Quantitative Methods – I
A.Y. 2020-21

Practice 3

Lorenzo Cavallo

For any clarification/meeting: cavallo@istat.it

Summary of the Practice

1. Sample space, event
2. Set operation, and Venn diagram
3. Counting principles, combination and permutation
4. Probability
5. Bayes' Theorem

THEME #1

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Sample space & Event

The basic concepts of probability

Experiment: a measurement process that produces quantifiable results (e.g. throwing two dice, dealing cards, at poker, measuring heights of people, recording proton-proton collisions)

Outcome: a single result from a measurement (e.g. the numbers shown on the two dice)

Sample space (S or Ω or ξ): the set of **all possible** outcomes from an experiment (e.g. the set of all possible five-card hands)

The number of all possible outcomes may be

(a) **finite** (e.g. all possible outcomes from throwing a single die; all possible 5-card poker hands)
(b) **countably infinite** (e.g. number of proton-proton events to be made before a Higgs boson event is observed)
or (c) **constitute a continuum** (e.g. heights of people)

In case (a), the sample space is said to be **finite**
in cases (a) and (b), the sample space is said to be **discrete**
in case (c), the sample space is said to be **continuous**

In this practice we consider discrete, mainly finite, sample spaces

An **event** is any subset of a sample set (including the empty set, and the whole set)

Two events that have no outcome in common are called **mutually exclusive** events.

In discussing discrete sample spaces, it is useful to use **Venn diagrams** and basic set-theory.

Therefore we will refer to the **union** (A U B), **intersection,** (A ∩ B) and **complement** ($\bar{A}$ or A^c) of events A and B.

THEME #2

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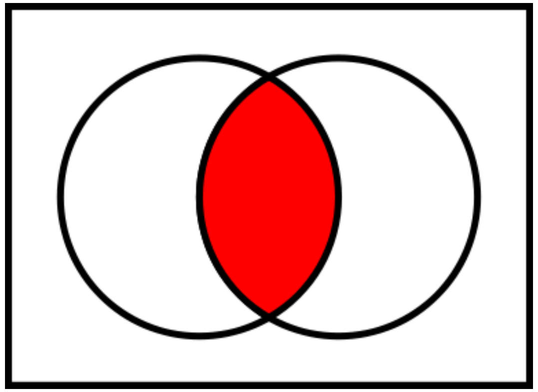
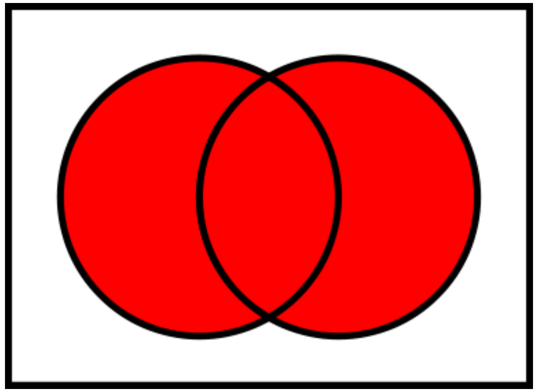
Venn Diagrams

Intersection ∩



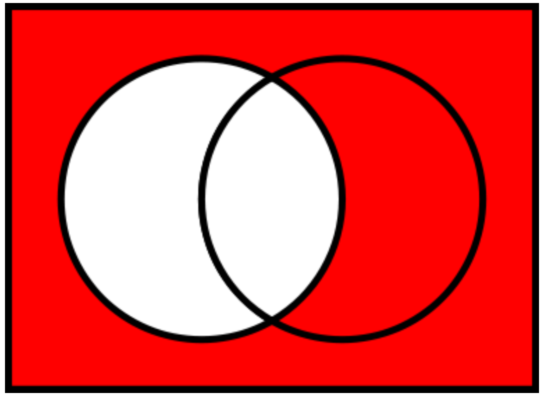
$A \cap B$

Union ∪

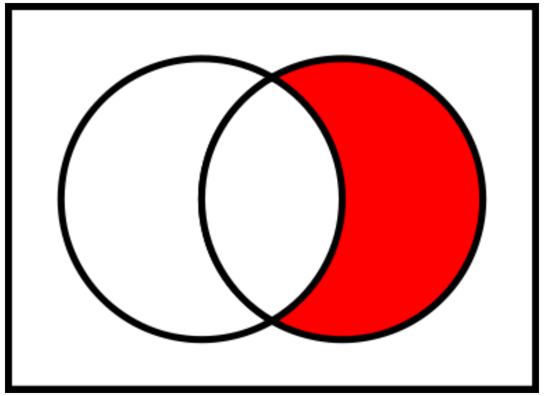


$A \cup B$

Complement $\bar{A}$ or A^{C}



$A^c = U \setminus A$

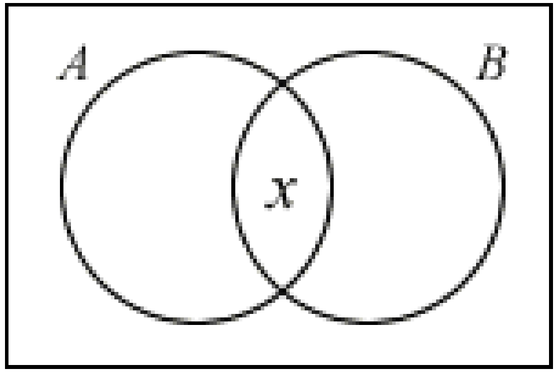


$A^c \cap B = B \setminus A$

Intersect

$A \cap B = x$

$A \cup B =$
$n(A) + n(B) - x$

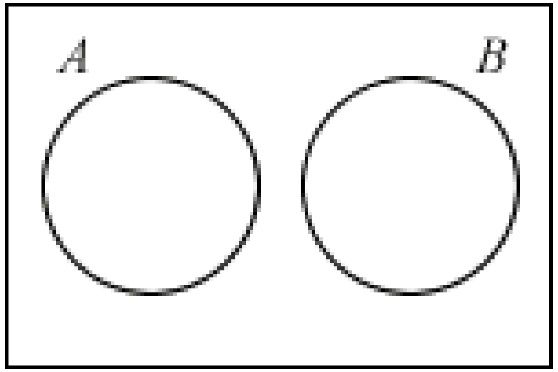


Because x is in A e B

Disjoint

$A \cap B = \varnothing$

$A \cup B =$
$n(A) + n(B)$

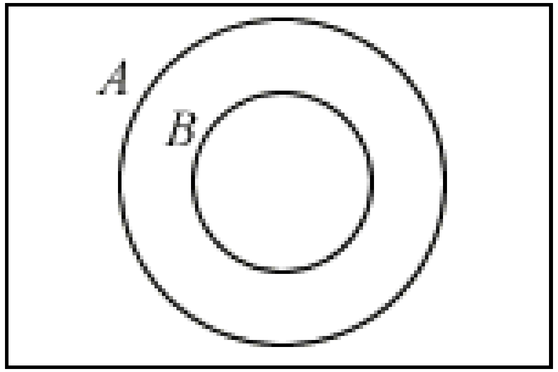


Subset

$B \subseteq A$

$A \cap B = B$

$A \cup B = A$



Properties of the operations between events		
	Union	Intersection
Idempotency	A ∪ A = A	A ∩ A = A
Neutral event	A ∪ Ø = A	A ∩ Ω = A
Commutativity	A ∪ B = B ∪ A	A ∩ B = B ∩ A
Associativity	(A ∪ B) ∪ C = A ∪ (B ∪ C)	(A ∩ B) ∩ C = A ∩ (B ∩ C)

Exercise 0: $S = (3, 4, 2, 8, 9, 10, 27, 23, 14)$

$A = (2, 4, 8)$ $B = (3, 4, 8, 27)$

Calculate

$\bar{A} = (3, 9, 10, 27, 23, 14)$

$\bar{B} = (2, 9, 10, 23, 14)$

$A \cup B = (2, 3, 4, 8, 27)$

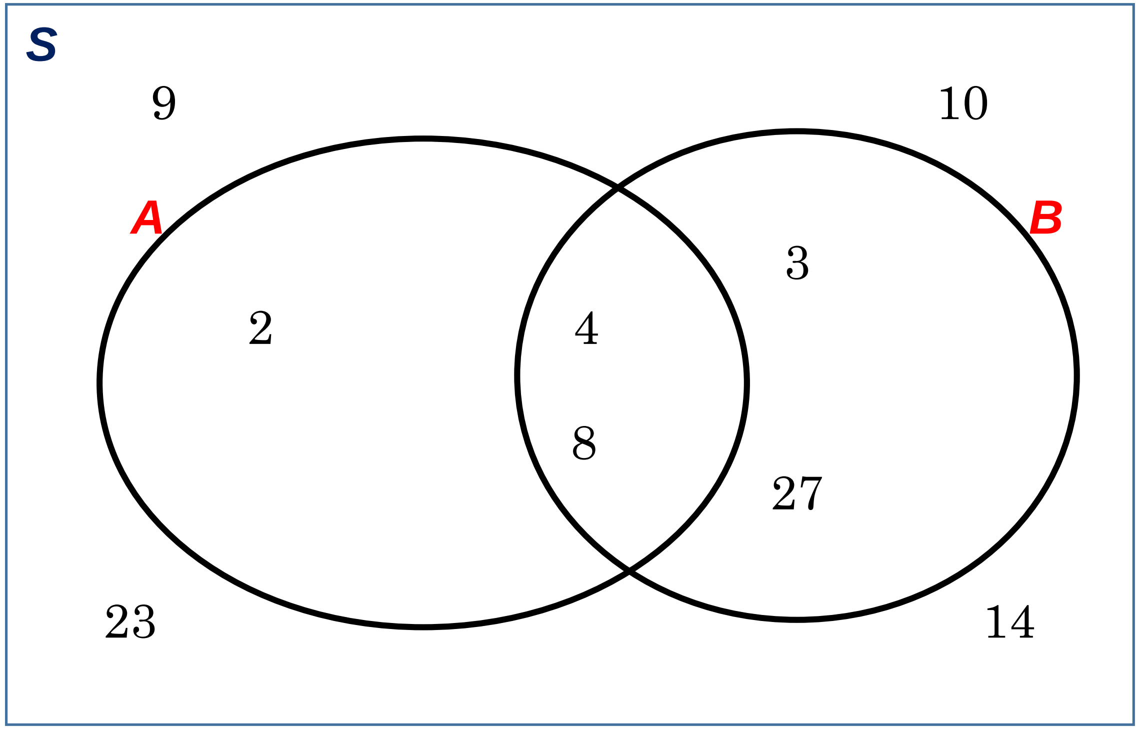
$A \cap B = (4, 8)$

$A - B = (2)$

$\overline{A \cup B} = (9, 10, 23, 14)$

$\overline{A \cap B} = (3, 9, 10, 27, 23, 14, 2)$

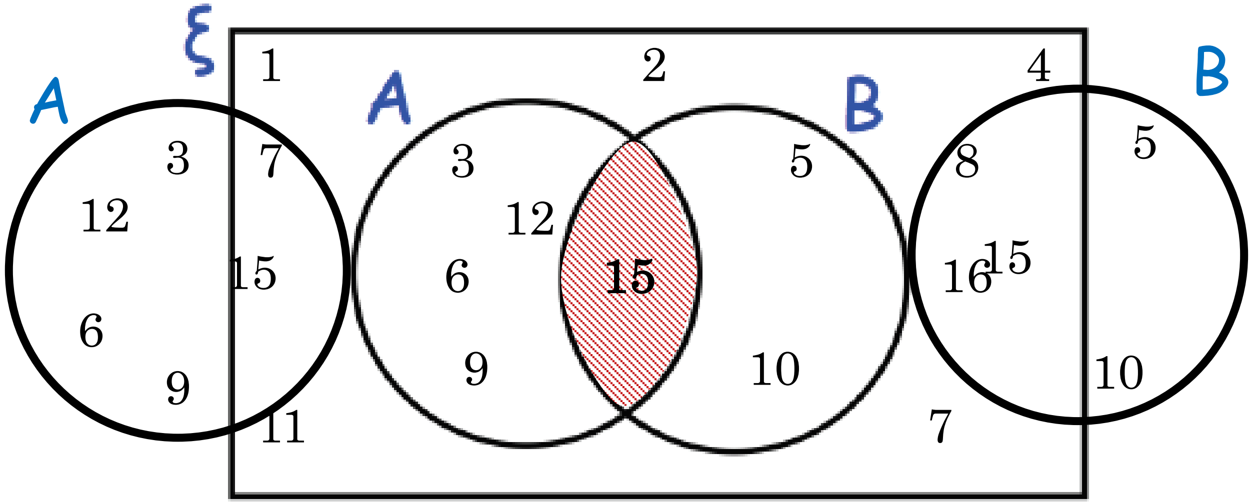
Draw the Venn diagram



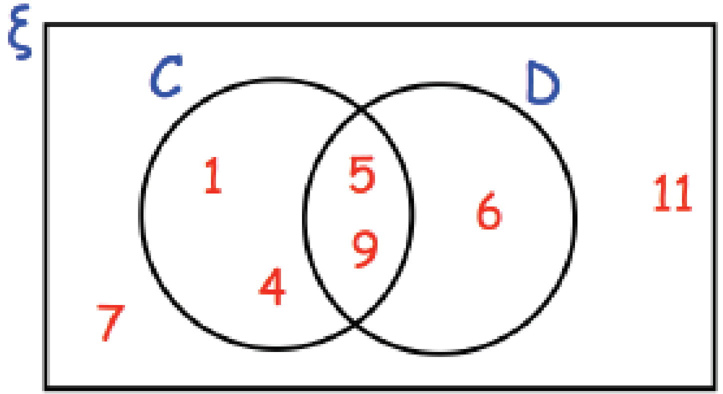
Exercise 1. $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

A = multiples of 3
B = multiples of 5

(a) Draw the Venn diagram
(b) Find A∩B



Exercise 2. Here is a Venn diagram



Write down the numbers that are in set

(a) D $(6, 5, 9)$

.

(b) $C \cup D$

$C \cup D = (1, 4 , 5, 9) \cup (5, 9, 6)$

$C + D - C \cap D = (1,4,5,9) + (5,9,6) - (5,9)$

$C \cup D = (1, 4, 5, 9, 6)$

(c) $\bar{C}$

$\bar{C} = \xi \cap C = (1, 4, 5, 9, 6, 7, 11) \cap (1, 4, 5, 9)$
$= (6, 7, 11)$

Exercise 3. There are 80 students in year 11.

9 students study French and German.
35 students only study French
2 students do not study French or German.

(a) Complete the Venn diagram

$n(\xi) = 80$

$n(G \cap F) = 9$

$n(F) - n(G \cap F) = 35 = n(\bar{G} \cap F)$

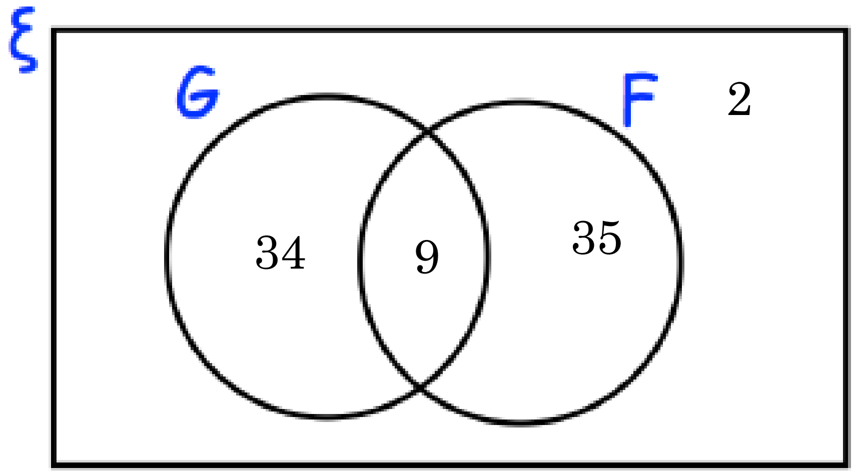
$n(F) = 35 + 9 = 44$

$n(\overline{G \cup F}) = 2$

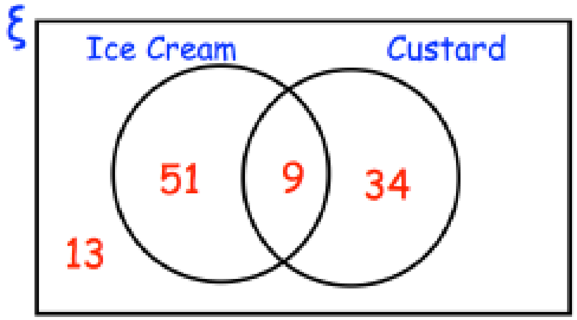
$n(G \cup F) = 80 - 2 = 78$

$n(G) = 78 - 35 = n(G \cup F) - n(\bar{G} \cap F) = 43$

$n(G) - n(G \cap F) = 43 - 9 = n(\bar{F} \cap G) = 34$



Exercise 4. At a wedding, the guests may have ice cream or custard with their dessert. The Venn diagram shows information about the choices the guests made.



(a) How many guests had custard?

35 + 9 = 44

Event «Custard»

(b) How many guests had ice cream and custard?

9

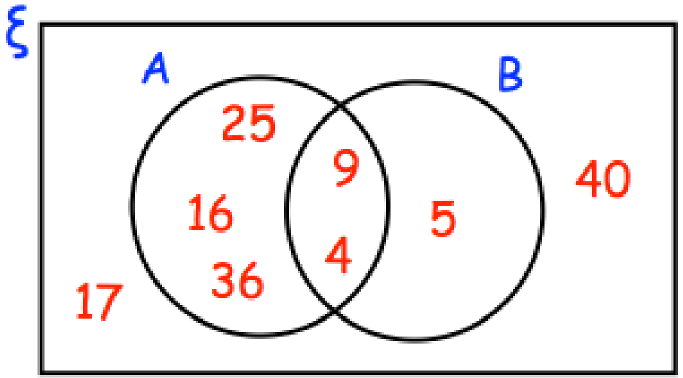
Intersection of Events «Custard» and «Ice Cream»

(c) How many guests went to the wedding?

51 + 9 + 34 + 13 = 107

Sample Size (S): «Desserts»

Exercise 5. Here is a Venn diagram.



Write down the numbers that are in set

(a) $A \cap B$ (9, 4)

(b) $A \cup B$ (25, 16, 36, 9, 4, 5)

(c) A^c (5, 17 , 40)

THEME #3

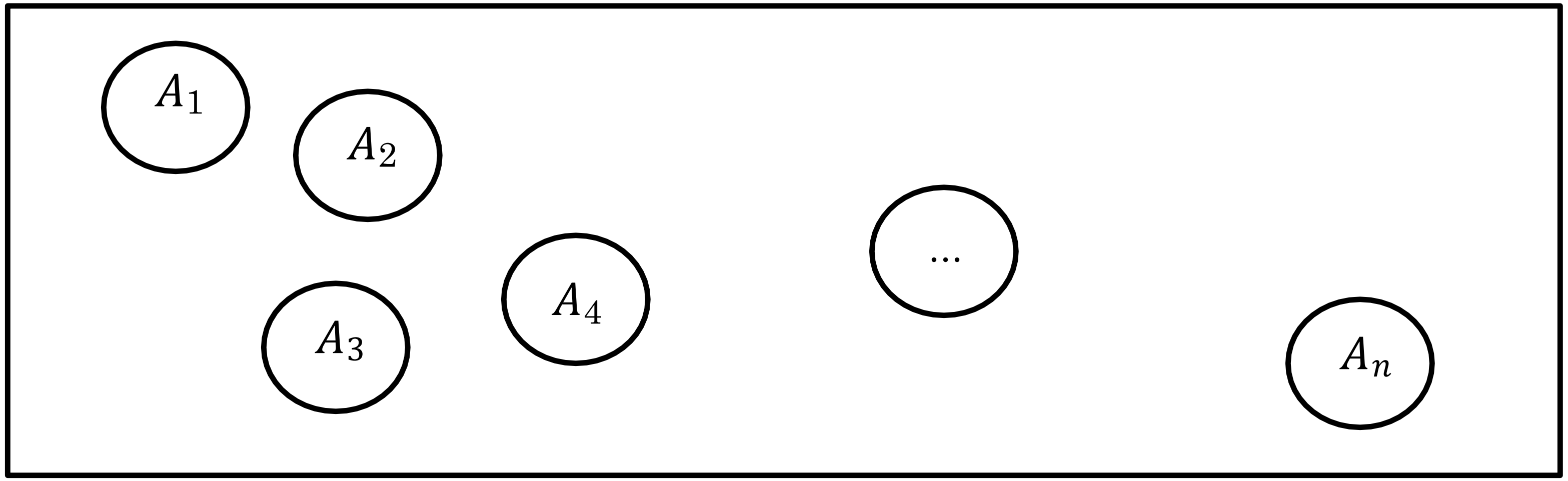
—

Counting principles

Counting principles

The **addition principle:** If there are n_1 outcomes in event A_1,
n_2 outcomes in event A_2,
…
n_k outcomes in event A_k

and the events $A_1, A_2, \ldots A_k$ are mutually distinct (share no outcomes in common), then the total number of outcomes in $A_1 \cup A_2 \cup \ldots \cup A_k$ is $n_1 + n_2 + \ldots + n_k$



The **multiplication principle**

If a composite outcome can be described by a procedure that can be broken into k successive (ordered) stages such that there are

n_1 outcomes in stage 1,
n_2 outcomes in event 2,
…
n_k outcomes in event k

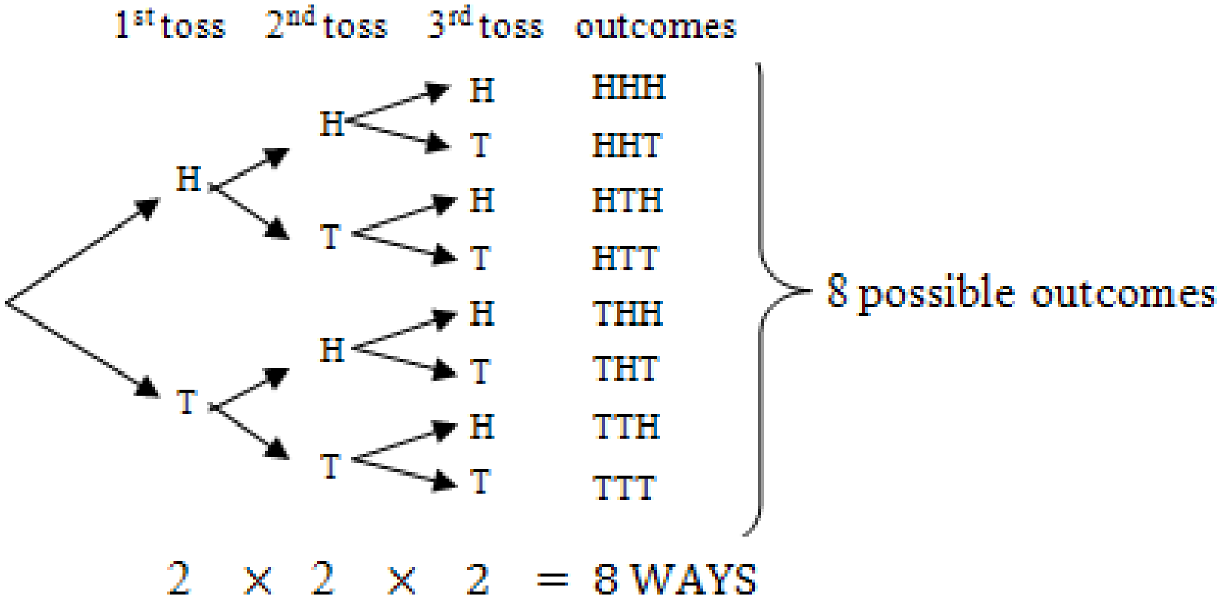
and if the number of outcomes in each stage is independent of the choices in previous stages and if the composite outcomes are all distinct then the number of possible composite outcomes is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

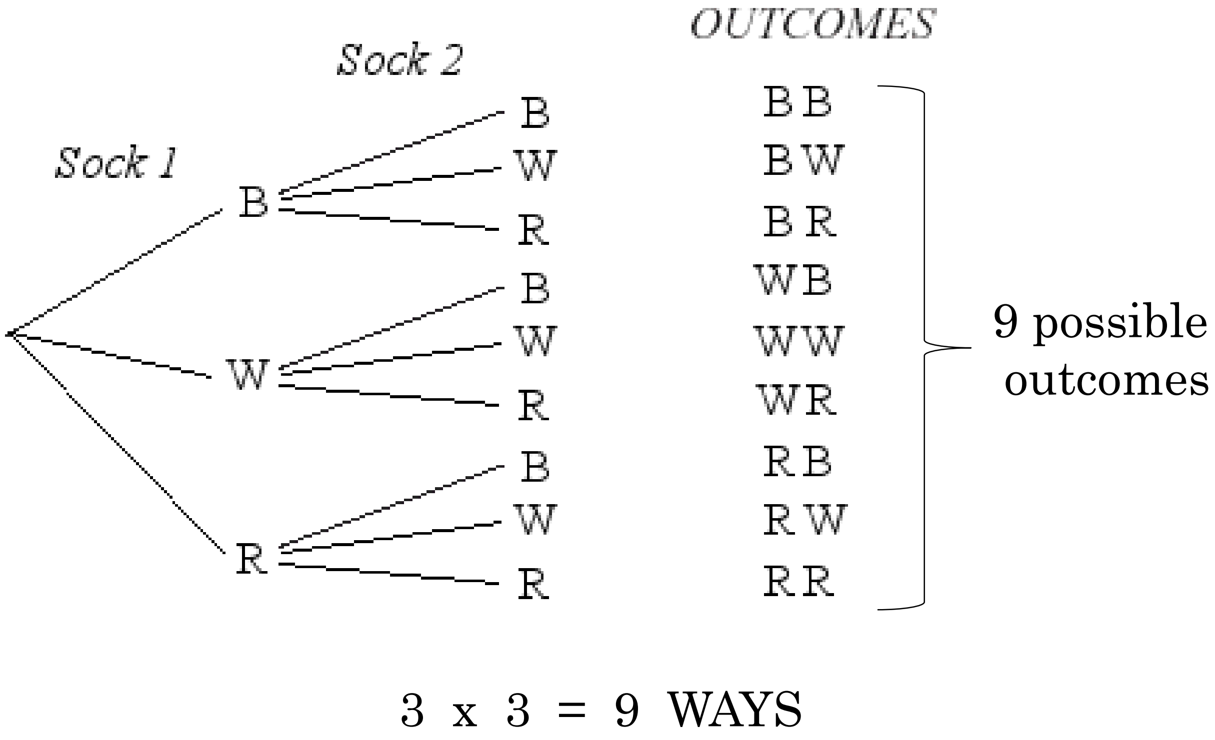
e.g. suppose the composite outcomes of the trio (A,B,C) of class values for cars, where

A denotes the mileage class (A_1, A_2, or A_3)
B denotes the price class (B_1, or B_2)
C denotes the operating cost class (C_1, C_2, or C_3)

The outcome is clearly written as a 3-stage value
There are 3 outcomes in class A, 2 in class B and 3 in class C
The number of outcomes in class B does not depend on the choice made for A, etc

Then there will be $3 \cdot 2 \cdot 3 = 18$ distinct composite outcomes for car classification.


1st toss
2nd toss
3rd toss
outcomes
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT
8 possible outcomes
2 × 2 × 2 = 8 WAYS


Sock 1
Sock 2
OUTCOMES
BB
BW
BR
WB
WW
WR
RB
RW
RR
9 possible outcomes
3 x 3 = 9 WAYS

Counting rules

We use this rule to count the number of possible and favourable outcomes of an experiment.

In particular, we will consider a set with n elements and will count the number of groups or subsets of m elements that can be drawn with or without replacement.

Consider an experiment that consists of m parts, and let n_k, $k = 1, 2, \ldots, m$, denote the number of possible outcomes of the k-th part.

The total number of possible outcomes is then: $n_1 \times n_2 \times \cdots \times n_m$.

This is sometimes referred to as the **multiplication principle**. It provides the basic principle of counting.

Factorials: n! is represents the product of all the integers from n to 1.

$$n! = n\,(n-1)\,(n-2)\,(n-3)\ldots 3 \cdot 2 \cdot 1$$

Dispositions with replacement

Total outcomes of an experiment = $n_1 \times n_2 \times n_3 \times \dots$

n_1 outcomes of the first step, n_2 outcomes of the second step, …

If $n_1 = n_2 = n_3 = \dots \rightarrow$ Total outcomes (k steps) = n^k

This provides in how many ways you can take m elements of n.

In case of selection of few elements from a group:

Combination: number of combination of n things, taken k at a time

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Permutation (Combination with order) **or Dispositions without replacement of n elements k at a time:** number of permutations of n things, taken k at a time

$$_nP_k = \frac{n!}{(n-k)!}$$

Question 15 A man just bought 4 suits, 8 shirts, and 12 ties. All of these suits, shirts, and ties coordinate with each other. If he is to randomly select one suit, one shirt, and one tie to wear on a certain day, how many different outcomes (selections) are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 4 \times 8 \times 12 = 384$

Question 16 A student is to select three classes for next semester. If this student decides to randomly select one course from each of eight economics classes, six mathematics classes, and five computer classes, how many different outcomes are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 8 \times 6 \times 5 = 240$

Question 17 An environmental agency will randomly select 4 houses from a block containing 25 houses for a radon check.

How many total selections are possible?

How many permutations are possible?

Solution

Combination $_{25}C_4 = \frac{n!}{k!(n-k)!}$ = 25!/(4!×21!)= (25×24×23×22×$\cancel{21\times\ldots\times1}$)/(4×3×2×$\cancel{21\times20\times\ldots\times1}$)= (25×24×23×22)/(4×3×2)= 12.650

Permutation $_{25}P_4 = \frac{n!}{(n-k)!}$ = (25×24×23×22×$\cancel{21\times\ldots\times1}$)/($\cancel{21\times20\times\ldots\times1}$)= 25×24×23×22=303.600

Question 18 You just got a free ticket for a boat ride, and you can bring along 2 friends! Unfortunately, you have 5 friends who want to come along.

How many different groups of friends could you take with you?

Solution

${}_5C_2 = \frac{n!}{k!(n-k)!}$ = 5!/(2!×3!)=(5×4×3×2×1)/(2×1×3×2×1)=5×2=10

Question 19 Emily is packing her bags for her vacation. She has 6 shirts, but only 3 fit in her bag.

How many different groups of 3 shirts can she take?

Solution

${}_6C_3$ = 6!/(3!×3!)=(6×5×4×3×2×1)/(3×2×1×3×2×1)=5×4=20

Question 20 How many ways can the positions of president and vice president be assigned from a group of 8 people?

Solution

${}_8P_2$=8!/6!= (8×7×6×5×4×2×1)/(6×5×4×3×2×1) = 8×7 = 56

Question 21 Suppose we have an office of 5 women and 6 men and need to select a 4 person committee. How many ways can we select

a) 2 men and 2 women?

Men (n=5) Woman (m=6)

k=2 Men: ${}_5C_2 = \frac{n!}{k!(n-k)!}$ = 5!/(2!×3!) = (5×4×3×2×1)/(2×1×3×2×1)=5×2=10

k=2 Woman: ${}_6C_2 = \frac{m!}{k!(m-k)!}$ = 6!/(2!×4!) = (6×5×4×3×2×1)/(2×1×4×3×2×1)=3×5=15

2 Men and 2 Woman: ${}_5C_2 \times {}_6C_2$ = 10 × 15 = 150

b) 3 men and 1 woman?

k=3 Men: ${}_5C_3 = \frac{n!}{k!(n-k)!}$ = 5!/(3!×2!) = (5×4×3×2×1)/(3×2×1×2×1)=5×2=10

k=1 Woman: ${}_6C_1 = \frac{m!}{k!(m-k)!}$ = 6!/(1!×5!) !)=(6×5×4×3×2×1)/(×1×5×4×3×2×1)=6

3 Men and 1 Woman: ${}_5C_3 \times {}_6C_1$ = 10 × 6 = 60

c) All women?

Men (n=5) Woman (m=6)

k=0 Men: ${}_5C_0 = \frac{n!}{k!(n-k)!}$ = 5!/(0!×5!)=1

k=4 Woman: ${}_6C_4 = \frac{m!}{k!(m-k)!}$ = 6!/(4!×2!) = (6×5×4×3×2×1)/(4×3×2×1×2×1)=3×5=15

All Woman: ${}_5C_0 \times {}_6C_4$ = 1 × 15 = 15

THEME #4

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Probability

Probability

Given the event A the probability of A is:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total of outcomes}}$$

The classical definition of probability

There are also:

Empirical or Statistical Probability or Frequency of occurrence

Let n_A be the number of times event A occurs after n trials. We define the *probability* of event A as

$$p_A = \lim_{n \to \infty} \frac{n_A}{n}$$

Axiomatic probability theory

The Axioms of Probability

First Axiom of Probability

$$0 \leq P(A) \leq 1$$

$P(A)=0 \rightarrow$ impossible event, $P(A)=1 \rightarrow$ certain event

Second Axiom of Probability

For an experiment with outcomes $S=\{E_1, E_2, E_3, \ldots,\}$, $\sum P(E_i) = P(E_1) + P(E_2) + P(E_3) + \ldots = 1$

Third Axiom of Probability

If A and B are mutually exclusive events in S, then

$$P(A \cup B)=P(A) + P(B)$$

A and B are mutually exclusive events if $P(A \cap B) = 0$

*If A and B are **NOT** mutually exclusive events in S*

$$P(A \cup B)=P(A) + P(B) - P(A \cap B)$$

Events **mutually exclusive** or **disjoint:** if they cannot occur at the same time.

Conditional probability: the probability that event A occurs, given that event B has occurred P(A|B).

Probability of the **intersection of events A and B:** P(A ∩ B) or P(A and B).

Probability of the **union of A and B:** P(A ∪ B) or P(A or B).

If the occurrence of event A changes the probability of event B, then **events** are **dependent**.

If the occurrence of event A does not change the probability of event B, then **events** are **independent**.

Conditional probability: $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ Intersection: $P(A \cap \mathrm{B}) = P(A)P(B|A)$

Union: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are indipendent, $P(A|\mathrm{B}) = P(A)$

Complementary event: The complement of event A, denoted by Ā, is the event that includes all the outcomes that are not in A.

P(A)+P(Ā)=1. Indeed, P(Ā)=1 – P(A)

Question 1

A die is rolled, find the probability that an even number is obtained.

Solution

Let us first write the sample space S of the experiment. S = {1,2,3,4,5,6}.

Let E be the event "*an even number is obtained*":   E = {2,4,6}

We now use the formula of the classical probability. P(E) = n(E) / n(S) = 3 / 6 = 1 / 2

Question 2

Two coins are tossed, find the probability that two heads are obtained.

Solution

The sample space S is given by. S = {(H,T),(H,H),(T,H),(T,T)}

Let E be the event "*two heads are obtained*": E = {(H,H)}

We use the formula of the classical probability. P(E) = n(E) / n(S) = 1 / 4

Question 3

Which of these numbers cannot be a probability?

a) -0.00001
b) 0.5
c) 1.001
d) 0
e) 1
f) 20%

Solution

A probability is always greater than or equal to 0 and less than or equal to 1.

Question 4

Two dice are rolled, find the probability that the sum is

a) equal to 1
b) equal to 4
c) less than 13

Solution

The sample space S of two dice is shown below.

S ={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

n(S)=36 -> Events of the 1th dice (6) × Events of the 2nd dice (6)

a) There are no outcomes which correspond to a sum equal to 1, hence P(E)=n(E)/n(S) = 0 / 36 = 0

b) Three possible outcomes give a sum equal to 4:
E = {(1,3),(2,2),(3,1)}, hence. P(E) = n(E) / n(S) = 3 / 36 = 1 / 12 = 0.083 = 8.3%

c) All possible outcomes, E = S, give a sum less than 13, hence. P(E) = n(E) / n(S) = 36 / 36 = 1

Question 5

A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution

The sample space S of the experiment described in question 5 is as follows

S = { (1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)}

n(S)=12 -> Events of the coin (2) × Events of the dice (6)

Let E be the event "the die shows an odd number and the coin shows a head".

Event E may be described as follows E={(1,H),(3,H),(5,H)}

The probability P(E) is given by P(E) = n(E) / n(S) = 3 / 12 = 1 / 4

Question 6

A card is drawn at random from a deck of cards. Find the probability of getting the 3 of diamond.

Solution
An examination of the sample space shows that there is one "3 of diamond"
so that n(E) = 1 and n(S) = 52.
Hence the probability of event E occurring is given by P(E) = 1 / 52

Question 7

A card is drawn at random from a deck of cards. Find the probability of getting a queen.

Solution
An examination of the sample space shows that there are 4 "Queens"
so that n(E) = 4 and n(S) = 52.
Hence the probability of event E occurring is given by P(E) = 4 / 52 = 1 / 13

Question 8

A jar contains 3 red marbles, 7 green marbles and 10 white marbles. If a marble is drawn from the jar at random, what is the probability that this marble is white?

Solution
We now use the empirical formula of the probability
P(E) = Frequency for white color / Total frequencies in the above table = 10 / 20 = 1 / 2 = 50%

Question 9

The blood groups of 200 people is distributed as follows: 50 have type A blood, 65 have B blood type, 70 have 0 blood type and 15 have type AB blood. If a person from this group is selected at random, what is the probability that this person has 0 blood type?

Solution

We use the empirical formula of the probability

P(E) = Frequency for 0 blood / Total frequencies = 70 / 200 = 0.35

Question 10

At the production of a certain item, two types of defects, A and B, can occur.

We know that P(A) = 0.1, P(B) = 0.2 and P(A∩B)=0.05.

Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

a) Probability of Union: P(AUB) = P(A)+P(B)-P(A∩B)=0.1+0.2-0.05=0.25
b) P(A)-P(A∩B)=0.1-0.05=0.05
c) Complementary of the Union of Event: P($\overline{A \cup B}$) = 1 – 0.25=0.75
d) P(AUB) - P(A∩B) = 0.2

Question 11

A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test.
What percent of those who passed the first test also passed the second test?

Solution
We have to calculate the Conditional Probability: $P(\text{Second} \mid \text{First}) = P(\text{First} \cap \text{Second}) / P(\text{First})$

$P(\text{First})=0.42$
$P(\text{First} \cap \text{Second})=0.25$
Hence, $P(\text{Second} \mid \text{First}) = P(\text{First} \cap \text{Second}) / P(\text{First}) = 0.25 / 0.42 = 0.5952 = 59.6\%$

Question 12

At a middle school, 18% of all students play football and basketball and 32% of all students play football. What is the probability that a student plays basketball given that the student plays also football?

Solution
We have to calculate the Conditional Probability: $P(\text{Basketball} \mid \text{Football})$

$P(F \cap B)=0.18$
$P(F)=0.32$
$P(B \mid F) = P(F \cap B) / P(F) = 0.18 / 0.32 = 0.5625 = 56.3\%$

Question 13

The two-way table below gives the thousands of commuters in Massachusetts in 2018 by transportation method and one-way length of commute.

	Less than 15 minutes	15-29 Minutes	30-44 Minutes	45-59 Minutes	>60 Minutes	**Total**
Private vehicle	636	908	590	257	256	**2647**
Public Transportation	9	54	96	62	108	**329**
Other	115	70	23	7	7	**222**
Total	**760**	**1032**	**709**	**326**	**371**	**3198**

a. Given that the commuter used public transportation, find the probability that the commuter had a commute of 60 or more minutes.
b. Given that the commuter used other method of transportation, find the probability that the commuter had a commute of less than 15 minutes.
c. Given that the commuter had a commute of 35 minutes, find the probability that the commuter used a private vehicle.

Solution

a. 108/329
b. 115/222
c. 590/709

Question 14 A random sample of 250 adults was taken, and they were asked whether they prefer watching sports or opera on television. The following table gives the two-way classification of these adults.

	Prefer Watching Sports	Prefer Watching Opera	
Male	104	18	122
Female	55	73	128
	159	91	250

If one adult is selected at random from this group, find the probability that this adult

a. prefers watching opera

b. is a male

c. prefers watching sports given that the adult is a female

d. is a male given that he prefers watching sports

e. is a female and prefers watching opera

f. prefers watching sports or is a male

a. P(Opera)=91/250=0.364

b. P(Male)=122/250=0.488

c. P(Sport|Female)=(Sport and Female)/Female= =55/128=0.4297

d. P(Male|Sport)=104/159=0.654

e. P(Female and Opera)=73/250=0.292

f. P(Sport or Male) = P(Sport) + P(Male) – P(Sport and Male) = 159/250 + 122/250 – 104/250 = 177/250 = 0.7080

Summary of probabilities

Event	Probability
A	$P(A) \in [0,1]$
not A	$P(A^{\complement}) = 1 - P(A)$
A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
A and B	$P(A \cap B) = P(A\|B)P(B) = P(B\|A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B\|A)P(A)}{P(B)}$

THEME #5

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Bayes' Theorem

The Bayes' Theorem

Bayes's theorem is stated mathematically as the following equation:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$.

$P(A|B)$ is a conditional probability: the likelihood of event A occurring given that B is true.

$P(B|A)$ is also a conditional probability: the likelihood of event B occurring given that A is true.

$P(A)$ and $P(B)$ are the probabilities of observing A and B respectively; they are known as the marginal probability.

Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?
c. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins ?

Solution

Denote by D the event that the coin is the one with two heads.

(a) P(D) = 1/10

(b) Denote by H the event that we get a head when we toss the coin. Then we want to find P(D|H).

By Bayes theorem, we have

$$P(D|H) = \frac{P(H|D) \cdot P(D)}{P(H)}$$

We have P(H|D) = 1 and P(D) = 1/10.

P(H): think of the bag as containing the possible tosses. As the bag contains 9 fair coins and one double-headed coin, it must contains 11 heads and 9 tails, so that the probability of choosing a head is

P(H) = 11/(11 + 9) = 11/20

We now obtain the answer $P(D|H) = \frac{\frac{1}{10}}{\frac{11}{20}} = \frac{2}{11}$

(c) 1. If it comes up tails, it can’t be the coin with two heads. Therefore it must be one of the other nine.