Quantitative Methods – I
A.Y. 2021-22

Practice 5

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THEME #1

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Sample space & Event

The basic concepts of probability

Experiment: a measurement process that produces quantifiable results (e.g. throwing two dice, dealing cards, at poker, measuring heights of people, recording proton-proton collisions)

Outcome: a single result from a measurement (e.g. the numbers shown on the two dice)

Sample space (S or Ω or ξ): the set of **all possible** outcomes from an experiment (e.g. the set of all possible five-card hands)

The number of all possible outcomes may be

(a) **finite** (e.g. all possible outcomes from throwing a single die; all possible 5-card poker hands)
(b) **countably infinite** (e.g. number of proton-proton events to be made before a Higgs boson event is observed)
or (c) **constitute a continuum** (e.g. heights of people)

In case (a), the sample space is said to be **finite**
in cases (a) and (b), the sample space is said to be **discrete**
in case (c), the sample space is said to be **continuous**

In this practice we consider discrete, mainly finite, sample spaces

An **event** is any subset of a sample set (including the empty set, and the whole set)

Two events that have no outcome in common are called **mutually exclusive** events.

In discussing discrete sample spaces, it is useful to use **Venn diagrams** and basic set-theory.

Therefore we will refer to the **union** ($A \cup B$)**, intersection,** ($A \cap B$) and **complement** ($\bar{A}$ or A^c) of events A and B.

THEME #2

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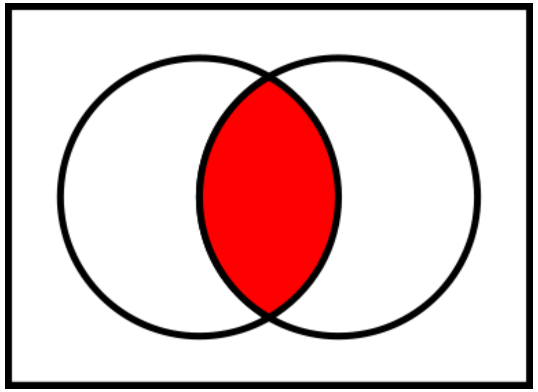
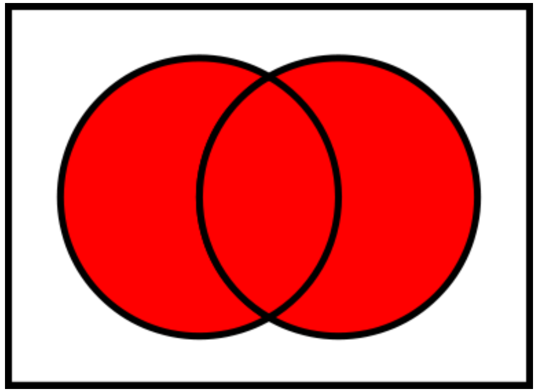
Venn Diagrams

Intersection ∩

Union ∪

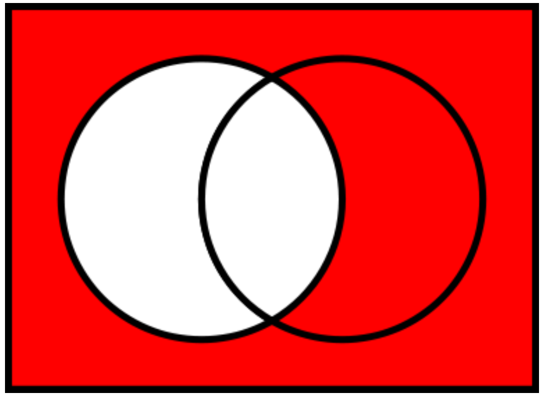


$A \cap B$

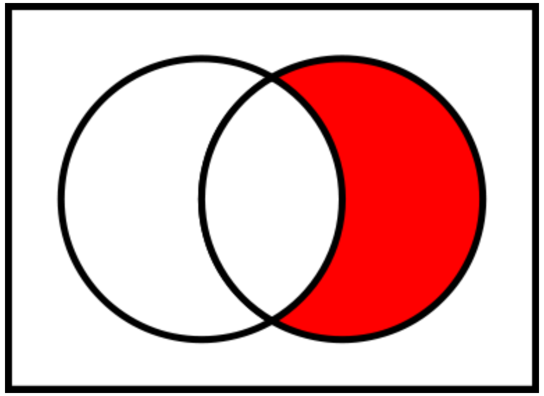


$A \cup B$

Complement $\bar{A}$ or A^{C}



$A^c = U \setminus A$

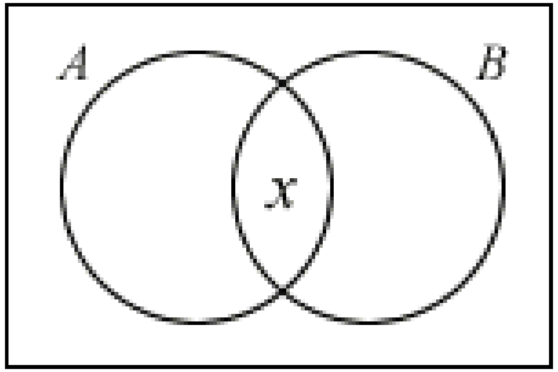


$A^c \cap B = B \setminus A$

Intersect

$A \cap B = x$

$A \cup B = n(A) + n(B) - x$

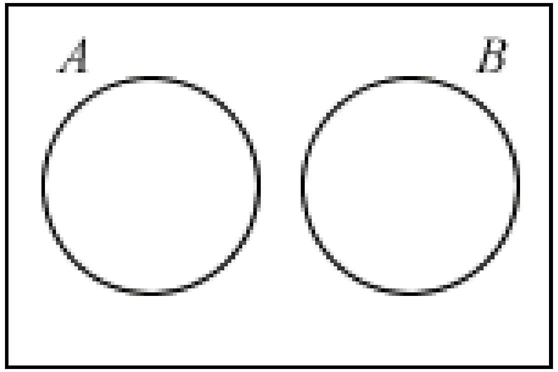


Because x is in A e B

Disjoint

$A \cap B = \varnothing$

$A \cup B = n(A) + n(B)$

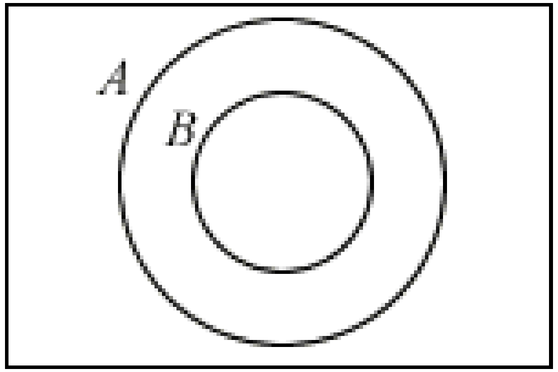


Subset

$B \subseteq A$

$A \cap B = B$

$A \cup B = A$



Properties of the operations between events		
	Union	Intersection
Idempotency	A ∪ A = A	A ∩ A = A
Neutral event	A ∪ Ø = A	A ∩ Ω = A
Commutativity	A ∪ B = B ∪ A	A ∩ B = B ∩ A
Associativity	(A ∪ B) ∪ C = A ∪ (B ∪ C)	(A ∩ B) ∩ C = A ∩ (B ∩ C)

Exercise 0: S = (3, 4, 2, 8, 9, 10, 27, 23, 14)

A = (2, 4, 8) B = (3, 4, 8, 27)

Calculate

$\bar{A}$ = (3, 9, 10, 27, 23, 14)

$\bar{B}$ = (2, 9, 10, 23, 14)

$A \cup B$ = (2, 3, 4, 8, 27)

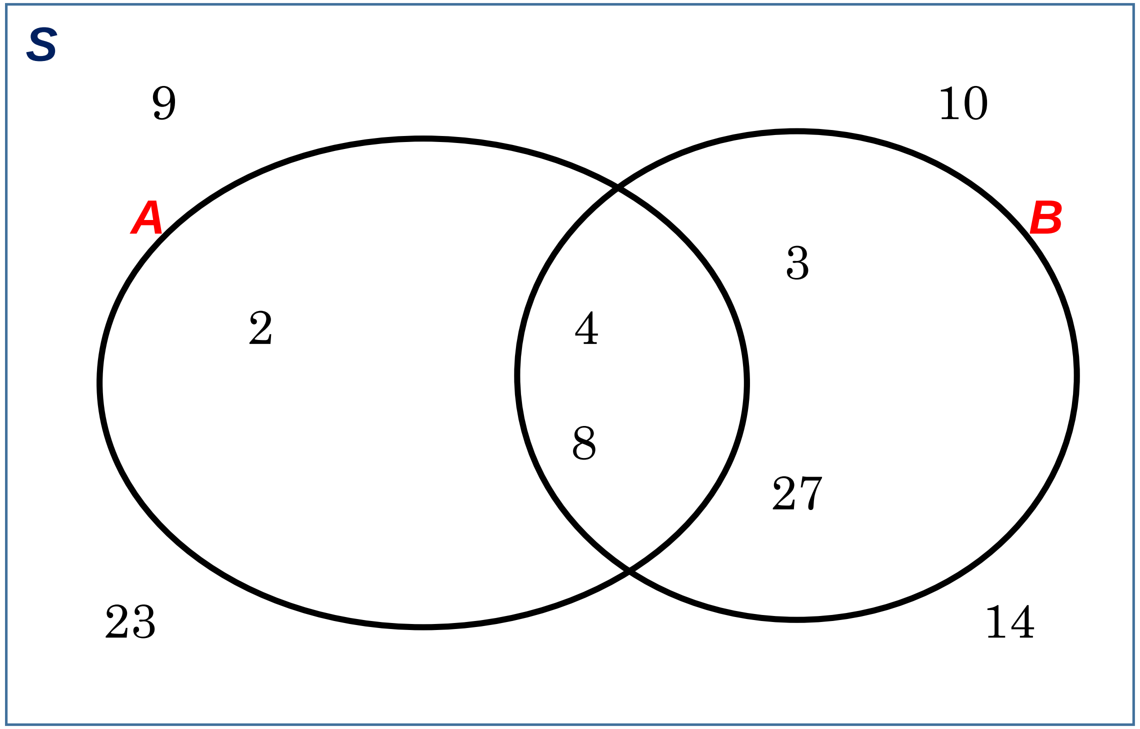
$A \cap B$ = (4, 8)

$A - B$ = (2)

$\overline{A \cup B}$ = (9, 10, 23, 14)

$\overline{A \cap B}$ = (3, 9, 10, 27, 23, 14, 2)

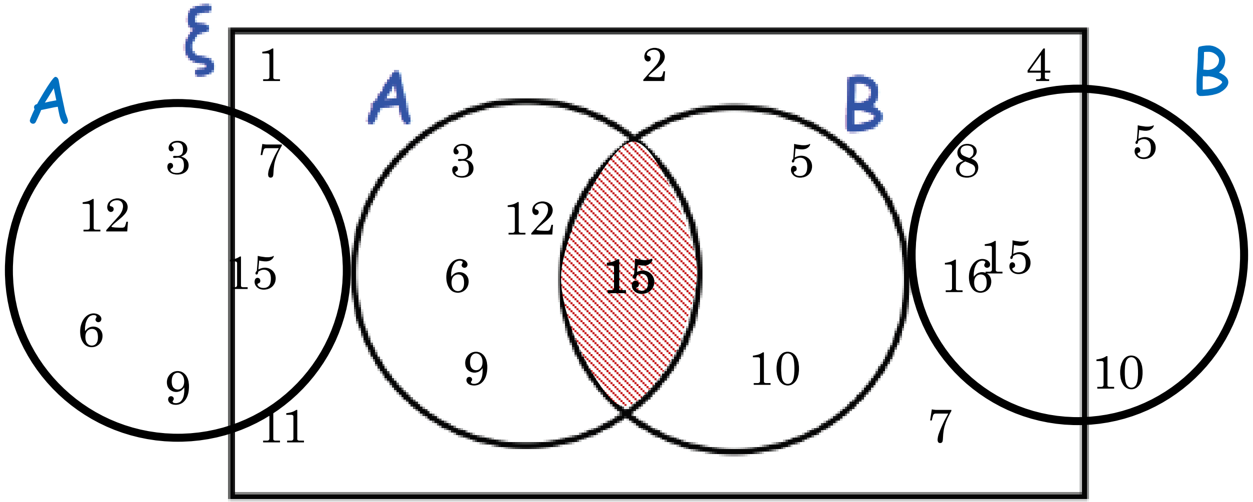
Draw the Venn diagram



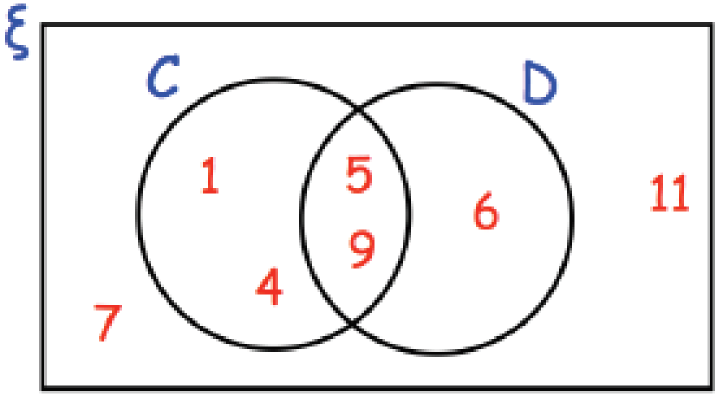
Exercise 1. ξ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16}

A = multiples of 3
B = multiples of 5

(a) Draw the Venn diagram
(b) Find A∩B



Exercise 2. Here is a Venn diagram



Write down the numbers that are in set

(a) D

(6, 5, 9)

(b) C ∪ D

C ∪ D = (1, 4 , 5, 9) ∪ (5, 9, 6)

C + D – C ∩ D = (1,4,5,9) + (5,9,6) – (5,9)

C ∪ D = (1, 4, 5, 9, 6)

(c) $\bar{C}$

$\bar{C} = \xi - C = (1, 4, 5, 9, 6, 7, 11) \cap (1, 4, 5, 9)$
$= (6, 7, 11)$

Exercise 3. There are 80 students in year 11.

9 students study French and German.
35 students only study French
2 students do not study French or German.

(a) Complete the Venn diagram

$n(\xi) = 80$

$n(G \cap F) = 9$

$n(F) - n(G \cap F) = 35 = n(\bar{G} \cap F)$

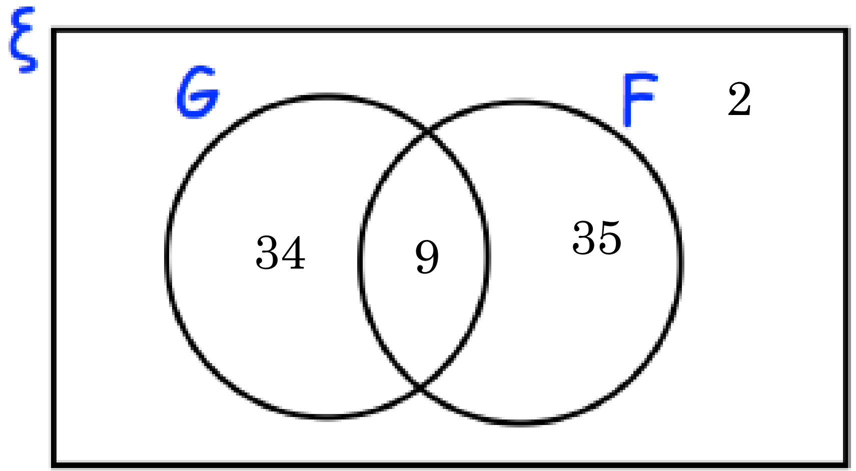
$n(F) = 35 + 9 = 44$

$n(\overline{G \cup F}) = 2$

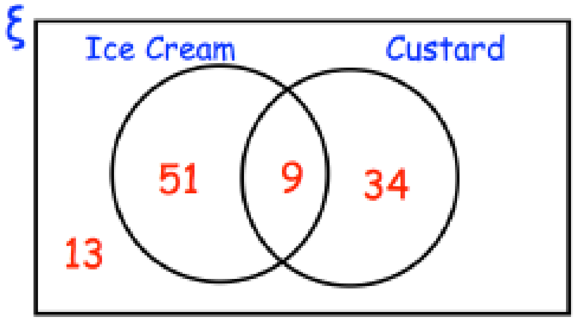
$n(G \cup F) = 80 - 2 = 78$

$n(G) = 78 - 35 = n(G \cup F) - n(\bar{G} \cap F) = 43$

$n(G) - n(G \cap F) = 43 - 9 = n(\bar{F} \cap G) = 34$



Exercise 4. At a wedding, the guests may have ice cream or custard with their dessert.
The Venn diagram shows information about the choices the guests made.



(a) How many guests had custard?

35 + 9 = 44

Event «Custard»

(b) How many guests had ice cream and custard?

9

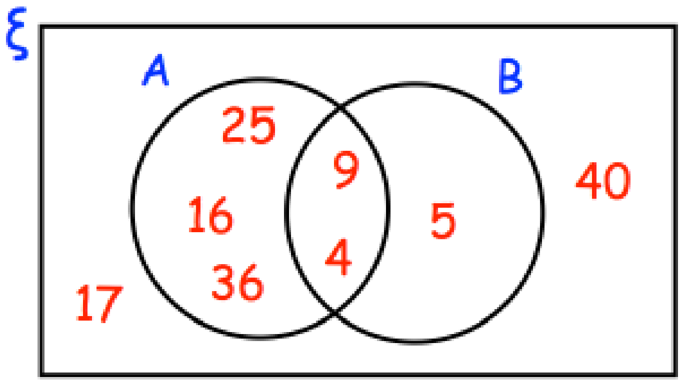
Intersection of Events «Custard» and «Ice Cream»

(c) How many guests went to the wedding?

51 + 9 + 34 + 13 = 107

Sample Size (S): «Desserts»

Exercise 5. Here is a Venn diagram.



Write down the numbers that are in set

(a) $A \cap B$ (9, 4)

(b) $A \cup B$ (25, 16, 36, 9, 4, 5)

(c) A^c (5, 17 , 40)

THEME #3

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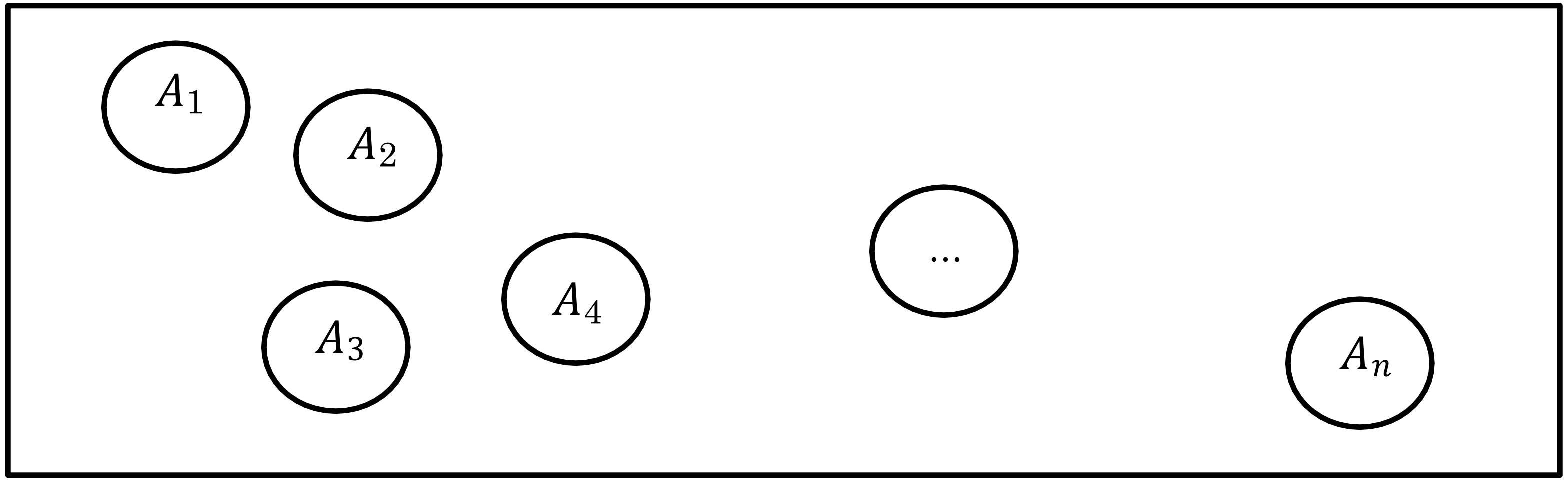
Counting principles

Counting principles

The **addition principle:** If there are n_1 outcomes in event A_1,
n_2 outcomes in event A_2,
…
n_k outcomes in event A_k

and the events $A_1, A_2, \dots A_k$ are mutually distinct (share no outcomes in common), then the total number of outcomes in $A_1 \cup A_2 \cup \dots \cup A_k$ is $n_1 + n_2 + \dots + n_k$



The **multiplication principle**

If a composite outcome can be described by a procedure that can be broken into k successive (ordered) stages such that there are

n_1 outcomes in stage 1,
n_2 outcomes in event 2,
…
n_k outcomes in event k

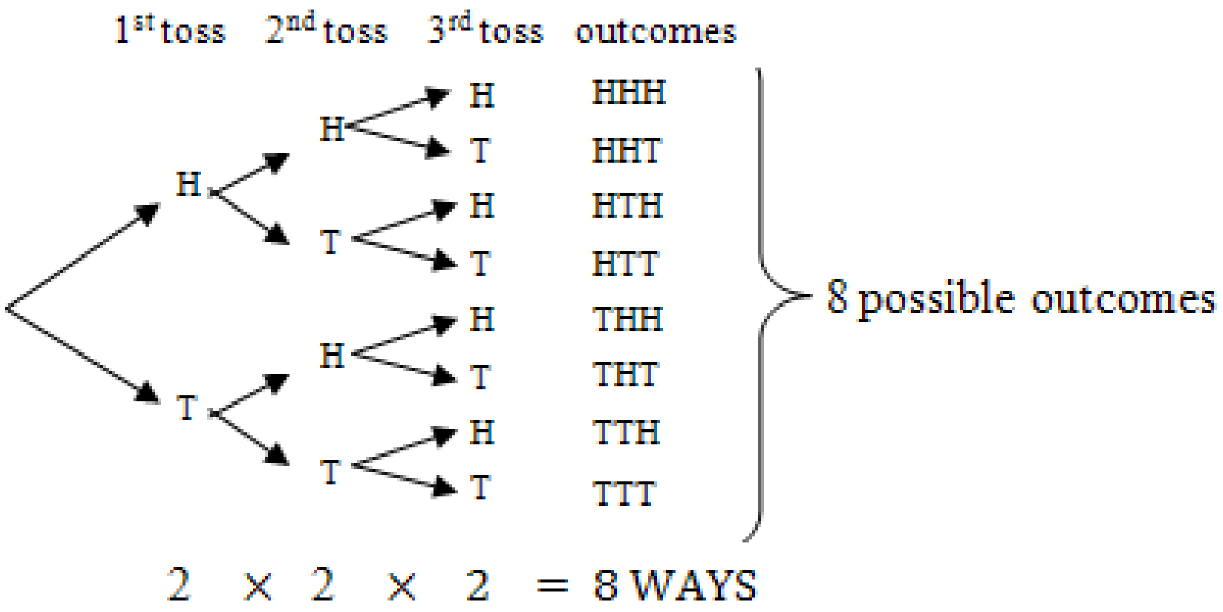
and if the number of outcomes in each stage is independent of the choices in previous stages and if the composite outcomes are all distinct then the number of possible composite outcomes is $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

e.g. suppose the composite outcomes of the trio (A,B,C) of class values for cars, where

A denotes the mileage class (A_1, A_2, or A_3)
B denotes the price class (B_1, or B_2)
C denotes the operating cost class (C_1, C_2, or C_3)

The outcome is clearly written as a 3-stage value
There are 3 outcomes in class A, 2 in class B and 3 in class C
The number of outcomes in class B does not depend on the choice made for A, etc

Then there will be $3 \cdot 2 \cdot 3 = 18$ distinct composite outcomes for car classification.


1st toss 2nd toss 3rd toss outcomes
H
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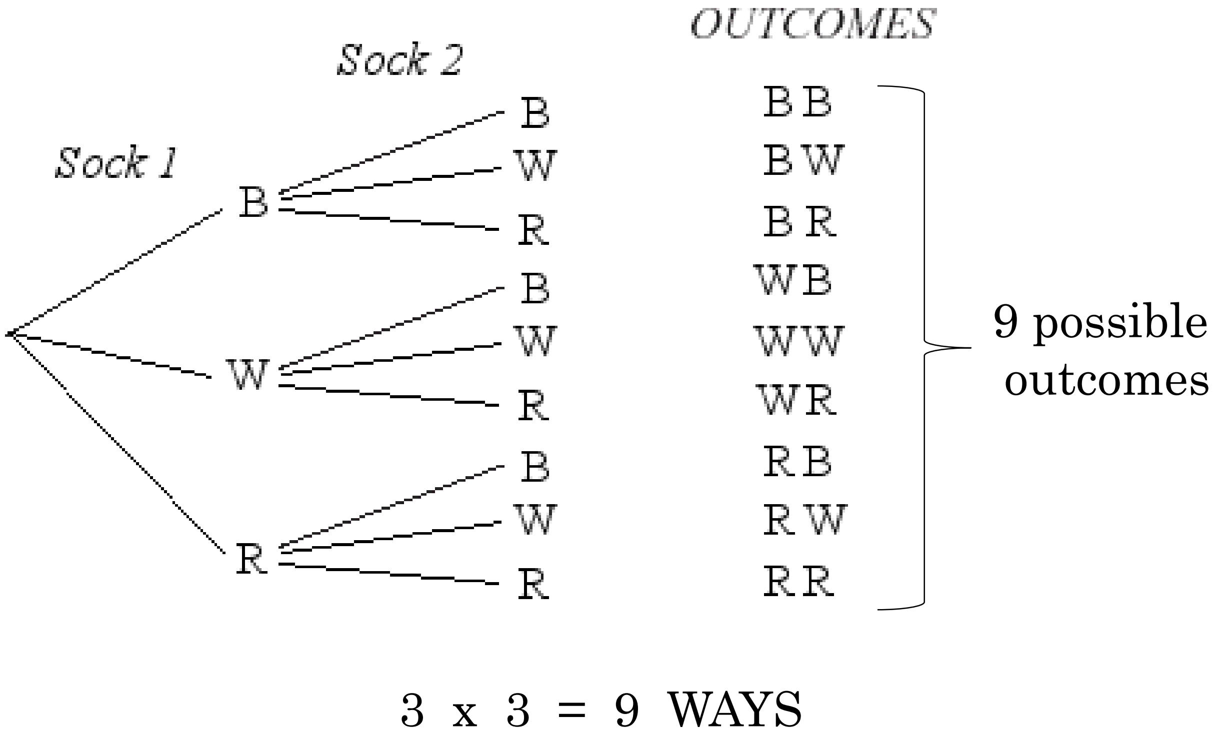
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT
8 possible outcomes
2 × 2 × 2 = 8 WAYS


Sock 1
Sock 2
OUTCOMES
B
W
R
B
W
R
B
W
R
B
W
R
BB
BW
BR
WB
WW
WR
RB
RW
RR
9 possible outcomes
3 x 3 = 9 WAYS

Counting rules

We use this rule to count the number of possible and favourable outcomes of an experiment.

In particular, we will consider a set with n elements and will count the number of groups or subsets of m elements that can be drawn with or without replacement.

Consider an experiment that consists of m parts, and let n_k, $k = 1, 2, . . . , m$, denote the number of possible outcomes of the k-th part.

The total number of possible outcomes is then: $n_1 \times n_2 \times \cdot\cdot\cdot \times n_m$.

This is sometimes referred to as the **multiplication principle**. It provides the basic principle of counting.

Factorials: n! is represents the product of all the integers from n to 1.

$$n! = n\,(n-1)\,(n-2)\,(n-3)\,\ldots\,3 \cdot 2 \cdot 1$$

Dispositions with replacement

Total outcomes of an experiment = $n_1 \times n_2 \times n_3 \times \ldots$

n_1 outcomes of the first step, n_2 outcomes of the second step, …

If $n_1 = n_2 = n_3 = \ldots \rightarrow$ Total outcomes (k steps) = n^k

This provides in how many ways you can take m elements of n.

In case of selection of few elements from a group:

Combination: number of combination of n things, taken k at a time

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Permutation (Combination with order) **or Dispositions without replacement of n elements k at a time:** number of permutations of n things, taken k at a time

$$_nP_k = \frac{n!}{(n-k)!}$$

Question 15 A man just bought 4 suits, 8 shirts, and 12 ties. All of these suits, shirts, and ties coordinate with each other. If he is to randomly select one suit, one shirt, and one tie to wear on a certain day, how many different outcomes (selections) are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 4 \times 8 \times 12 = 384$

Question 16 A student is to select three classes for next semester. If this student decides to randomly select one course from each of eight economics classes, six mathematics classes, and five computer classes, how many different outcomes are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 8 \times 6 \times 5 = 240$

Question 17 An environmental agency will randomly select 4 houses from a block containing 25 houses for a radon check.

How many total selections are possible?

How many permutations are possible?

Solution

Combination $_{25}C_4 = \frac{n!}{k!(n-k)!}$ = 25!/(4!×21!)= (25×24×23×22×~~21×…×1~~)/(4×3×2×~~21×20×…×1~~)= (25×24×23×22)/(4×3×2)= 12.650

Permutation $_{25}P_4 = \frac{n!}{(n-k)!}$ = (25×24×23×22×~~21×…×1~~)/(~~21×20×…×1~~)= 25×24×23×22=303.600

Question 18 You just got a free ticket for a boat ride, and you can bring along 2 friends! Unfortunately, you have 5 friends who want to come along.

How many different groups of friends could you take with you?

Solution

${}_5C_2 = \frac{n!}{k!(n-k)!} = 5!/(2!\times3!)=(5\times4\times3\times2\times1)/(2\times1\times3\times2\times1)=5\times2=10$

Question 19 Emily is packing her bags for her vacation. She has 6 shirts, but only 3 fit in her bag.

How many different groups of 3 shirts can she take?

Solution

${}_6C_3 = 6!/(3!\times3!)=(6\times5\times4\times3\times2\times1)/(3\times2\times1\times3\times2\times1)=5\times4=20$

Question 20 How many ways can the positions of president and vice president be assigned from a group of 8 people?

Solution

${}_8P_2=8!/6!= (8\times7\times6\times5\times4\times2\times1)/(6\times5\times4\times3\times2\times1) = 8\times7 = 56$

Question 21 Suppose we have an office of 5 women and 6 men and need to select a 4 person committee. How many ways can we select

a) 2 men and 2 women?

Men (n=5) Woman (m=6)

k=2 Men: ${}_5C_2 = \frac{n!}{k!(n-k)!}$ = 5!/(2!×3!) = (5×4×3×2×1)/(2×1×3×2×1)=5×2=10

k=2 Woman: ${}_6C_2 = \frac{m!}{k!(m-k)!}$ = 6!/(2!×4!) = (6×5×4×3×2×1)/(2×1×4×3×2×1)=3×5=15

2 Men and 2 Woman: ${}_5C_2 \times {}_6C_2$ = 10 × 15 = 150

b) 3 men and 1 woman?

k=3 Men: ${}_5C_3 = \frac{n!}{k!(n-k)!}$ = 5!/(3!×2!) = (5×4×3×2×1)/(3×2×1×2×1)=5×2=10

k=1 Woman: ${}_6C_1 = \frac{m!}{k!(m-k)!}$ = 6!/(1!×5!) !)=(6×5×4×3×2×1)/(×1×5×4×3×2×1)=6

3 Men and 1 Woman: ${}_5C_3 \times {}_6C_1$ = 10 × 6 = 60

c) All women?

Men (n=5) Woman (m=6)

k=0 Men: ${}_5C_0 = \frac{n!}{k!(n-k)!}$ = 5!/(0!×5!)=1

k=4 Woman: ${}_6C_4 = \frac{m!}{k!(m-k)!}$ = 6!/(4!×2!) = (6×5×4×3×2×1)/(4×3×2×1×2×1)=3×5=15

All Woman: ${}_5C_0 \times {}_6C_4$ = 1 × 15 = 15