrac{x-1}{x+6}\right)^{2x}=\dfrac{1}{e^{14}}$

$(p)\ \lim_{x\to 0}\dfrac{\log(1+3x)}{2x^2+x}=3$

$(q)\ \lim_{x\to 0}\dfrac{e^{2x^2+x}-1}{2x}=\dfrac{1}{2}$

$(r)\ \lim_{x\to 0}\dfrac{e^{\sin(x)}-1}{x}=1$

$(s)\ \lim_{x\to +\infty}\left(\dfrac{1+x^3}{x^3+3x}\right)^{x^2}=\dfrac{1}{e^3}$

$(t)\ \lim_{x\to +\infty}\left(1+\dfrac{1}{x^2+1}\right)^{2x^2+1}=e^2$

$(u)\ \lim_{x\to +\infty}\log(x+3)-\log(x-1)=-\infty$

$(v)\ \lim_{x\to +\infty}x(\log(x+3)-\log(x+1))=2$

$(w)\ \lim_{x\to 0}\dfrac{3\sin(x)-2x-4x^2}{5\sin(x)-4x}=1$

$(x)\ \lim_{x\to 0}\dfrac{\sin^2(2x)+3x^2}{x^2+x\log(1+5x)-4x^2}=\dfrac{7}{2}$

$(y)\ \lim_{x\to 0}\dfrac{1-\cos^2(x)}{x\sin(x)\cos(x)}=1$

$(z)\ \lim_{x\to 0}\dfrac{1-\cos^3(x)}{x\sin(x)\cos(x)}=\dfrac{3}{2}$

2. Continuous Functions

(1) For each function determine the set where the function is continuous and identify the nature of discontinuity points, if any.

(a)

$$f(x)=\frac{|4-x^2|}{x+2}$$

The function is continuous for all $x\neq -2$. $x=-2$ (jump discontinuity)

(b)

$$f(x) = \frac{\log(x+1)}{x}$$

The function is continuous for all $x \neq \{-1, 0\}$. $x = -1$ (essential discontinuity) and $x = 0$ (removable discontinuity)

(c)

$$f(x) = 5 + \log(|x|)$$

The function is continuous for all $x \neq 0$. $x = 0$ (essential discontinuity)

(d)

$$f(x) = \frac{e^x - 1}{x}$$

The function is continuous for all $x \neq 0$. $x = 0$ (removable discontinuity)

(e)

$$\frac{1}{2e^x - x + 1}$$

The function is continuous in $\mathbb{R}$

(f)

$$f(x) = e^{-\frac{1}{x}}$$

The function is continuous for all $x \neq 0$. $x = 0$ (essential discontinuity)

(g)

$$\frac{x^2 + 2x}{x^3 + 8}$$

The function is continuous for all $x \neq -2$. $x = -2$ (removable discontinuity)

(h)

$$f(x) = \frac{e^{2x} - 1}{e^x - 1}$$

The function is continuous for all $x \neq 0$. $x = 0$ (removable discontinuity)

(i)

$$f(x) = \frac{e^x + e^{-x}}{x}$$

The function is continuous for all $x \neq 0$. $x = 0$ (essential discontinuity)

(j)

$$f(x) = \begin{cases} \log(-x) & \text{if } x \leq -1 \\ x + 1 & \text{if } x > -1 \end{cases}$$

The function is continuous for all $x \neq -1$. $x = -1$ (essential discontinuity)

(k)

$$(k)\ f(x) = \begin{cases} e^{-x-2} & \text{if } x \leq -2 \\ \sqrt{x+2} & \text{if } x > -2 \end{cases}$$

The function is continuous for all $x \neq -2$. $x = -2$ (jump discontinuity)

(l)

$$f(x) = \begin{cases} \dfrac{1+x^3}{x+1} & \text{if } \ x \neq -1 \\ 4 & \text{if } \ x = -1 \end{cases}$$

The function is continuous for all $x \neq -1$. $x = -1$ (removable discontinuity)

(m)

$$f(x) = \begin{cases} \dfrac{e^{2x}-1}{x} & \text{if } \ x \leq 0 \\ 2 & \text{if } \ x = 0 \end{cases}$$

The function is continuous for all $x \leq 0$.

(n)

$$f(x) = \frac{x+1}{|x+2|-1}$$

The function is continuous for all $x \neq \{-3, -1\}$. $x = -3$ (infinite discontinuity) and $x = -1$ (removable discontinuity)

(o)

$$f(x) = \frac{x^2-1}{\sqrt{x-1}}$$

The function is continuous for all $x > 1$. $x = 1$ (removable discontinuity). Notice that in this case we cannot approach the point $x = 1$ from the left but only from the right. Since $\lim_{x\to 1^+} \frac{x^2-1}{\sqrt{x-1}} = 0$ (i.e. the limit is finite), but $x = 1$ does not belong to the domain, then the point $x = 1$ is a removable discontinuity.

(p)

$$f(x) = \frac{e^x - 1}{\log(x+1)}$$

The function is continuous for all $x \in (-1, 0) \cup (0, +\infty)$. $x = -1$ (removable discontinuity), $x = 0$ (removable discontinuity). Notice that in this case we cannot approach the point $x = -1$ from the left but only from the right. Since $\lim_{x\to -1^+} \frac{e^x-1}{\log(x+1)} = 0$ (i.e. the limit is finite), but $x = -1$ does not belong to the domain, then the point $x = -1$ is a removable discontinuity.

(q)

$$f(x) = \frac{e^{\frac{1}{x}} + e^{\frac{2}{x}}}{e^{\frac{1}{x}} + 5}$$

The function is continuous for all $x \neq 0$. $x = 0$ (essential discontinuity)

(2) For each function, say if Weierstrass theorem applies in the indicated interval

$$(a) \ f(x) = \frac{x^3}{4-x^2} \qquad\qquad I = [-1, 0]$$

$f(x) = \dfrac{x^3}{4-x^2}$ is continuous in $D = \{x \in \mathbb{R} : x \neq \pm 2\}$, hence it is continuous in $I = [-1, 0]$. Therefore Weierstrass theorem applies.

$$(b)\ f(x) = \frac{1}{\log(x)} \qquad\qquad I = \left[\frac{1}{3}, 2\right]$$

$f(x) = \dfrac{1}{\log(x)}$ is continuous in $D = (0,1) \cup (1, +\infty)$, hence it is not continuous in $I = \left[\frac{1}{3}, 2\right]$. Therefore Weierstrass theorem does not apply.

$$(c)\ f(x) = \frac{1}{|x|} \qquad\qquad I = [-1, 1]$$

$f(x) = \dfrac{1}{|x|}$ is continuous in $D = \mathbb{R} \setminus \{0\}$, hence it is not continuous in $I = [-1, 1]$. Therefore Weierstrass theorem does not apply.

$$(d)\ f(x) = \frac{\sin(x)}{1-\cos(x)} \qquad\qquad I = [0, \pi]$$

$f(x) = \dfrac{\sin(x)}{1-\cos(x)}$ is continuous in $D = \mathbb{R} \setminus \{2k\pi,\ k \in \mathbb{Z}\}$, hence it is not continuous in $I = [0, \pi]$. Therefore Weierstrass theorem does not apply.

$$(e)\ f(x) = \sqrt{|x-1|} \qquad\qquad I = [0, 2]$$

$f(x) = \sqrt{|x-1|}$ is continuous in $D = \mathbb{R}$, hence it is continuous in $I = [0, 2]$. Therefore Weierstrass theorem applies.

$$(f)\ f(x) = \log(|x-1|) \qquad\qquad I = [0, 2]$$

$f(x) = \log(|x-1|)$ is continuous in $D = \mathbb{R} \setminus \{1\}$, hence it is not continuous in $I = [0, 2]$. Therefore Weierstrass theorem does not apply.

(3) For each equation, prove that a solution exists in the indicated interval and identify the solution graphically (as intersection point of suitable functions)

$$(a)\ \sin(x) - x = 0 \qquad\qquad I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Let $f(x) = \sin(x) - x$. $f(x)$ is continuous for all $x \in mathbbR$. Then it is continuous in $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Moreover

$$f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} > 0$$
$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2} < 0$$

Then by the Theorem on existence of zeros, there is a point $x_0 \in I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $f(x_0) = 0$.

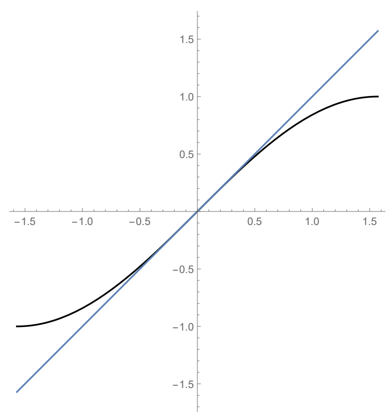

FIGURE 1. Black: $h(x) = \sin(x)$, Blue: $g(x) = x$

(b) $\dfrac{x-1}{x} + x^3 = 0$ $\qquad I = \left[\frac{1}{2}, 2\right]$

Let $f(x) = \dfrac{x-1}{x} + x^3$. $f(x)$ is continuous for all $x \in mathbbR \setminus \{0\}$. Then it is continuous in $I = \left[\frac{1}{2}, 2\right]$. Moreover

$$f\left(\frac{1}{2}\right) = -1 + \frac{1}{8} < 0$$
$$f(2) = \frac{1}{2} + 8 > 0$$

Then by the Theorem on existence of zeros, there is a point $x_0 \in I = \left[\frac{1}{2}, 2\right]$ such that $f(x_0) = 0$.

(c) $e^x + 3x = 0$ $\qquad I = [-1, 0]$

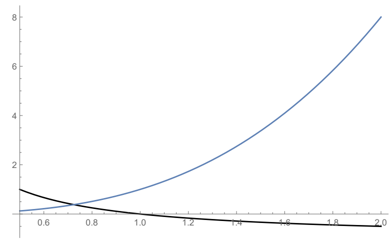

FIGURE 2. Black: $h(x) = -\frac{x-1}{x}$, Blue: $g(x) = x^3$

Let $f(x) = e^x + 3x$. $f(x)$ is continuous for all $x \in mathbbR$. Then it is continuous in $I = [-1, 0]$. Moreover

$$f(-1) = e^{-1} - 3 < 0$$
$$f(0) = 1 > 0$$

Then by the Theorem on existence of zeros, there is a point $x_0 \in I = [-1, 0]$ such that $f(x_0) = 0$.

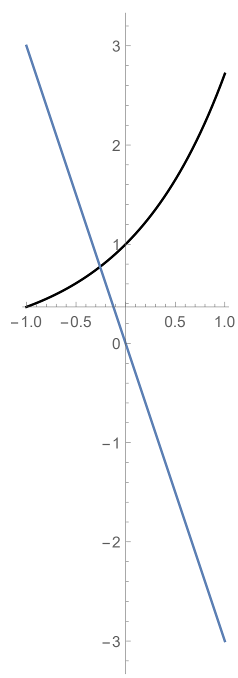

FIGURE 3. Black: $h(x) = e^x$, Blue: $g(x) = -3x$

(d) $x^3 - 3x^2 + 1 = 0$ $\qquad I = [0, 2]$

Let $f(x) = x^3 - 3x^2 + 1$. $f(x)$ is continuous for all $x \in mathbbR$. Then it is continuous in $I = [0, 2]$. Moreover

$$f(0) = 1 > 0$$
$$f(2) = -3 < 0$$

Then by the Theorem on existence of zeros, there is a point $x_0 \in I = [0, 2]$ such that $f(x_0) = 0$.

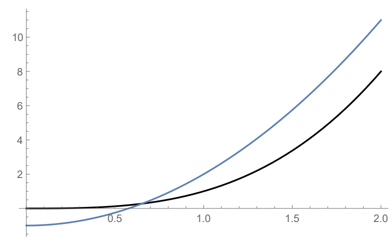

FIGURE 4. Black: $h(x) = x^3$, Blue: $g(x) = 3x^2 - 1$

(e) $3 - x - \log(x)$ $\qquad$ $I = [0, 3]$

Let $f(x) = 3 - x - \log(x)$. $f(x)$ is continuous for all $x \in (0, +\infty)$. Then it is continuous in $[a, 3]$ for all $a > 0$. Let, for instance $a = 1$. We get that

$$f(1) = 2 > 0$$
$$f(3) = -\log(3) < 0$$

Then by the Theorem on existence of zeros, there is a point $x_0 \in I = [1, 3]$ such that $f(x_0) = 0$.

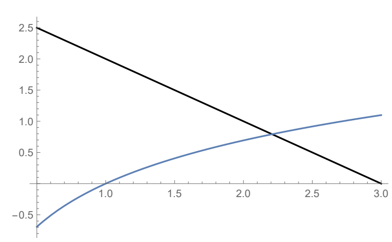

FIGURE 5. Black: $h(x) = 3 - x$, Blue: $g(x) = \log(x)$

(4) Compute all horizontal and vertical asymptotes of the functions below

(a) $f(x) = \dfrac{1}{x^3} - x$ $\qquad$ (b) $f(x) = \dfrac{1}{x^2 + 2}$

(c) $f(x) = \dfrac{x^2 - 4x - 1}{x^2 - 4}$ (d) $f(x) = \dfrac{x^2}{x^2 - x}$

(e) $f(x) = x + \sqrt{x}$ (f) $f(x) = \dfrac{\sqrt{x}}{x + 1}$

(g) $f(x) = \sqrt{3 - x^2} - 4x$ (h) $f(x) = \dfrac{x^2 - x - 3}{x + 1}$

(i) $f(x) = e^{-\frac{x+3}{x}}$ (j) $f(x) = x\left(e^{\frac{1}{x}} - 1\right)$

(k) $f(x) = \dfrac{3}{x^3(x - 1)}$ (l) $f(x) = \dfrac{1}{\sqrt{x} - \sqrt{x - 1}}$

(m) $f(x) = \dfrac{\log(x + 1)}{x^2}$ (n) $f(x) = \dfrac{1}{\log(x)}$

(a) Vertical asymptote $x = 0$, no horizontal asymptote
(b) Horizontal asymptote $y = 0$, no vertical asymptote
(c) Horizontal asymptote $y = 1$, vertical asymptotes $x = 2$, $x = -2$
(d) Horizontal asymptote $y = 1$, vertical asymptote $x = 1$
(e) No vertical nor horizontal asymptotes
(f) Horizontal asymptote $y = 0$, no vertical asymptote
(g) No vertical nor horizontal asymptotes
(h) Vertical asymptote $x = -1$, no horizontal asymptote
(i) Vertical asymptote $x = 0$, horizontal asymptote $y = \frac{1}{e}$
(j) Vertical asymptote $x = 0$, horizontal asymptote $y = 1$
(k) Vertical asymptotes $x = 0$, $x = 1$, horizontal asymptote $y = 0$
(l) No vertical nor horizontal asymptotes
(m) Vertical asymptotes $x = 0$, $x = -1$, horizontal asymptote $y = 0$
(n) Vertical asymptote $x = 1$, horizontal asymptote $y = 0$