MATHEMATICS 1
ADDITIONAL EXERCISES N. 6

KATIA COLANERI

Notation: log stands for the natural logarithm (i.e. the logarithm with the basis e)

1. Derivatives and Applications

(1) Use the definition of the derivative to compute the derivative of the following functions at the given point x_0

$$f(x) = 3x^2 + 1, \qquad x_0 = 0$$
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{3h^2 + 1 - 1}{h} = \lim_{h\to 0} 3h = 0$$
$$f(x) = 5x + 7, \qquad x_0 = 1$$
$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{5(1+h) + 7 - 5 - 7}{h} = 5$$
$$f(x) = \sqrt{x}, \qquad x_0 = 2$$
$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{2+h} + \sqrt{2})} = \frac{1}{2\sqrt{2}}$$
$$f(x) = \frac{x+1}{x}, \qquad x_0 = -3$$
$$\lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0} \frac{\frac{-3+h+1}{-3+h} - \frac{-3+1}{-3}}{h} = -\frac{1}{9}$$

Katia Colaneri, Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, 00133 Rome, Italy.

E-mail address: katia.colaneri@uniroma2.it.

$$f(x) = \log(x+3), \qquad x_0 = 0$$
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\log(h+3)-\log(3)}{h} = \frac{1}{3}$$
$$f(x) = \sqrt{x^2+2}, \qquad x_0 = 3$$
$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{\sqrt{(3+h)^2+2}-\sqrt{3^2+2}}{h} = \frac{3}{\sqrt{11}}$$
$$f(x) = 3x^2+1, \qquad x_0 = 0$$
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{3h^2+1-1}{h} = 0$$
$$f(x) = |x^2-4|, \qquad x_0 = -1$$
$$\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{|(-1+h)^2-4|-|-3|}{h} = 2$$
$$f(x) = \frac{5}{x+1}, \qquad x_0 = -2$$
$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}\frac{\frac{5}{-2+h+1}-\frac{5}{-2+1}}{h} = -5$$
$$f(x) = e^{2x}, \qquad x_0 = 0$$
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{2h}-1}{h} = 2$$

(2) Compute the derivatives of the following functions

1. $f'(x) = 1 + \cos(x) - 3\sin(x)$

2. $f'(x) = e^x + \dfrac{1}{x} + 6x$

3. $f'(x) = \dfrac{1}{2\sqrt{x}} + \dfrac{9\sqrt{x}}{2} + \dfrac{4}{3x^{\frac{1}{3}}}$

4. $f'(x) = -\dfrac{14}{x^3} - 2e^{-x}$

5. $f'(x) = 3x^2 - 3\cos(x)$

6. $f'(x) = 2xe^x + e^x x^2$

7. $f'(x) = 2\left(\dfrac{\log(x)}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}\right)$

8. $f'(x) = \log(x) + 1$

9. $f'(x) = \dfrac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x) + \cos(x)$

10. $f'(x) = 2x\sin(x) + \cos(x)\, x^2$

11. $f'(x) = 2e^x - 2e^{2x} + \cos(x) - 3\sin(x)$

12. $f'(x) = e^x x^2 + e^x x - e^x$

13. $f'(x) = 2e^x \cos(x)$

14. $f'(x) = \dfrac{1}{\cos^2(x)}$

15. $f'(x) = \dfrac{1}{\cos^2(x)}(1 - 2\tan(x))$

16. $f'(x) = 2\sin(x)\tan(x) + \dfrac{1}{\cos^2(x)}(1 - 2\cos(x))$

17. $f'(x) = 2x\arctan(x) + \dfrac{x^2}{x^2+1}$

18. $f'(x) = -\dfrac{x\arcsin(x)}{\sqrt{1-x^2}} + 1$

19. $f'(x) = 3x^2\sin(x)\ln(x) + \left(\cos(x)\ln(x) + \dfrac{\sin(x)}{x}\right)x^3$

20. $f'(x) = 12(3x+1)^3$

21. $f'(x) = 6\cos(3x)$

22. $f'(x) = e^{2x+1} \cdot 2 + \dfrac{2x+1}{x^2+x}$

23. $f'(x) = (x^2+2x)^3\left[8(x+1)\left(\dfrac{3}{x} - 2x\right) - \left(\dfrac{3}{x^2} + 2\right)(x^2+2x)\right]$

24. $f'(x) = \dfrac{3x+1}{\sqrt{3x^2+2x+5}}$

25. $f'(x) = \dfrac{x(2x^2+1)}{\sqrt{\log(x^4+x^2+1)}\,(x^4+x^2+1)}$

26. $f'(x) = \dfrac{2\log(x)}{x} + 3$

27. $f'(x) = \log^3(x) + \dfrac{3\log^2(x)(x+1)}{x}$

28. $f'(x) = \dfrac{4x^2+8x+5}{(x+1)^2}$

29. $f'(x) = \dfrac{x+2}{(x^2+1)\sqrt{x^2+1}}$

30. $f'(x) = -\dfrac{1}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}$

31. $f'(x) = \dfrac{-x+2}{2\sqrt{x}\,(x+2)^2}$

32. $f'(x) = \dfrac{e^x - e^{-x} + 2}{(1-e^x)^2}$

33. $f'(x) = \dfrac{1}{x\,(1+\log(x))^2}$

34. $f'(x) = \dfrac{2\sin^2(x) - \cos(x)\,(1-2\cos(x))}{\sin^2(x)}$

35. $f'(x) = \dfrac{-\frac{1}{\cos^2(x)}\,(\sin(x)-\cos(x)) - (\cos(x)+\sin(x))\,(1-\tan(x))}{1-\sin(2x)}$

36. $f'(x) = -\sin\left(x^4+4x\right)\left(4x^3+4\right)$

37. $f'(x) = -2e^{5-x^2}x$

38. $f'(x) = -\dfrac{1}{x\,(x-1)}$

39. $f'(x) = \dfrac{x}{x^2+4}$

40. $f'(x) = \dfrac{6\cos(3x+1)}{\sin(3x+1)}$

41. $f'(x) = \dfrac{e^{2x}-1}{e^{2x}+1}$

42. $f'(x) = \dfrac{x}{\cos^2(x)}$

43. $f'(x) = e^{\frac{x-1}{x}} + \dfrac{e^{\frac{x-1}{x}}}{x}$

44. $f'(x) = \dfrac{1}{x^3\,(x+1)}$

45. $f'(x) = \dfrac{2}{4x^2+1}$

46. $f'(x) = \dfrac{e^x}{\sqrt{1-e^{2x}}}$

47. $f'(x) = 5\left(e^{3x^2+2} - 2x\right)^4 \left(e^{3x^2+2}\cdot\, 6x - 2\right)$

48. $f'(x) = \dfrac{\log(x)}{4\sqrt{x}} + \dfrac{1}{2\sqrt{x}}$

49. $f'(x) = e^{\sqrt{x}} + \dfrac{e^{\sqrt{x}}\sqrt{x}}{2}$

50. $f'(x) = 2x\log(x+1) + \dfrac{x^2}{x+1}$

(3) For each of the following functions, compute the equation of the tangent line at the given point x_0

$$f(x) = 3x^2 - 9x + 4, \qquad x_0 = 1$$

Tangent line: $y = 1 - 3x$

$$f(x) = x^2 + 1, \qquad x_0 = 1$$

Tangent line: $y = 2x$

$$f(x) = \frac{x+1}{x}, \qquad x_0 = -1$$

Tangent line: $y = -x - 1$

$$f(x) = 5 + \log(x), \qquad x_0 = 1$$

Tangent line: $y = x + 4$

$$f(x) = \sqrt{1 - x^3}, \qquad x_0 = 1$$

No tangent line exists at $x_0 = 1$

$$f(x) = e^{4x-1}, \qquad x_0 = -\frac{1}{2}$$

Tangent line: y $= \frac{4x}{e^3} + \frac{3}{e^3}$

$$f(x) = \frac{x^2}{x+3}, \qquad x_0 = -1$$

Tangent line: $y = -\frac{5x}{4} - \frac{3}{4}$

$$f(x) = \log(3x + 2), \qquad x_0 = 0$$

Tangent line: $y = \frac{3x}{2} + \log(2)$

(4) For each of the following functions, say if they continuous and differentiable, and if not, identify the nature of discontinuity and non-differentiability points

$$1. \qquad f(x) = |9 - x^2|$$

Continuous in $\mathbb{R}$, differentiable in $\mathbb{R} \setminus \{-3, 3\}$. $x = 3$ and $x = -3$ are angle points.

$$2. \qquad f(x) = |x^2 + 3|$$

Continuous and differentiable in $\mathbb{R}$

$$3. \quad f(x) = \sqrt{x^2 - 4x + 3}$$

Continuous in $(-\infty\,,\ 1] \cup\ [3,\ \infty\,)$. Differentiable in $(-\infty\,,\ 1) \cup\ (3,\ \infty\,)$.

$$4. \qquad f(x) = \sqrt[3]{2x^2 - 8}$$

Continuous in $\mathbb{R}$. Differentiable in $\mathbb{R} \setminus \{-2, 2\}$. $x = -2$ and $x = 2$ are

$$5. \quad f(x) = \sqrt[3]{3x-1}$$

Continuous in $\mathbb{R}$. Differentiable in $\mathbb{R} \setminus \{1/3\}$. $x = 1/3$ is an inflection point with vertical tangent.

$$6. \qquad f(x) = x\sqrt{x}$$

Continuous in $[0, +\infty)$, differentiable in $[0, +\infty)$.

$$7. \quad f(x) = |x| + |x+1|$$

Continuous in $\mathbb{R}$, differentiable in $\mathbb{R} \setminus \{-1, 0\}$. $x = 0$ and $x = -1$ are angle points.

$$8. \qquad f(x) = \frac{1}{\log(x)}$$

Continuous in $(0,1) \cup (1, +\infty)$. Differentiable in $(0,1) \cup (1, +\infty)$.

$$9. \quad f(x) = x\sqrt[3]{x}$$

Continuous in $\mathbb{R}$, differentiable in $\mathbb{R}$.

$$10. \qquad f(x) = \frac{|x^2 - x|}{x}$$

Continuous in $(-\infty\ ,\ 0) \cup (0,\ \infty\ )$. Differentiable in $(-\infty\ ,\ 0) \cup (0,1) \cup (1\ \infty\ )$. $x = 1$ is an angle point.

(5) For each of the following functions, compute the first order Taylor approximation $P(x)$ at the point x_0. Evaluate also the error (absolute R and in percentage r) that is made by

approximating the function $f(x_1)$ with $P(x_1)$.

(1). $f(x)=\sqrt{x+1}, \qquad x_0=0, \qquad x_1=0.1$

$P(x)=1+\frac{1}{2}x \qquad R(0.1)=\sqrt{1.1}-1.05\sim -0.0012, \quad r(0.1)=\frac{R(0.1)}{\sqrt{1.1}}\sim 0.0011=-0.11\%$

(2). $f(x)=x^3, \qquad x_0=2, \qquad x_1=2.01$

$P(x)=8+12\,(x-2) \qquad R(2.01)\sim 0.240601, \quad r(2.01)\sim 2.9\%$

(3). $f(x)=\frac{1}{x+1}, \qquad x_0=0, \qquad x_1=0.01$

$P(x)=1-x \qquad R(0.01)\sim 0.0000990099, \quad r(0.01)\sim 0.01\%$

(4). $f(x)=\frac{x+1}{x}, \qquad x_0=-1, \qquad x_1=-0.9$

$P(x)=-\,(x+1) \qquad R(-0.9)\sim -0.0111111, \quad r(-0.9)\sim 10\%$

(5). $f(x)=\log(x+1), \qquad x_0=0, \qquad x_1=0.2$

$P(x)=x \qquad R(0.2)\sim -0.0176784, \quad r(0.2)\sim -9\%$

(6). $f(x)=\sqrt{1-x^3}, \qquad x_0=1, \qquad x_1=1.01$

$\nexists P(x)$

(7). $f(x)=e^x, \qquad x_0=0, \qquad x_1=-0.2$

$P(x)=1+x \qquad R(-0.2)\sim 0.0187308, \quad r(-0.2)=2.2\%$

(8). $f(x)=\sin(x), \qquad x_0=0, \qquad x_1=\frac{\pi}{6}$

$P(x)=x \qquad R(\pi/6)\sim -0.023, \quad r(\pi/6)\sim -4.7\%$

(6) Determine the intervals in which the functions are increasing and decreasing

$$1. \qquad f(x)=x^2+2x$$

Increasing: $x>-1$ Decreasing: $x<-1$

$$2. \qquad f(x)=x^2-2x+3$$

Increasing:$x>1$ Decreasing: $x<1$

$$3. \quad f(x)=2x^4-27x$$

Increasing: $x>\frac{3}{2}$ Decreasing: $x<\frac{3}{2}$

$$4. \qquad f(x)=\frac{1}{x+1}$$

Increasing: never Decreasing: $x\in(-\infty,-1)$ or $x\in(-1,+\infty)$

$$5. \quad f(x)=\frac{x^2-1}{x}$$

Increasing:$x \in (-\infty, 0)$ or $x \in (0, +\infty)$ Decreasing: never

$$6. \qquad f(x) = \sqrt{x+1}$$

Increasing: $x > -1$ Decreasing: never

$$7. \quad f(x) = x\sqrt{2x+1}$$

Increasing: $x > -\frac{1}{3}$ Decreasing: $-\frac{1}{2} < x < -\frac{1}{3}$

$$8. \qquad f(x) = x + \sin(x)$$

Increasing: for all $x \in \mathbb{R}$ Decreasing: never

$$9. \quad f(x) = x + \cos(x)$$

Increasing: for all $x \in \mathbb{R}$ Decreasing: never

$$10. \qquad f(x) = \log(x) - \frac{e}{x}$$

Increasing: for all $x > 0$ Decreasing: never

$$11. \quad f(x) = x^2 - \log(x^2 - 1)$$

Increasing: $-\sqrt{2} < x < -1$ or $x > \sqrt{2}$ Decreasing: $x < -\sqrt{2}$ or $1 < x < \sqrt{2}$

$$12. \qquad f(x) = \frac{1 + \log(x)}{x}$$

Increasing: $0 < x < 1$ Decreasing: $x > 1$

$$13. \quad f(x) = \log^3(x) - \log^2(x)$$

Increasing: $0 < x < 1$ or $x > e^{2/3}$ Decreasing: $1 < x < e^{2/3}$

$$14. \qquad f(x) = \log\left(\frac{x^2+4}{x^2-4}\right)$$

Increasing: $x < -2$ Decreasing: $x > 2$

$$15. \quad f(x) = e^x - e^{-x}$$

Increasing: for all $x \in \mathbb{R}$ Decreasing: never

$$16. \qquad f(x) = (x+1)e^x$$

Increasing: $x > -2$ Decreasing: $x < -2$

$$17. \quad f(x) = \frac{x-1}{e^x}$$

Increasing: $x < 2$ Decreasing: $x > 2$

$$18. \qquad f(x) = \frac{x+1}{x-2}$$

Increasing: $x < 2$ Decreasing: $x > 2$

$$19. \quad f(x) = 1 + 2\sqrt{x^2 - 1}$$

Increasing: $x > 1$ Decreasing: $x < -1$

$$20. \qquad f(x) = \frac{1}{1-\sqrt{x}}$$

Increasing: $0 < x < 1$ Decreasing: $x > 1$

(7) Determine the intervals in which the functions are concave and convex

$$1. \qquad f(x) = 4x^3 - 1$$

$$\text{Concave } : -\infty \ < x < 0, \text{ Convex } : 0 < x < \infty$$

$$2. \qquad f(x) = x^3 - 3x^2 + 3$$

$$\text{Concave } : -\infty \ < x < 1, \text{ Convex } : 1 < x < \infty$$

$$3. \quad f(x) = 2x^4 - 2x^3$$

$$\text{Convex } : -\infty \ < x < 0 \text{ and } \frac{1}{2} < x < \infty, \text{ Concave } : 0 < x < \frac{1}{2}$$

$$4. \qquad f(x) = \frac{1}{x-1}$$

$$\text{Concave } : -\infty \ < x < 1, \text{ Convex } : 1 < x < \infty$$

$$5. \quad f(x) = \frac{x}{x+2}$$

$$\text{Convex } : -\infty \ < x < -2, \text{ Concave } : -2 < x < \infty$$

$$6. \qquad f(x) = x\sqrt{x^2+1}$$

$$\text{Concave } : -\infty \ < x < 0, \text{ Convex } : 0 < x < \infty$$

$$7. \quad f(x) = e^{-x^2+2x}$$

$$\text{Convex } : -\infty \ < x < \frac{2-\sqrt{2}}{2} \text{ and } \sqrt{\frac{1}{2}} + 1 < x < \infty \text{ , Concave } : \frac{2-\sqrt{2}}{2} < x < \sqrt{\frac{1}{2}} + 1$$

$$8. \qquad f(x) = x^2 - x\log(x)$$

$$\text{Concave } : 0 < x < \frac{1}{2}, \text{ Convex } : \frac{1}{2} < x < \infty$$

$$9. \quad f(x) = \frac{x^3-1}{x^3+1}$$

$$\text{Convex } : -\infty \ < x < -1 \text{ and } 0 < x < \frac{2^{\frac{2}{3}}}{2}, \text{ , Concave } : -1 < x < 0 \text{ and } \frac{2^{\frac{2}{3}}}{2} < x < \infty$$

$$10. \qquad f(x) = \log^3(x) - 9\log(x)$$

$$\text{Concave } : 0 < x < \frac{1}{e} \text{ and } e^3 < x < \infty \text{ , Convex } : \frac{1}{e} < x < e^3$$

$$11. \quad f(x) = x(x+2)^3$$

$$\text{Convex } : -\infty \; < x < -2 \text{ and } -1 < x < \infty \, , \; \text{Concave } : -2 < x < -1$$

$$12. \qquad f(x) = \frac{1}{4}x^4 - 2x^2$$

$$\text{Convex } : -\infty \; < x < -\frac{2}{\sqrt{3}} \text{ and } \frac{2}{\sqrt{3}} < x < \infty \, , \; \text{Concave } : -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$$

$$13. \quad f(x) = \frac{x^3+1}{x}$$

$$\text{Convex } : -\infty \; < x < -1 \text{ and } 0 < x < \infty \, , \; \text{Concave } : -1 < x < 0$$

$$14. \qquad f(x) = x^2 - \sin(x)$$

$$\text{Convex } : -\infty \; < x < \infty$$

$$15. \quad f(x) = e^{x^2-1}$$

$$\text{Convex } : -\infty \; < x < \infty$$

$$16. \qquad f(x) = \log(x^3 - 1)$$

$$\text{Concave } : 1 < x < \infty$$

$$17. \quad f(x) = \frac{x}{e^x}$$

$$\text{Concave } : -\infty \; < x < 2, \; \text{Convex } : 2 < x < \infty$$

$$18. \qquad f(x) = \log^2(x+1)$$

$$\text{Concave } : -1 < x < \infty$$

$$19. \quad f(x) = \sqrt{2x-1}$$

$$\text{Concave } : \frac{1}{2} < x < \infty$$

$$20. \qquad f(x) = x^2 - \sqrt{x}$$

$$\text{Convex } : 0 < x < \infty$$

(8) Compute the stationary points of the following functions and determine if these are local maxima, local minima or inflection points with horizontal tangents

$$1. \qquad f(x) = x^3 - 9x$$

$x = -\sqrt{3}$ (local maximum), $x = \sqrt{3}$ (local minimum)

$$2. \qquad f(x) = x^4 - 2x^2$$

$x = -1$ (local minimum), $x = 0$ (local maximum), $x = 1$ (local minimum)

$$3. \quad f(x) = x^4 - 4x^3 + 5$$

$x = 0$ (inflection point with horizontal tangent), $x = 3$ (local minimum)

$$4. \qquad f(x) = \frac{1}{x-1}$$

no stationary points

$$5. \quad f(x) = \frac{x}{x+1}$$

no stationary points

$$6. \qquad f(x) = \frac{3-x}{x+2}$$

no stationary points

$$7. \quad f(x) = \frac{x^2}{x^2+1}$$

$x = 0$ (local minimum)

$$8. \qquad f(x) = x\sqrt[3]{x}$$

$x = 0$ (local minimum)

$$9. \quad f(x) = e^{\frac{1-x}{x^2}}$$

$x = 2$ (local minimum)

$$10. \qquad f(x) = \log(x^2 + 1)$$

$x = 0$ (local minimum)

$$11. \quad f(x) = \frac{\log(x)}{x}$$

$x = e$ (local maximum)

$$12. \qquad f(x) = \log(2x - x^2)$$

$x = 1$ (local maximum)

$$13. \quad f(x) = xe^{-x^2}$$

$x = -1/\sqrt{2}$ (local minimum), $x = 1/\sqrt{2}$ (local maximum)

$$14. \qquad f(x) = e^{x^3-6x^2}$$

$x = 0$ (local maximum), $x = 4$ (local minimum)

$$15. \quad f(x) = \frac{x^2-3}{x-5}$$

$x = 5 - \sqrt{(}22)$ (local maximum) $x = 5 + \sqrt{(}22)$ (local minimum)

$$16. \qquad f(x) = x^2 + \frac{1}{x}$$

$x = 1/2^(1/3)$ (local minimum)

$$17. \quad f(x) = \frac{x^2}{x^2+3x-3}$$

$x = 0$ (local maximum), $x = 2$ (local minimum)

$$18. \qquad f(x) = \frac{x^2+4x}{x^2+6x+5}$$

no stationary points

$$19. \quad f(x) = \log^4(x) - \log^2(x)$$

$x = 1$ (local maximum), $x = e^{1/\sqrt{2}}$ and $x = e^{-1/\sqrt{2}}$ (local minima)

$$20. \qquad f(x) = x - \sqrt{x}$$

$x = \frac{1}{4}$ (local minimum)

(9) Compute the maxima and minima of the following functions in the given intervals

$$1. \qquad f(x) = \sqrt{x^2+1} \quad \text{in} \quad [-1,1]$$

Maximum at $x = -1$ and $x = 1$. Minimum at $x = 0$

$$2. \qquad f(x) = \frac{x^2}{x^2+1} \quad \text{in} \quad [-1,1]$$

Maximum at $x = -1$ and $x = 1$. Minimum at $x = 0$

$$3. \qquad f(x) = x^3 - 3x + 7 \quad \text{in} \quad [0,4]$$

Maximum at $x = 4$. Minimum at $x = 1$

$$4. \qquad f(x) = x\sqrt[3]{x} \quad \text{in} \quad [-2,-1]$$

Maximum at $x=-2$. Minimum at $x=-1$

$$5. \qquad f(x)=\frac{x^2+x+1}{(x-1)^2} \quad \text{in} \quad [-2,2]$$

Minimum at $x=-2$, the maximum does not exist.

Explanation:

Here one cannot apply the Weierstrass Theorem because the function is not continuous in $[-2,2]$. To conclude that the maximum does not exist we observe that $\lim_{x\to 1^+}\frac{x^2+x+1}{(x-1)^2}=+\infty$ (it also holds that $\lim_{x\to 1^-}\frac{x^2+x+1}{(x-1)^2}=+\infty$).

To get that the function has a minimum at $x=-2$ we observe that the function is increasing in $[-2,1)$ and decreasing in $(1,2]$, hence if a minimum exists it has to be at $x=-2$ or at $x=2$. Since $f(-2)=\frac{1}{3}$ and $f(2)=6$, then the minimum is at $x=-2$.

$$6. \qquad f(x)=e^{|x-1|} \quad \text{in} \quad [-3,3]$$

Maximum at $x=-3$. Minimum at $x=1$

$$7. \qquad f(x)=\frac{\log(x)}{x} \quad \text{in} \quad [1,e^2]$$

Maximum at $x=e$. Minimum at $x=1$

$$8. \qquad f(x)=\frac{1}{x^2-3} \quad \text{in} \quad [-1,2]$$

No maximum nor minimum exists

Explanation:

Here one cannot apply the Weierstrass Theorem because the function is not continuous in $[-1,2]$. To conclude that nor the minimum nor the maximum exist we observe that $\lim_{x\to\sqrt{3}^+}\frac{1}{x^2-3}=+\infty$ and $\lim_{x\to\sqrt{3}^-}\frac{1}{x^2-3}=-\infty$.

(10) Use De L'Hopital rule to compute the following limits

$$1. \qquad \lim_{x\to 0}\frac{x\cos(x)-\sin(x)}{x^3}=\lim_{x\to 0}\frac{\cos(x)-x\sin(x)-\cos(x)}{3x^2}=\lim_{x\to 0}\frac{-\sin(x)}{3x}=-\frac{1}{3}$$

$$2. \qquad \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\frac{\frac{1}{3}x^{-2/3}}{1}=\lim_{x\to 1}\frac{1}{3}x^{-2/3}=\frac{1}{3}$$

$$3. \qquad \lim_{x\to 0}\frac{e^x-e^{\sin(x)}}{x+\sin(x)}=\lim_{x\to 0}\frac{e^x-\cos(x)e^{\sin(x)}}{1+\cos(x)}=0$$

$$4. \qquad \lim_{x\to 1} \frac{e^x - e}{x-1} = \lim_{x\to 1} \frac{e^x}{1} = e$$

$$5. \quad \lim_{x\to 0} \frac{x - \log(1+x)}{x} = \lim_{x\to 0} \frac{1 - \frac{1}{1+x}}{1} = 0$$

$$6. \qquad \lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = \lim_{x\to -\infty} -e^x = 0$$

$$7. \quad \lim_{x\to 0} \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

Difficult (multiple application of De l'Hopital rule) We make the common denominator

$$= \lim_{x\to 0} \left(\frac{x^2 - \sin^2(x)}{x^2 \sin^2(x)} \right)$$

We apply De l'Hopital rule

$$= \lim_{x\to 0} \left(\frac{2x - \sin(2x)}{2x \sin^2(x) + \sin(2x)\, x^2} \right)$$

We apply De l'Hopital rule

$$\begin{aligned} &= \lim_{x\to 0} \left(\frac{2 - \cos(2x) \cdot 2}{2x^2 \cos(2x) + 4x \sin(2x) + 2\sin^2(x)} \right) \\ &= \lim_{x\to 0} \left(\frac{2\sin^2(x)}{x^2 \cos(2x) + 2x\sin(2x) + \sin^2(x)} \right) \end{aligned}$$

We apply De l'Hopital rule

$$= \lim_{x\to 0} \left(\frac{2\sin(2x)}{-2x^2 \sin(2x) + 6x\cos(2x) + 3\sin(2x)} \right)$$

We apply De l'Hopital rule

$$\begin{aligned} &= \lim_{x\to 0} \left(\frac{4\cos(2x)}{-4x^2\cos(2x) - 16x\sin(2x) + 12\cos(2x)} \right) \\ &= \frac{4\cos(2\cdot 0)}{-4\cdot 0^2 \cos(2\cdot 0) - 16\cdot 0 \cdot \sin(2\cdot 0) + 12\cos(2\cdot 0)} = \frac{1}{3} \end{aligned}$$

$$8. \qquad \lim_{x\to +\infty} \frac{\log(x^2+3x+5)}{x^2+1} = \lim_{x\to +\infty} \frac{\frac{1}{x^2+3x+5}(2x+3)}{2x} = \lim_{x\to +\infty} \frac{2x+3}{2x(x^2+3x+5)} = 0$$

9.
$$\begin{aligned}\lim_{x\to 0}\frac{2\cos(x)-2+x^2}{x^4} &= \lim_{x\to 0}\frac{-2\sin(x)+2x}{4x^3}\\ &= \lim_{x\to 0}\frac{-2\cos(x)+2}{12x^2}\\ &= \lim_{x\to 0}\frac{2\sin(x)}{24x}\\ &= \lim_{x\to 0}\frac{2\cos(x)}{24} = \frac{1}{12}\end{aligned}$$

10.
$$\begin{aligned}\lim_{x\to 0^+}\frac{1}{x}+\log(x) = \lim_{t\to+\infty} t-\log(t) &= \lim_{t\to+\infty} t\left(1-\frac{\log(t)}{t}\right)\\ &= \lim_{t\to+\infty}\frac{1-\frac{\log(t)}{t}}{\frac{1}{t}}\\ &= \lim_{t\to+\infty}\frac{-\frac{1-\log(t)}{t^2}}{-\frac{1}{t^2}} = +\infty\end{aligned}$$

Notice that this can be solved in a much simpler way by using the notable limits.

2. Sketch of the plot of a function

(1) for each of the following functions determine, if possible
  (a) the domain
  (b) the sign
  (c) the asymptotes
  (d) the intervals in which functions are increasing and decreasing
  (e) the intervals where the functions are concave and convex
  (f) local maxima, local minima and inflection points

  Finally use the information collected above to sketch the graph of the functions.

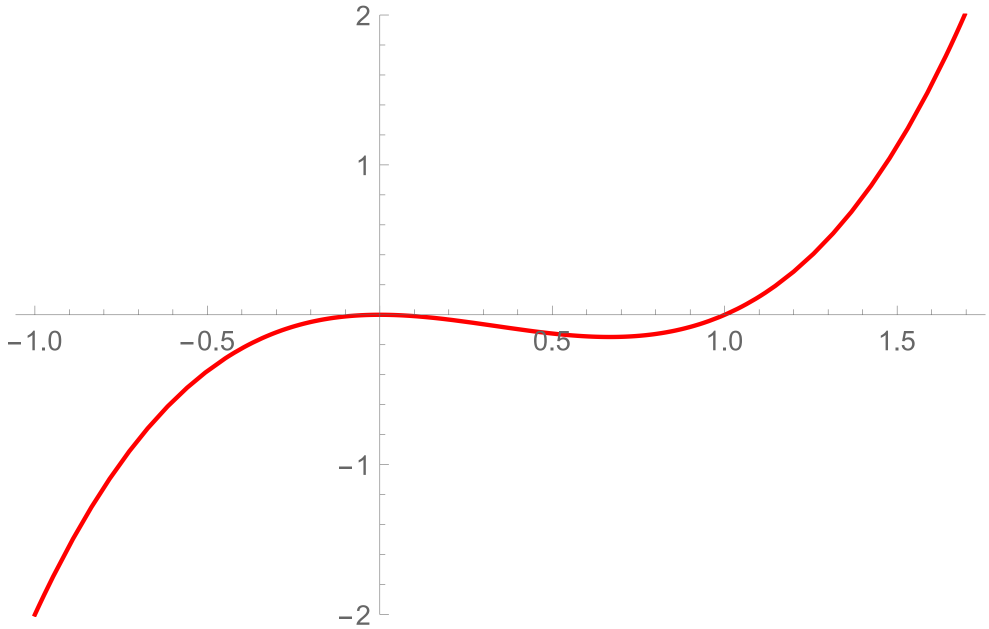

FIGURE 1. $f(x) = x^3 - x^2$

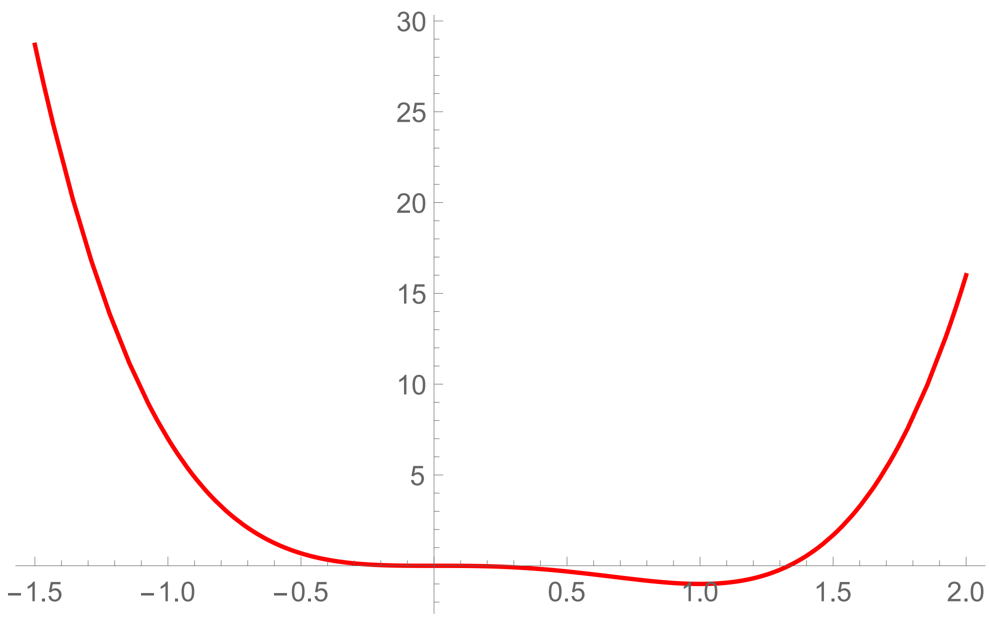

FIGURE 2. $f(x) = 3x^4 - 4x^3$

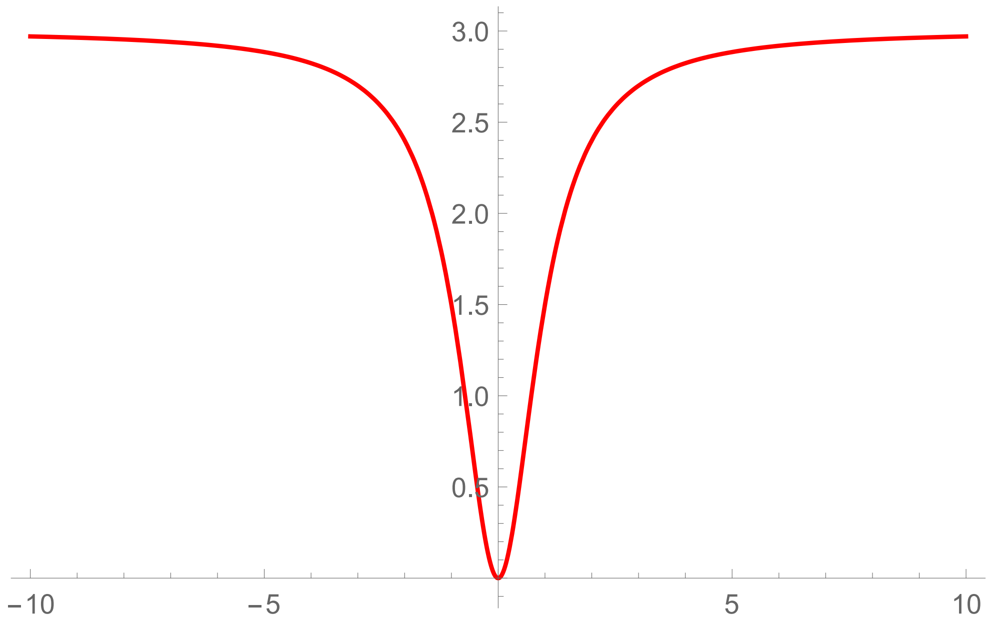

FIGURE 3. $f(x) = \dfrac{3x^2}{x^2+1}$

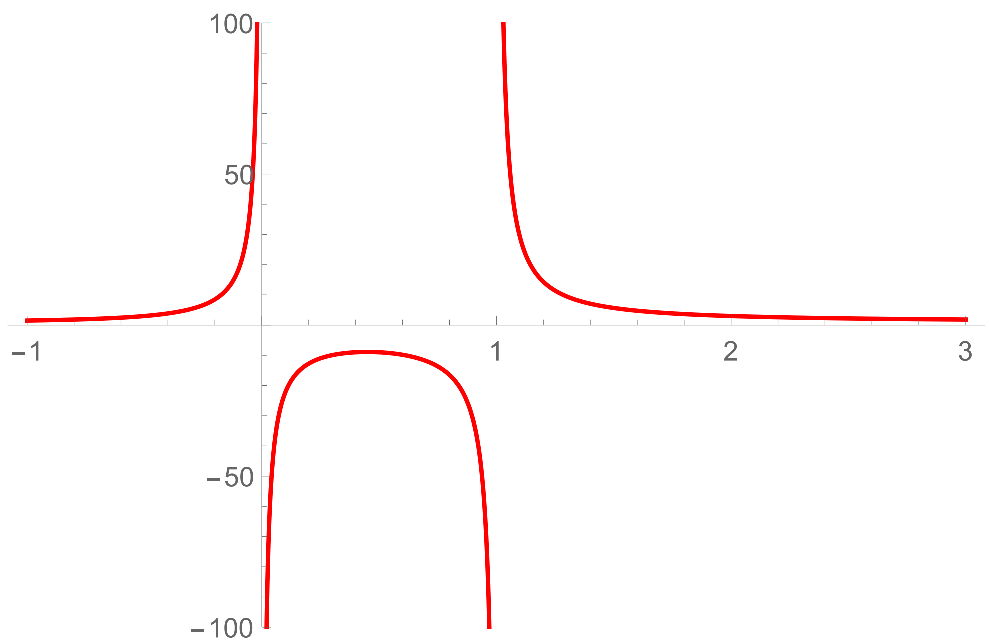

FIGURE 4. $f(x) = \dfrac{x^2+2}{x^2-x}$

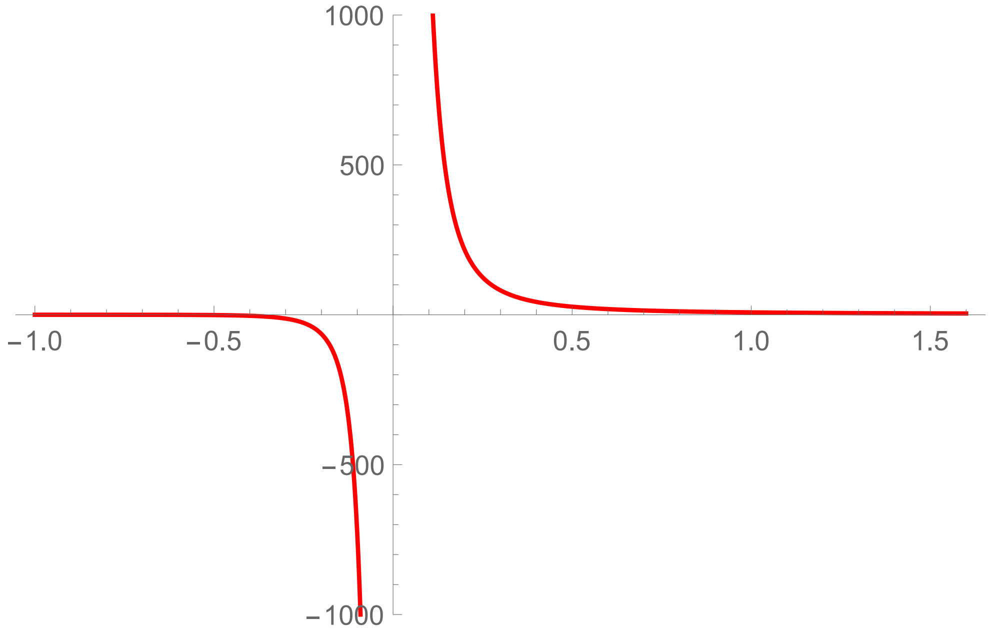

FIGURE 5. $f(x) = \left(1 + \dfrac{1}{x}\right)^3$

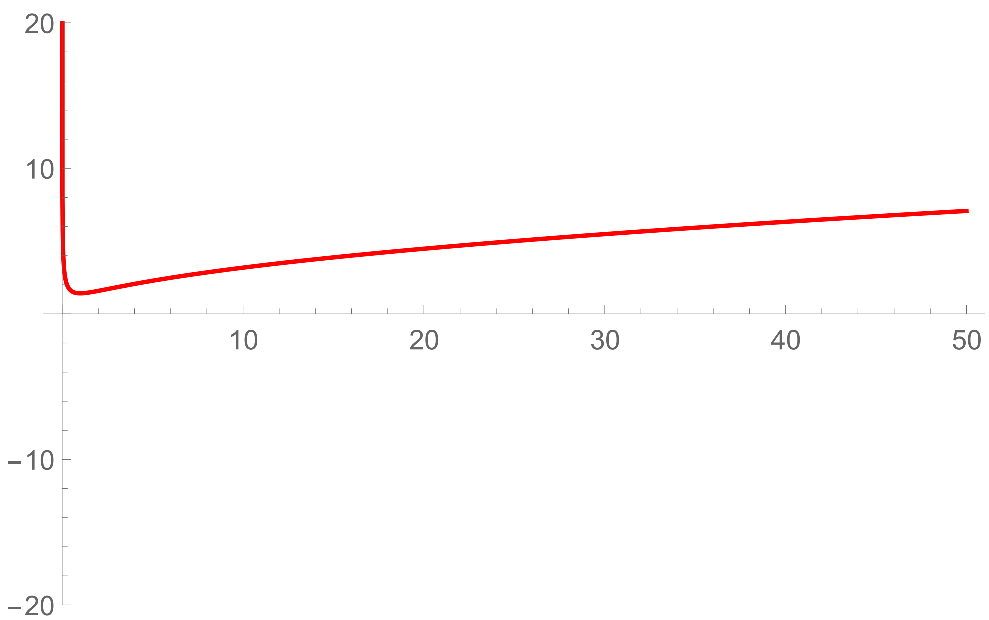

FIGURE 6. $f(x) = \sqrt{x + \dfrac{1}{x}}$

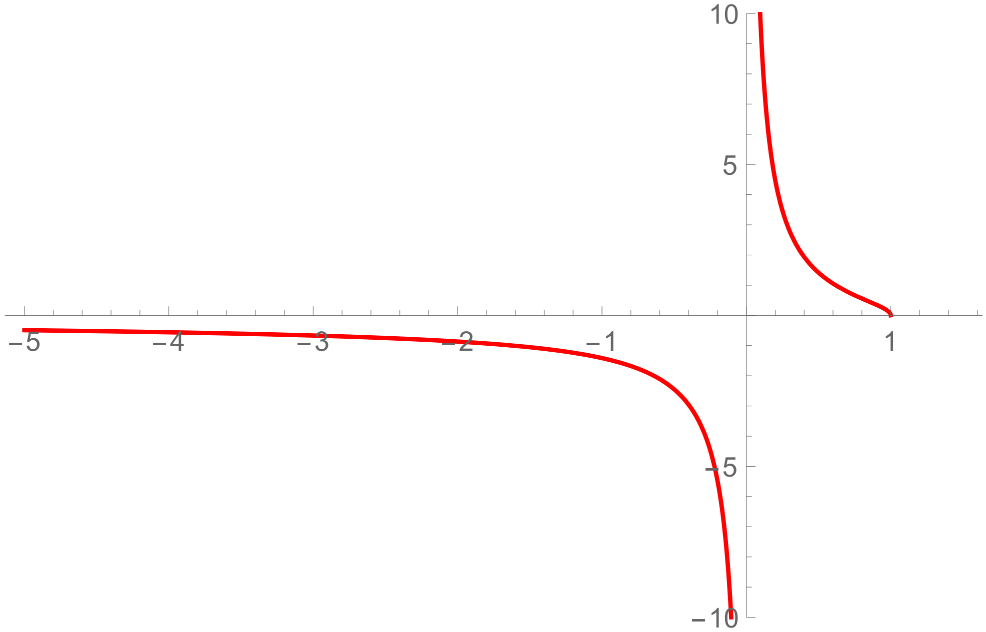

FIGURE 7. $f(x) = \dfrac{\sqrt{1-x}}{x}$

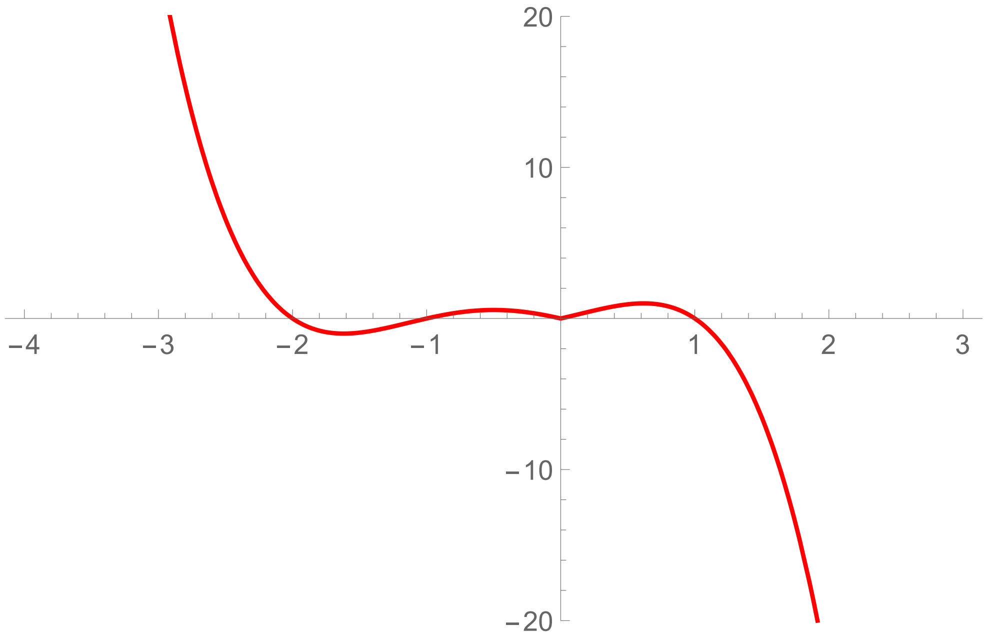

FIGURE 8. $f(x) = \dfrac{(x+2)|x|}{1-x^2}$

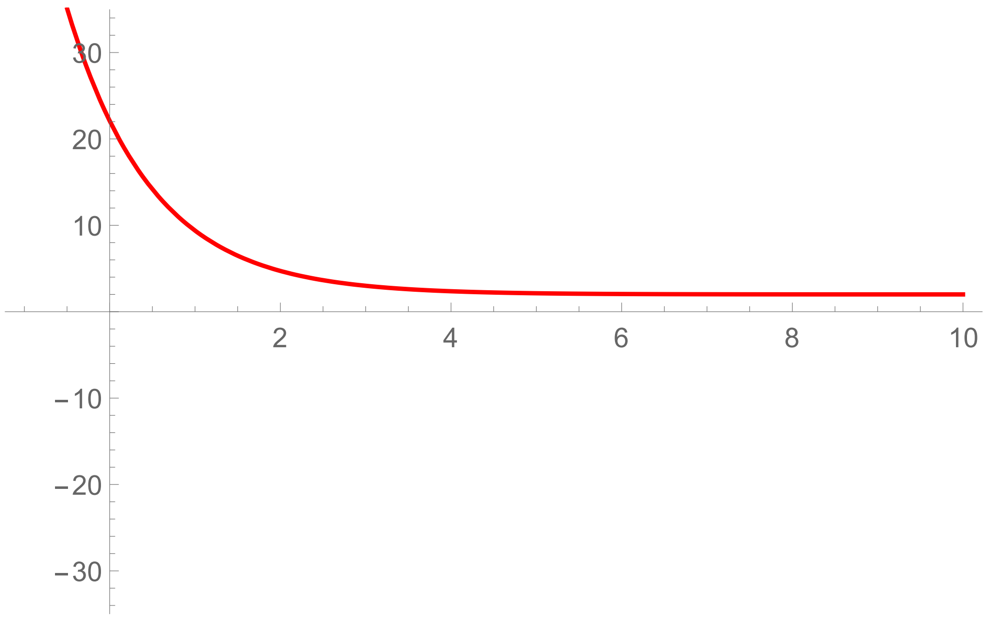

FIGURE 9. $f(x) = 2 + e^{-x+3}$

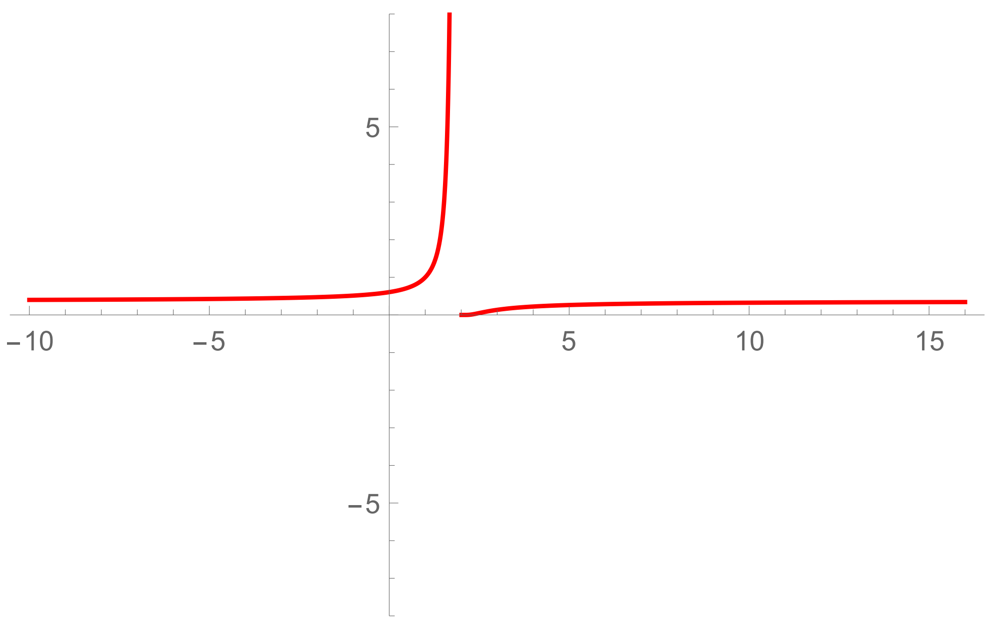

FIGURE 10. $f(x) = e^{\frac{x-1}{2-x}}$

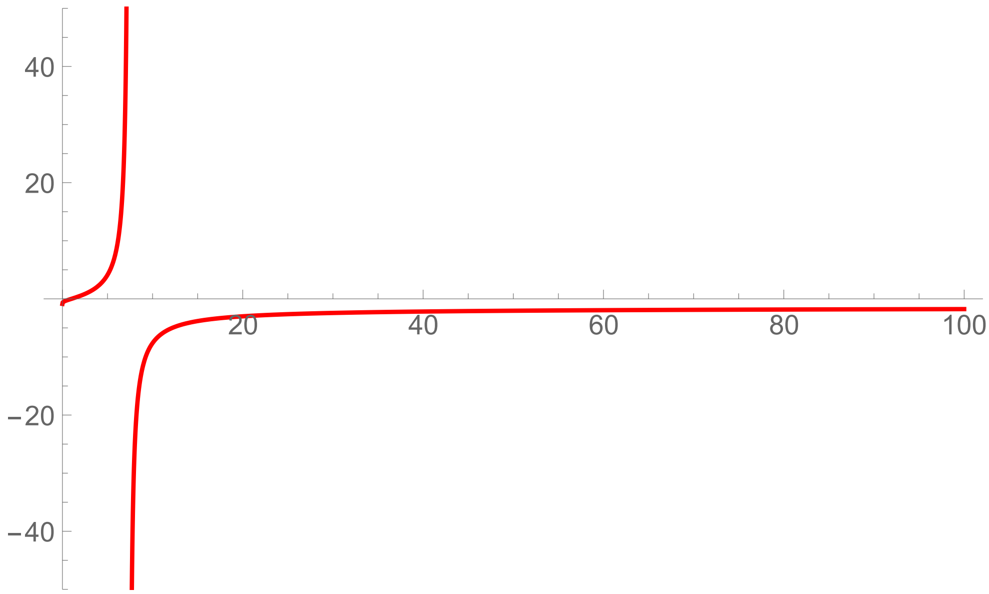

FIGURE 11. $f(x) = \dfrac{\log(x)}{2 - \log(x)}$

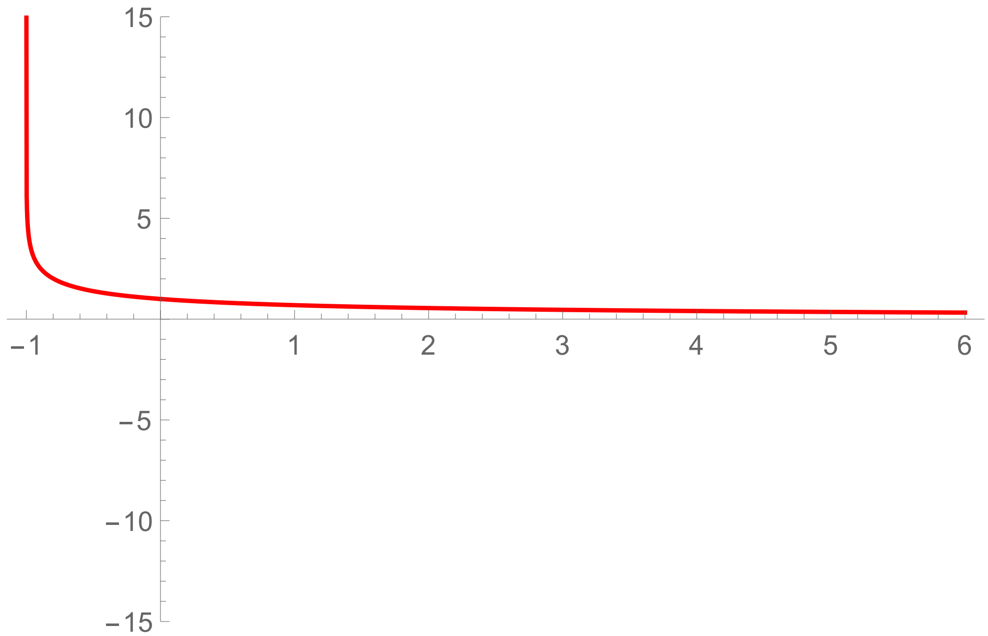

FIGURE 12. $f(x) = \dfrac{\log(x+1)}{x}$

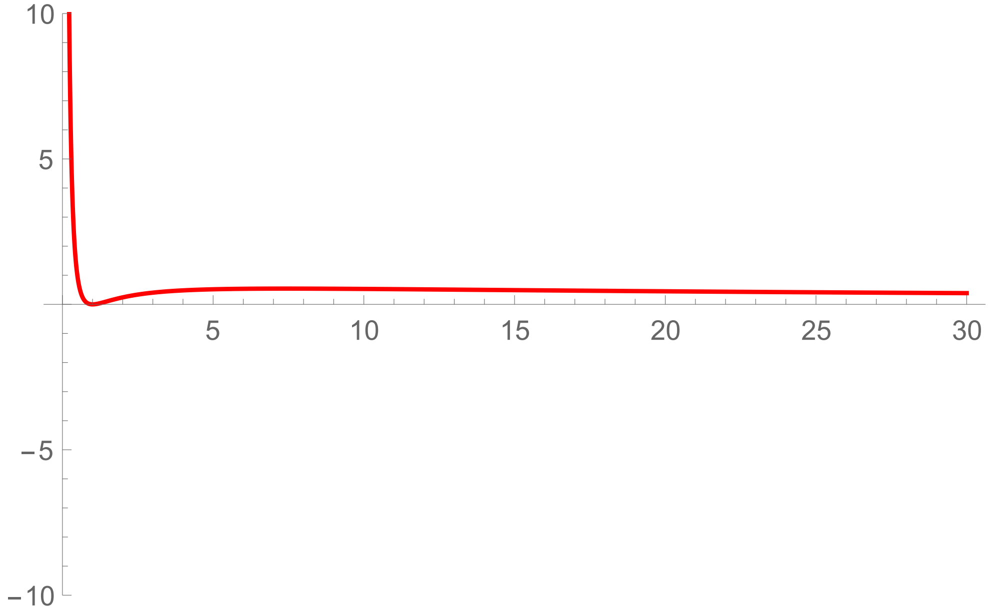

FIGURE 13. $f(x) = \dfrac{\log^2(x)}{x}$

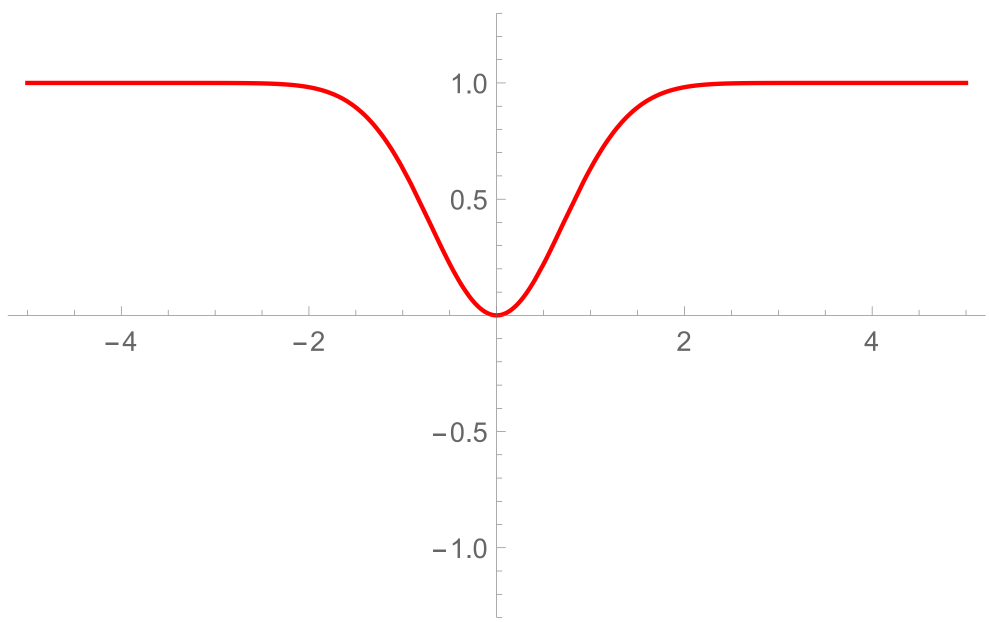

FIGURE 14. $f(x) = 1 - e^{-x^2}$

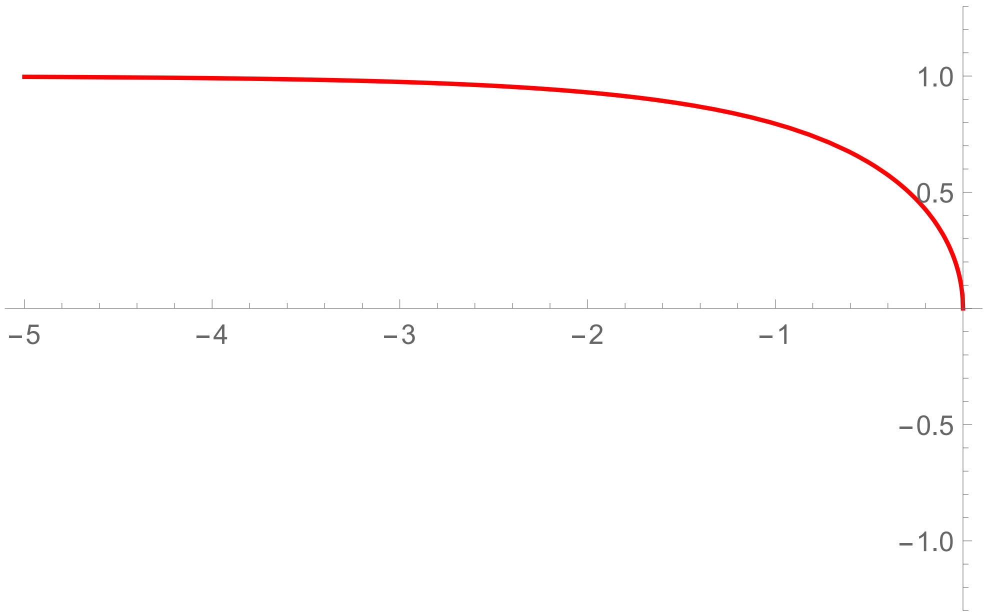

FIGURE 15. $f(x) = \sqrt{1 - e^x}$

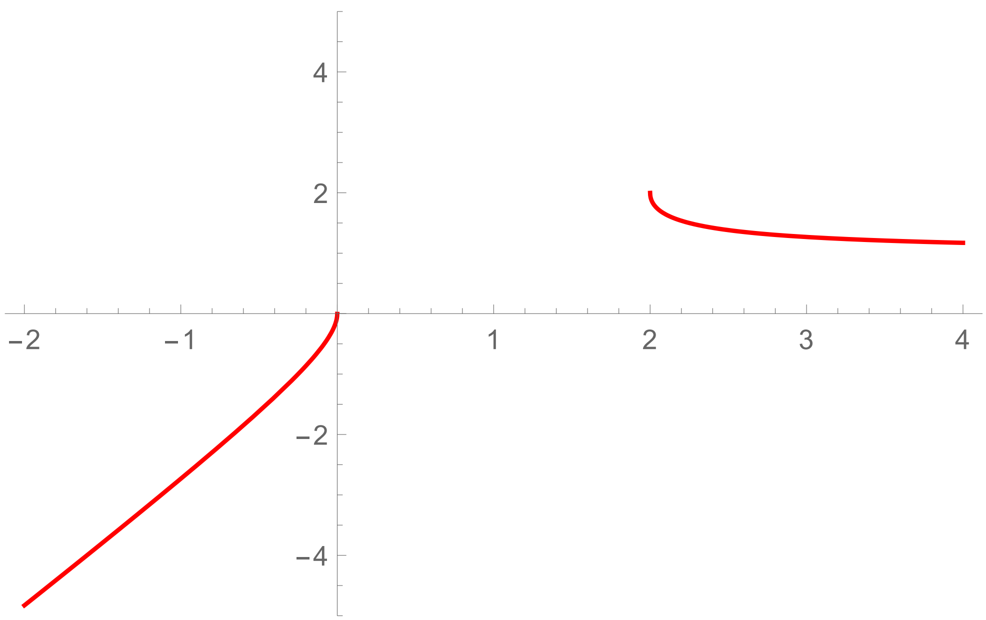

FIGURE 16. $f(x) = x - \sqrt{x^2 - 2x}$

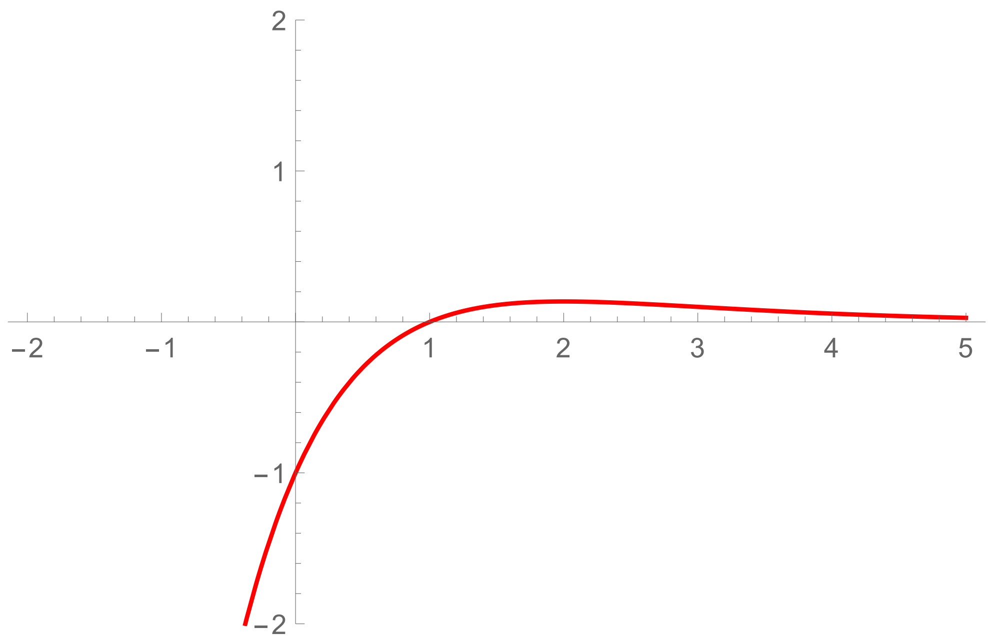

FIGURE 17. $f(x) = \dfrac{x-1}{e^x}$

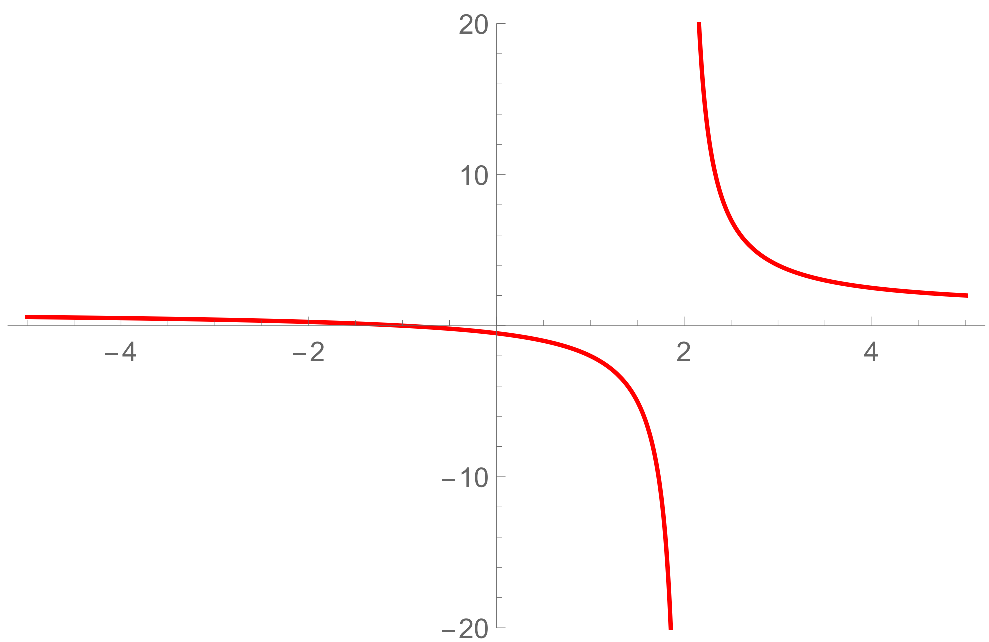

FIGURE 18. $f(x) = \dfrac{x+1}{x-2}$

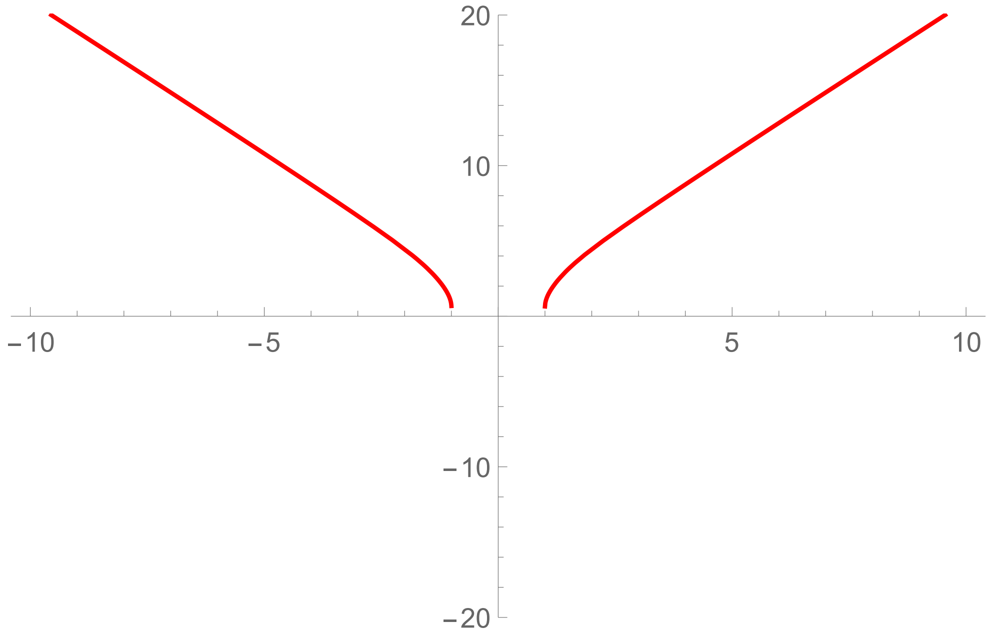

FIGURE 19. $f(x) = 1 + 2\sqrt{x^2 - 1}$

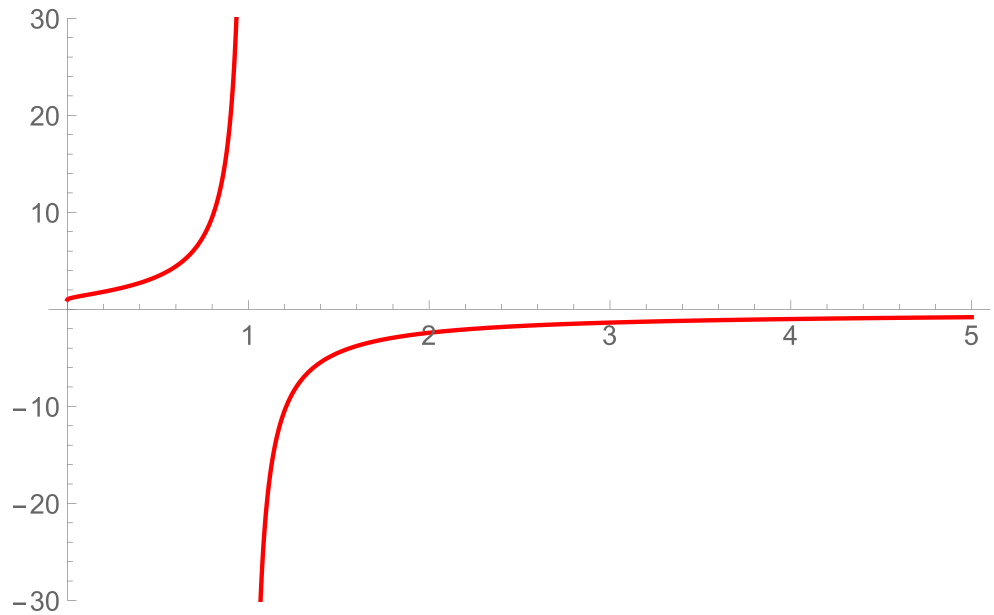

FIGURE 20. $f(x) = \dfrac{1}{1 - \sqrt{x}}$