Quantitative Methods – I
A.Y. 2020-21

Practice 4

Lorenzo Cavallo

For any clarification/meeting: cavallo@istat.it

Summary of the Practice

1. Recap on Probability
2. Bayes' Theorem
3. Random Variables
4. Probability distributions of discrete variables

THEME #1

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Recap on Probability

The classical definition of probability

Given the event A the probability of A is:

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total of outcomes}}$$

Probability of the **intersection of events A and B**: P(A ∩ B) or P(A and B).

Probability of the **union of A and B**: P(A ∪ B) or P(A or B).

Events **mutually exclusive** or **disjoint**:
if they cannot occur at the same time.

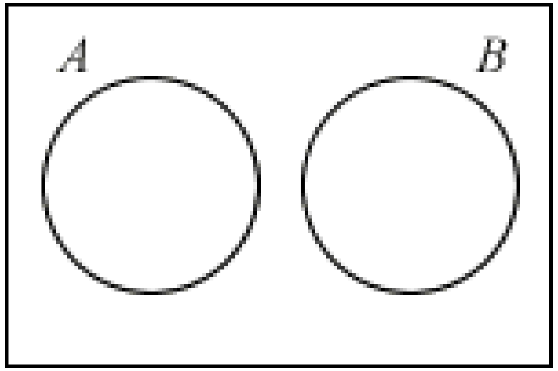
If A and B are disjoint event:

$P(A \cap \mathrm{B}) = \mathrm{P(A\ and\ B)} = 0$

Disjoint

$A \cap B = \varnothing$

$A \cup B =$
$n(A) + n(B)$



Union of Events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are disjoint $P(A \cap \mathrm{B}) = 0$, so $P(A \cup B) = P(A) + P(B)$

Conditional probability: the probability that event A occurs, given that event B has occurred P(A | B).

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Intersection of the 2 events

Conditioning Event

Intersection of Events: $P(A \cap B) = P(A|B)P(B)$

Independent Events: if the occurrence of event A does not change the probability of event B, then **events** are **independent**.

If A and B are indipendent: $P(A|B) = P(A)$

Complementary Event: The complement of event A, denoted by Ā, is the event that includes all the outcomes that are not in A.

P(A)+P(Ā)=1. Indeed, P(Ā)=1 – P(A)

Summary of probabilities

	Event	Probability
	A	$P(A) \in [0,1]$
Complementary Event	not A	$P(A^C) = 1 - P(A)$
Union of Events	A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
Intersection of Events	A and B	$P(A \cap B) = P(A\|B)P(B) = P(B\|A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
Conditional probability	A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B\|A)P(A)}{P(B)}$

Question 1

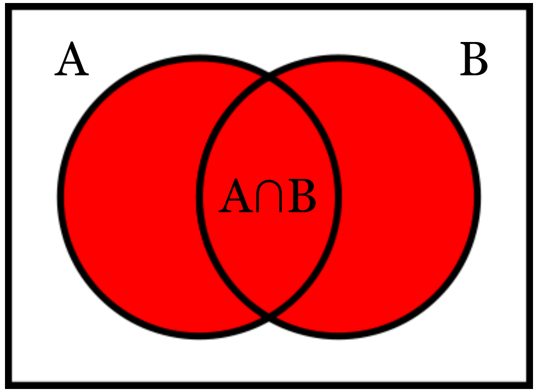
At the production of a certain item, two types of defects, A and B, can occur.
We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B)=0.05$.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

a) To calculate the probability that a produced unit has "at least one of the defect" we have to calculate the probability that a produced unit has "the defect A OR the defect B".
This is the probability of the union of the two events A or B, P(AUB).
A and B are not disjoint (because $P(A \cap B) \neq 0$).

So, $P(A \cup B) = P(A)+P(B)-P(A \cap B) = 0.1 + 0.2 - 0.05 = 0.25$



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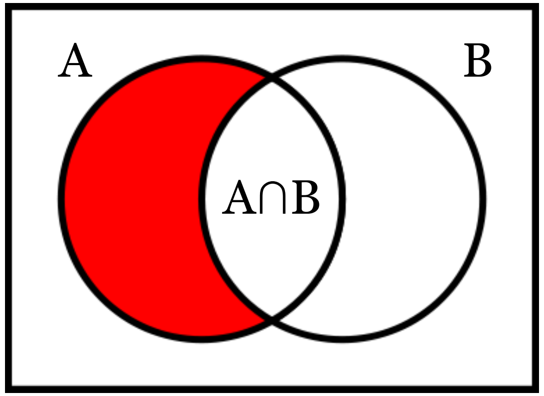
At the production of a certain item, two types of defects, A and B, can occur.
We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B)=0.05$.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

b) We have to calculate the probability that a produced unit has "the defect A but not the defect B".
We can calculate this probability removing from the probability that a produced unit has the defect A, $P(A)$, the probability that this produced unit has also the defect B (the part of the event A in common with B), represented by the intersection of the two events, $P(A \cap B)$.

So the probability that a produced unit has "the defect A but not the defect B" is: $P(A)-P(A \cap B)=0.1-0.05=0.05$



Question 1

At the production of a certain item, two types of defects, A and B, can occur.
We know that P(A) = 0.1, P(B) = 0.2 and P(A∩B)=0.05.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

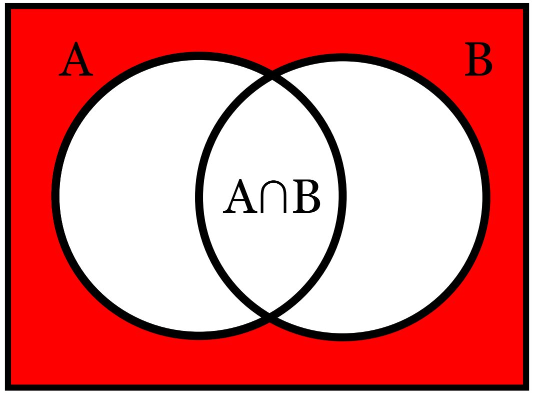
Solution

c) To calculate the event "none of the defects", we have to calculate the complementary event of both events, A and B.
This event is the Complementary event of the union of the events A or B: P($\overline{A \cup B}$).

P(AUB) = 0.25

From the property of the complementary events:

$$P(\overline{A \cup B}) = 1 - P(AUB) = 1 - 0.25 = 0.75$$



Question 1

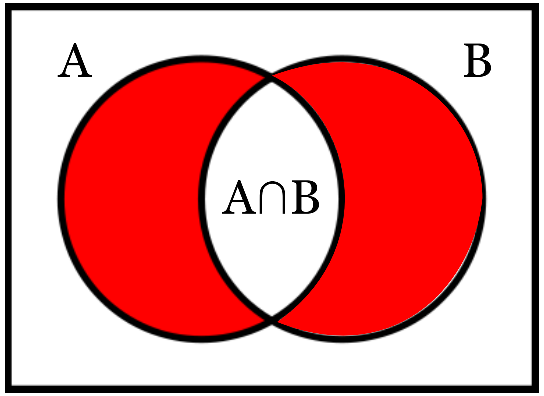
At the production of a certain item, two types of defects, A and B, can occur.
We know that $P(A) = 0.1$, $P(B) = 0.2$ and $P(A \cap B)=0.05$.
Compute the probability that a produced unit has:

a) at least one of the defects
b) defect A but not defect B
c) none of the defects
d) precisely one of the defects A and B

Solution

d) To calculate the probability of the event that a produced unit has "precisely one of the defects A and B" we have to calculate first of all the probability union of the events A and B.
But in this probability we have also the probability that a product has both the defects, A and B. For this reason, from the probability of the union of the two events we have to remove the intersection of the events.
$P(A \cup B) = 0.25$ and $P(A \cap B)=0.05$

So, P("precisely one of the defect A and B")$=P(A \cup B) - P(A \cap B) = 0.2$



Question 2

A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test.
What percent of those who passed the first test also passed the second test?

Solution
First, we have to consider that:
P(A)=P(Second Test), the probability that a student passed the second test,
P(B)=P(First Test), the probability that a student passed the first test,
P(A ∩ B)=P(Second ∩ First), the probability that a student passed both the tests.

So, the probability that a student who passed the first test also passed the second test is given by the conditional probability, P(Second|First), and this probability is:

$$\mathrm{P(Second|First)} = \frac{\mathrm{P(Second \cap First)}}{\mathrm{P(First)}}$$

where P(First) is the "conditioning event" (select a students that passed the second test given that has passed the first test).

P(First)=0.42
P(First ∩ Second)=0.25

Hence, $\mathrm{P(Second|First)} = \frac{\mathrm{P(Second \cap First)}}{\mathrm{P(First)}} = 0.25 \; / \; 0.42 = 0.5952 = 59.6\%$

Question 3

At a middle school, 18% of all students play football and basketball and 32% of all students play football. What is the probability that a student plays basketball given that the student plays also football?

Solution

We have to calculate the Conditional Probability: P(Basketball | Football)

We have to consider that:

P(F), the probability that a student plays football,

P(B), the probability that a student pays basketball,

P(F ∩ B), the probability that a student plays both the sports.

So, the conditional probability is:

$$P(B|F) = \frac{P(F \cap B)}{P(F)}$$

P(F ∩ B)=0.18

P(F)=0.32

$$P(B|F) = \frac{P(F \cap B)}{P(F)} = 0.18 \ / \ 0.32 = 0.5625 = 56.3\%$$

Question 4

The two-way table below gives the thousands of commuters in Massachusetts in 2018 by transportation method and one-way length of commute.

	Less than 15 minutes	15-29 Minutes	30-44 Minutes	45-59 Minutes	>60 Minutes	**Total**
Private vehicle	636	908	590	257	256	**2647**
Public Transportation	9	54	96	62	108	**329**
Other	115	70	23	7	7	**222**
Total	**760**	**1032**	**709**	**326**	**371**	**3198**

a. Given that the commuter used public transportation, find the probability that the commuter had a commute of 60 or more minutes.
b. Given that the commuter used other method of transportation, find the probability that the commuter had a commute of less than 15 minutes.
c. Given that the commuter had a commute of 35 minutes, find the probability that the commuter used a private vehicle.

Solution

a. 108/329
b. 115/222
c. 590/709

Question 5 A random sample of 250 adults was taken, and they were asked whether they prefer watching sports or opera on television. The following table gives the two-way classification of these adults.

	Prefer Watching Sports	Prefer Watching Opera	
Male	104	18	122
Female	55	73	128
	159	91	250

If one adult is selected at random from this group, find the probability that this adult

a. prefers watching opera

b. is a male

c. prefers watching sports given that the adult is a female

d. is a male given that he prefers watching sports

e. is a female and prefers watching opera

f. prefers watching sports or is a male

a. P(Opera)=91/250=0.364

b. P(Male)=122/250=0.488

c. P(Sport|Female)=(Sport and Female)/Female=
=55/128=0.4297

d. P(Male|Sport)=104/159=0.654

e. P(Female and Opera)=73/250=0.292

f. P(Sport or Male) = P(Sport) + P(Male) – P(Sport and Male) = 159/250 + 122/250 – 104/250 = 177/250 = 0.7080

THEME #2

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Bayes' Theorem

The Bayes' Theorem

Bayes's theorem is stated mathematically as the following equation:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$.

$P(A|B)$ is a conditional probability: the likelihood of event A given B.

$P(B|A)$ is also a conditional probability: the likelihood of event B given A.

$P(A)$ and $P(B)$ are the probabilities of observing A and B respectively; they are called marginal probability.

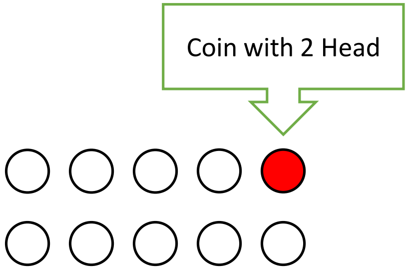
Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution

Denote by D the event that the coin is the one with two heads.



(a) P(D) is the probability to select the coin with two heads from all the 10 coins.

For this reason P(D) = $\frac{1}{10}$

(b) If it comes up tails, it can’t be the coin with two heads.
Therefore it must be one of the other nine.

For this reason the probability that it is one of the nine ordinary coins is 1 (certain event).

Question 15.

I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

a. If I take one of the coins at random from my pocket, what is the probability that it is the coin with two heads?
b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution

(c) Denote by H the event that we get a head when we toss the coin.

Then we have to find P(D|H) the probability that, having a head, is from the coin with two head.
By Bayes theorem, we have

$$P(D|H) = \frac{P(H|D) \cdot P(D)}{P(H)}$$

The probability P(H|D) is 1, because selecting the coin with two head, the result "HEAD" is a certain event.

The probability P(D) from the point (a) is 1/10.

We have to calculate the probability of H.

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b. If I toss the coin one further time and it comes up tails, what is the probability that it is one of the nine ordinary coins?
c. If I toss the coin and it comes up heads, what is the probability that it is the coin with two heads ?

Solution

We have to calculate the probability of H.

Think of the Sample Space of all the possible tosses.

The bag contains 9 fair coins and 1 double-headed coin,

for this reason in the bag we have 11 heads (9 from the fair coin and 2 from the double-head coin) and 9 tails (all from the 9 fair coin), so that the probability of choosing a head is:

$$P(H) = \frac{\text{n.of head}}{\text{n.of head and tail}} = \frac{11}{(11+9)} = \frac{11}{20}$$

We now obtain the answer: $P(D|H) = \frac{P(H|D) \cdot P(D)}{P(H)} = \frac{1 \cdot \frac{1}{10}}{\frac{11}{20}} = \frac{2}{11} = 18.18\%$

THEME #3

—

Random Variables & Probability Distribution of discrete variables

Random Variable

A variable whose value is determined by the outcome of an experiment.

Probability distributions of Discrete Variables

Lists all the possible values that the random variable can assume and their corresponding probabilities.

2 characteristics (conditions):

1. For each value of X, $0 \leq P(Xi) \leq 1$
2. $i=1,\dots,n \rightarrow \Sigma P(Xi) = 1$

Exercise 1

We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Also calculate the probability of having :

1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

The 8 possible outcomes, w, of the experiment and the probabilities with which they occur, P(w), are:

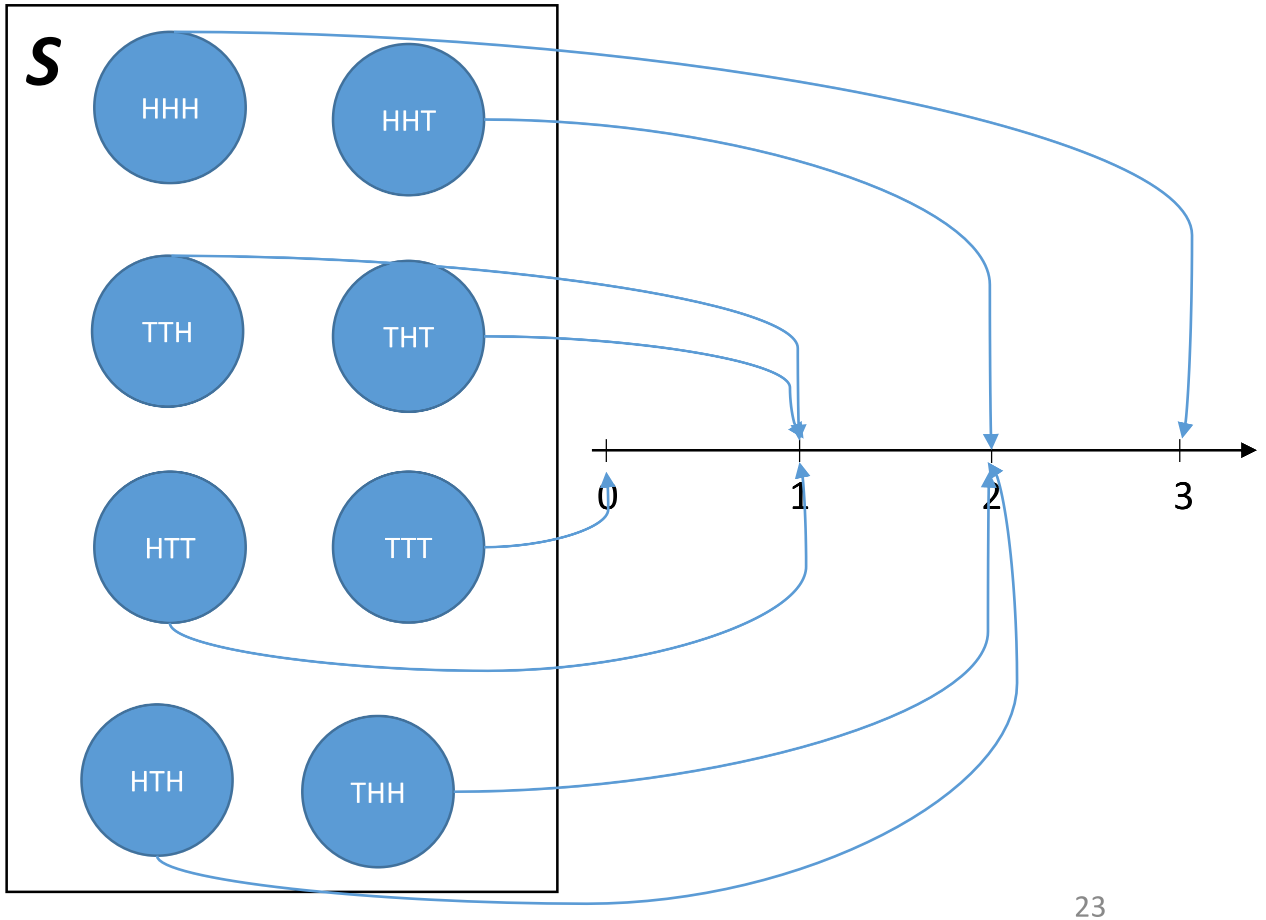
w	P (w)
HHH	1/8
HHT	1/8
HTH	1/8
THH	1/8
HTT	1/8
THT	1/8
TTH	1/8
TTT	1/8

Exercise 1

We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Solution
X is the random variable that counts the number of heads.

w	P (w)	X
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0



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1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

w	P (w)	X
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

So, the probability distribution of X is:

x_i	$P(X = x_i)$
0	1/8
1	3/8
2	3/8
3	1/8

Exercise 1

We toss 3 coins. Let X be random variable that counts the number of heads. Obtain the probability distribution of X.

Also calculate the probability of having :

1. two heads;
2. no head;
3. more than one head;
4. at least one head;
5. less than three heads;
6. at most one head.

Solution

From this probability distribution we can calculate the probability from 1. to 6.:

x_i	$P(X = x_i)$
0	1/8
1	3/8
2	3/8
3	1/8

1. $P(X = 2) = 3/8$
2. $P(X = 0) = 1/8$
3. $P(X > 1) = P(X = 2) + P(X = 3) = 3/8 + 1/8 = 4/8 = 1/2$
4. $P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = 1/8 + 3/8 + 3/8 = 7/8$
5. $P(X < 3) = 1 - P(X = 3) = 1 - 1/8 = 7/8$
6. $P(X \leq 1) = P(X = 0) + P(X = 1) = 1/8 + 3/8 = 4/8 = 1/2$

Exercise 2

Each of the following tables lists certain values of x and their probabilities. Determine whether or not each table represents a valid probability distribution.

a)

x	$P(x)$
0	0.18
1	0.01
2	0.29
3	0.37

b)

x	$P(x)$
2	0.35
3	0.24
4	0.18
5	0.23

c)

x	$P(x)$
7	0.65
8	0.50
9	-0.15

Solution

a) Each probability listed in this table is in the range 0 to 1, it satisfies the first condition of a probability distribution.
But, the sum of all probabilities is not equal to 1.0 because $\Sigma P(x)=0.18+0.01+0.29+0.37=0.85$ and the second condition is not satisfied.
This table **does not represent** a valid probability distribution.

b) Each probability listed in this table is in the range 0 to 1. Also, $\Sigma P(x)=0.35+0.24+0.18+0.23=1.0$
Consequently, this table **represents** a valid probability distribution.

c) The sum of all probabilities listed in this table is equal to 1.0, but one of the probabilities is negative.
This violates the first condition of a probability distribution. Therefore, this table **does not represent** a valid probability distribution.

Exercise 3

The following table lists the probability distribution of the number of breakdowns per week for a machine based on past data.

Breakdowns per week	0	1	2	3
Probability	0.15	0.20	0.35	0.30

Find the probability that the number of breakdowns for this machine during a given week is:

a) exactly 2
b) less than 2
c) more than 1
d) at most 1

Solution

a) $P(2) = 0.35 = 35\%$
b) $P(<2)=P(0)+P(1) = 0.15+0.20=0.35=35\%$
c) $P(>1)=P(2)+P(3)=0.35+0.30=0.75=75\%$
d) $P(\leq 1)=P(0)+P(1)=0.15+0.2=0.35=35\%$

THEME #4

—

Expected value and Variance of random variables

Expected Value

E(X) is the mean of the probability distribution

$$E(X) = \mu = \Sigma x_i \cdot P(x_i)$$

Variance and Standard Deviation

They measure the spread of the probability distribution:

$$Var\ (X) = \sigma^2 = E(X - E(X))^2 = E(X^2) - E(X)^2$$

For discrete, $V(X) = \sigma^2 = \Sigma x_i^2 \cdot P(x_i) - \mu^2$

Standard Deviation

$$SD(X) = \sigma = \sqrt{\sigma^2}$$

Exercise 3

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	?	0.45	0.24	0.12	0.09	0.05

a. What is the of the missing probability?
b. What is the probability that a course will require 2 or more textbooks?
c. What is the probability that a course require 2 or 3 textbooks?
d. Find the expected value of the random variable X.
e. Find the variance and the standard deviation of the random variable X.

Exercise 3

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	?	0.45	0.24	0.12	0.09	0.05

Solution

a. What is the of the missing probability?

The probability distributions P(X = x) have 2 characteristics:

1. For each value of X, $0 \leq P(Xi) \leq 1$

2. $i=1,\ldots,n \rightarrow \Sigma P(Xi) = 1$

From the second characteristic we have:

$P(X = 0) + \sum P(X = x_i) = 1$ then $P(X = 0) = 1 - \sum P(X = x_i) = 1 - 0.95 = 0.05$

The probability of P(X=0) is 0.05

Exercise 3

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

b. What is the probability that a course will require 2 or more textbooks?

$P(X \geq 2)$ is a sum of probability:

$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 0.24 + 0.12 + 0.09 + 0.05 = 0.5 = 50\%$

or the probability of the complementary event: $1 - P(X < 2) = 1 - P(X=0) - P(X=1) = 1 - 0.45 - 0.05 = 0.5$

c. What is the probability that a course require 2 or 3 textbooks?

The two event are disjoint so:

$P(X=2 \text{ or } 3) = P(X=2) + P(X=3) = 0.24 + 0.12 = 0.36 = 36\%$

Exercise 3

Let X be the random variable indicating the number of textbooks required by a randomly chosen statistical course.

The r.v. X can assume only the values between 0 and 5.

The table below provides the probability distribution of the random variable:

X_i	0	1	2	3	4	5
$P(X = x_i)$	0.05	0.45	0.24	0.12	0.09	0.05

Solution

d. Find the expected value of the random variable X.

For a discrete random variable the expected value E(X) is: $\mu = E(X) = \sum x_i \cdot P(X = x_i)$

for this reason: $E(X)=\mu= 0\cdot 0.05+1\cdot 0.45+2\cdot 0.24+3\cdot 0.12+4\cdot 0.09+5\cdot 0.05=0+0.45+0.48+0.36+0.36+0.25=1.9$

e. Find the variance and the standard deviation of the random variable X

For a discrete random variable the variance is:
$\sigma^2 = Var(X) = \sum(x_i-\mu)^2 \cdot P(X = x_i)$ or $\sigma^2 = E(X^2) - \mu^2 = \sum x_i^2 \cdot P(X = x_i) - \mu^2$

Using the second notation $\sigma^2 = E(X^2) - \mu^2 = \sum x_i^2 \cdot P(X = x_i) - \mu^2 =$
$=(0^2\cdot 0.05+1^2\cdot 0.45+2^2\cdot 0.24+3^2\cdot 0.12+4^2\cdot 0.09+5^2\cdot 0.05)-1.9^2 = 0+0.45+4\cdot 0.24+9\cdot 0.12+16\cdot 0.09+25\cdot 0.05-3.61=$
$=0+0.45+0.96+1.08+1.44+1.25-3.61=5.18-3.61=1.57$

σ is the square root of $\sigma^2 = 1.25$

THEME #5

—

Bernoulli, Binomial and Poisson probability distribution

Bernoulli Probability Distribution

Only 2 values (1="success", 0 = "failure"). 1 with probability p, 0 with probability $(1 - p)$

Probability function: $$P(x_i) = p^{x_i} \times (1 - p)^{(1-x_i)}$$

$$E(X) = p \quad \text{and} \quad V(X) = p \times (1 - p)$$

ex. Accept or decline an investment, vote yes or no on a ballot, etc.

Binomial Random Variable

Represents the number of successes in n Bernoulli experiments : a) independent b) with equal $\boldsymbol{p}$

Probability function: $$P(x) = \binom{n}{x} p^x \times (1 - p)^{(n-x)}$$

$$E(X) = n \times p \quad \text{and} \quad V(X) = n \times p \times (1 - p)$$

ex. Number of heads obtained tossing a coin 10 times.

Exercise 4

A club membership is renewed with probability 0.65. Let X be the random variable that represents the decision of each member to renew the subscription.

What is the distribution of X? Obtain E(X) and V(X).

Solution

The distribution is a Bernoulli probability distribution because the possible event are only 2:

"renew the membership" (event 0) or "not renew the membership" (event 1).

The probability distribution is:

$$P(X=x)=p^{x} \cdot (1-p)^{(1-x)} = 0.65^{x} \cdot (1-0.65)^{(1-x)}$$

The expected value of X for a Bernoulli distribution is:

$E(X) = p$

So, $E(X) = p = 0.65 = 65\%$

The variance of X for a Bernoulli distribution is:

$Var(X)= p \cdot (1-p)$

So, $Var(X)= p \cdot (1-p) = 0.65 \cdot (1-0.65) = 0.65 \cdot 0.35 = 0.2275 = 22.75\%$

Exercise 5

Based on his experience a travel agent believes that only 60% of the tickets booked then are bought.

If 8 customers have gone independently to the travel agency for a reservation, determine the probability that:

a. five buy a ticket;

b. everyone buys a ticket;

c. at least 2 buy a ticket.

Solution

The X is a Binomial because the customers can only BUY (event 0) or NOT BUY (event 1) a ticket.

The r.v. X is a Binomial distribution X ~ Bin (n, p) = Bin (8; 0.6)

where n = 8 (the customers) and p=0.6 (the probability of buying a ticket)

The probability function is:

$$P\ (X = x) = \binom{n}{x}\ p^x \cdot (1 - p)^{(n-x)}$$

a. The probability that x = 5 customers buy a ticket is:

$$P\ (X = 5) = \binom{n}{x}\ p^x \cdot (1 - p)^{(n-x)} = \binom{8}{5}\ 0.60^5 \cdot (1 - 0.60)^{(8-5)} =$$

$$= 8!/(5! \cdot 3!) \cdot 0.078 \cdot 0.064 = 56 \cdot 0.078 \cdot 0.064 = 0.28 = 28\%$$

Exercise 5

Based on his experience a travel agent believes that only 60% of the tickets booked then are bought.

If 8 customers have gone independently to the travel agency for a reservation, determine the probability that:

a. five buy a ticket;

b. everyone buys a ticket;

c. at least 2 buy a ticket.

Solution

b. The probability that all customers buy a ticket is:

$$P\ (X = 8) = \binom{n}{x}\ p^{x} \cdot (1 - p)^{(n-x)} = \binom{8}{8}\ 0.60^{8} \cdot (1 - 0.60)^{(8-8)} =$$

$$= 8!/(8! \cdot 0!) \cdot\ 0.0168 \cdot 1 = 1 \cdot 0.0168 = 1.68\%$$

From the complementary statement of probability

c. The probability that at least two customers buy a ticket, $P(X \geq 2)$, is:

$$P\ (X \geq 2) = 1 - P\ (X < 2) =$$

$$= 1 - P(X{=}1) - P(X{=}0) =$$

$$= 1 - \binom{8}{1}\ 0.60^{1} \cdot (1 - 0.60)^{(8-1)} - \binom{8}{0}\ 0.60^{0} \cdot (1 - 0.60)^{(8-0)} =$$

$$= 1 - 8!/(7! \cdot 1!) \cdot 0.6 \cdot 0.4^{7} - 8!/(0!8!) \cdot 0.6^{0} \cdot 0.4^{8} =$$

$$= 1 - 8 \cdot 0.6 \cdot 0.0016 - 1 \cdot 1 \cdot 0.0006 = 1 - 0.0077 - 0.0006 = 0.9917 = 99.17\%$$

Poisson Probability Distribution

Represents the number of occurrences in a given interval. Occurrences have to be random and independent

Probability function: $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

with λ as average number of occurrences in a given interval and x number of occurrences (in the same interval)

$$E(X) = V(X) = \lambda$$

ex. Number of telemarketing phone calls received in a day

Exercise 6

On average a household receives 3 telemarketing phone calls per month. Find the probability that a randomly selected household receives:

a. exactly 2 calls during a given month;

b. exactly 1 call during a given week (assume 4 weeks each month);

c. Find the expected value

Solution

To calculate the points a. b. and c. we have to use the Poisson probability distribution

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where λ is the number of telemarketing calls received per month (λ =3)

a. We have to calculate $P(X = 2)$.

Using the Poisson distribution, $P(X = 2) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{3^2 e^{-3}}{2!} =$ 9 · 0.049 / 2 = 0.2205 = 22.5%

b. If the calls per month are 3, the calls per week are:

λ = 3 calls per month /4 weeks = 0.75 calls per week

So, $P(X = 1)$ = 0.75^1 · e^(-0.75) / 1! = 0.75 · 0.4724 = 0.3543 = 35.43%

c. For the Poisson distribution: E(X) = V(X) = λ = 3

Extra Exercise

Exercise 7

The ratio of boys to girls at birth in Singapore is quite high at 1.09:1 [P(X=boy)=0.5215].

a. What proportion of Singapore families with 3 children will have exactly 3 boys? (Ignore the probability of multiple births.)

b. What proportion of Singapore families with exactly 6 children will have at least 3 boys?

Solution

Let X = number of boys in the family.

a. Here n=3, p=0.5215, 1-p=1–0.52153=0.4785

When x=3: P(X=3) =C(n,x) · p^x · (1-p)^(n-x) = C(3,3) · 0.5215^3 · 0.4785^0=3!/3!·0! · 0.1418 · 1 = 1 · 0.1418 ·1 = 0.1418 = 14.18%

b. Here, n=6, p=0.5215, 1-p=1–0.52153=0.4785

When x=3: P(X=3) =C(n,x) · p^x · (1-p)^(n-x) = C(6,3) · 0.5215^3 · 0.4785^3=6!/3!·3! · 0.1418 · 0.1096 = 20 · 0.0155 = 0.31 = 31%

When x=4: P(X=4) =C(n,x) · p^x · (1-p)^(n-x) = C(6,4) · 0.5215^4 · 0.4785^2=6!/4!·2! · 0.074 · 0.229 = 15 · 0.0169 = 0.2535 = 25.35%

When x=5: P(X=5) =C(n,x) · p^x · (1-p)^(n-x) = C(6,5) · 0.5215^5 · 0.4785^1=6!/5!·1! · 0.039 · 0.4785 = 6 · 0.0187 = 0.112 = 11.2%

When x=6: P(X=6) =C(n,x) · p^x · (1-p)^(n-x) = C(6,6) · 0.5215^6 · 0.4785^0=6!/6!·0! · 0.0201 · 1 = 1 · 0.0201 = 0.0201 = 2.01%

So P(X>=3)= 0.31+0.2535+0.112+0.0201=0.6956=69.56%

Exercise 8

2% of the product sold by Microsoft have some bugs. Find the probability that for 10 products sold:

a. 1 have bugs

b. at most 1 have bugs

Solution

The r.v. X has a Binomial distribution X ~ Bin (n, p)

a. $P(X = 1)$ = C(10,1) · 0.02^1 · (0.98)^(10–1) = 10 · 0.02 · 0.8337 = 0.16674 = 16.67%

b. $P(X \le 1) = P(X = 0) + P(X = 1)$ = C(10,0) · 0.020^0 · (0.98)^10 + 0.16674 = 1 · 1 · 0.8171 + 0.1667 = 0.9838 = 98.38%