Integration

A. Fabretti

Mathematics 2
A.Y. 2015/2016

Table of contents

Indefinite Integral

Suppose to a have a function f and we want to look for a function F such that

$$F'(x) = f(x)$$

we say that we are looking for an *anti-derivative* of f and we call F an indefinite integral of f.

Definition

If $F(x)$ is an antiderivative of f, the most general anti-derivative of f is called the *indefinite integral* and denoted

$$\int f(x)dx = F(x) + c$$

where c is a constant, called the *constant of integration*. Function f is called the *integrand*.

Indefinite Integral: observations

Roughly speaking Integration is the “inverse” process wrt Differentiation.
The solution to the problem of integration is not a one definite function F but a class of function $F(x) + c$ all having the same derivative f

Some Important Integrals (1)

Let $a \neq -1$, since we know that $D(x^n) = nx^{n-1}$, we have this integration formula which follow immediately from derivatives rules

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

If $a = -1$, $x^a = \frac{1}{x}$ and we know that $D(\ln x) = \frac{1}{x}$, hence

$$\int \frac{1}{x} dx = \ln |x| + C$$

we need to insert the absolute value of x, since the function $\frac{1}{x}$ is definite for all $x \neq 0$, while ln is definite only for $x > 0$.

Some Important Integrals(1):Examples

1. $\int x dx = \frac{1}{2}x^2 + C$
2. $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2x^2} + C$
3. $\int \sqrt{x} dx = \int x^{1/2} dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C = -\frac{2}{3}x^{\frac{3}{2}} + C$

Some Important Integrals (2)

Consider the exponental function e^x, we know that $D(e^x) = e^x$, which follows immediately

$$\int e^x dx = e^x + C$$

or more generally

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

since we can write $a^x = e^{(\ln a)x}$ with $a > 0$ and $a \neq 1$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

Some Immediate Integrals

Consider the chain rule $D[f(g(x))] = f'(g(x))g'(x)$, it follows immediately

$$\int f'(x)f^n(x)dx = \frac{1}{n+1}f^{n+1}(x) + C$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Some Immediate Integrals:Examples

1. $\int \frac{\ln(x)}{x} dx = \frac{1}{2} \ln^2 x + C$
2. $\int 3x^2 e^{x^3} dx = e^{x^3} + C$
3. $\int \frac{5x^4 - 2x}{x^5 - x^2 + 3} dx = \ln |x^5 - x^2 + 3| + C$

Integral Properties

As derivatives, Integral are linear hence

$$\int af(x)dx = a\int f(x)dx \qquad a \quad \text{is a constant}$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

jointing the two properties, we have the more general one

$$\int [a_1 f_1(x) + \cdots + a_n f_n(x)]dx = a_1 \int f_1(x)dx + \cdots + a_n \int f_n(x)dx$$

Examples

1. $\int (3x^4 + 5x^2 + 2)dx =$

2. $\int \left(\frac{3}{x} - 8e^{-4x}\right) dx =$

3. $\int (a + bq + cq^2)dq =$

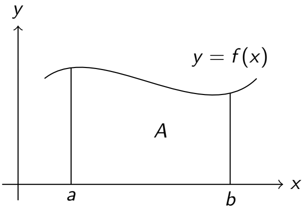
4. In the manufacture of a product, the marginal cost of producing x units is $C'(x)$ and fixed costs are $C(0)$. Find the total cost function C(x) when

$$C'(x) = 3x + 4 \qquad C(0) = 40$$

$$C'(x) = ax + b \qquad C(0) = C_0$$

Area and Definite Integral

How to compute the area A under the graph of a continuous and nonnegative function f over the interval $[a, b]$?



Riemann Integral

Let f be a bounded function in the interval $[a, b]$ and let n be a natural number. Subdivide $[a, b]$ into n parts by choosing points $a = x_0 < x_1 < \cdots < x_n = b$.

We have defined a partition of the interval $[a, b]$, denoted by P.

Put $\Delta x_i = x_{i+1} - x_i$, $i = 0, 1, ..., n - 1$.

Denote by m_i the infimum value and by M_i the superior value of f under the interval $[x_i, x_{i+1}]$.

We want to "approximate" the area under the plot of f between a and b using "small" rectangles.

Lower Riemann sum

Denote by m_i the infimum value of f under the interval $[x_i, x_{i+1}]$. We define the lower Riemann sum of f with respect to the partition P by

$$s(f, P) = \sum_{i=0}^{n-1} m_i \Delta x_i$$

$s(f, P)$ is the sum of the signed areas of rectangles that lie below the graph of f.

Upper Riemann sum

Denote by M_i the supremum value of f under the interval $[x_i, x_{i+1}]$. We define the upper Riemann sum of f with respect to the partition P by

$$S(f, P) = \sum_{i=0}^{n-1} M_i \Delta x_i$$

$S(f, P)$ is the sum of the signed areas of rectangles that lie above the graph of f.

Riemann Integral

Denoting with A the area under the graph of f we have

$$m(b-a) \leq s(f,P) \leq A \leq S(f,P) \leq M(b-a)$$

Suppose to take any possible partition P of $[a,b]$ and consider the supremum value of $s(f,P)$ and the infimum value of $S(f,P)$ over all possible partitions and denote them the lower integral $L(f)$ and the upper integral $U(f)$ respectively.

Riemann Integral

Definition

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if it is bounded and its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a, b]$, denoted by

$$\int_a^b f(x)dx$$

is the value of $U(f)$ or $L(f)$.

Definite Integral Properties

Given f a continuous function in an interval that contains a, b and c, then

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
4. $\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx$

The fundamental Theorem of calculus

Theorem

Given f a continuous function on an interval I, we define the integral function

$$G(x) = \int_a^x f(t)dt;$$

the integral function G is differentiable and

$$G'(x) = f(x)$$

and

$$\int_a^b f(x)dx = G(b) - G(a)$$

Examples

1. $\int_1^2 2x+1\ dx = x^2+x\,\big]_1^2 = 4+2-1-1=4$

2. $\int_0^1 \frac{e^x}{e^x+1}\ dx = \ln(e^x+1)\,\big]_0^1 = \ln(e^1+1)-\ln(e^0+1) = \ln\left(\frac{e+1}{2}\right)$

3. $\int_2^{1+e} \frac{t-1}{t}\ dt = \int_2^{1+e} 1-\frac{1}{t}\ dt = t-\ln|t|\,\big]_2^{1+e} = e-1\ln\left(\frac{e+1}{2}\right)$

4. $\int_1^2 \frac{1+x^2e^{x^2}}{x} dx =$

5. $\int_e^{2e} \frac{\ln(\sqrt{x})}{x} dx =$

Integration by parts

A convenient rule for computing antiderivatives considers the converse of the production rule

$$D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

which gives

$$\int (f'(x)g(x) + f(x)g'(x))dx = f(x)g(x)$$

hence we get the rule, know as *integration by parts*,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

which holds both for definite and indefinite integral.

Integration by parts:Examples

Solve $\int xe^{2x}dx$.

We can see e^{2x} as the derivative of $\frac{1}{2}e^{2x}$, hence

$$f'(x) = e^{2x}, \qquad g(x) = x, \qquad f(x) = \frac{1}{2}e^{2x}, \qquad g'(x) = 1$$

applying the rule of integration by parts we get

$$\begin{aligned}\int xe^{2x}dx &= x\ \tfrac{1}{2}e^{2x} - \int 1\ \tfrac{1}{2}e^{2x}dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C\end{aligned}$$

Integration by substitution

Consider the integral $\int f(x)\ dx$ and the substitution $x = u(t)$, we get

$$\int f(u(t))u'(t)\ dt$$

suppose we get the solution

$$\int f(u(t))u'(t)\ dt = G(t) + C.$$

If $g(t)$ is invertible, we have $t = g^{-1}(x)$ and the original integral is solved

$$\int f(x)\ dx = G(g^{-1}(x)) + C.$$

Integration by substitution: Observations

Integration by substitution is a useful methods to solve integral which can appear as complicated, however when substitution is "smart" this method lead straightforward solutions, because substitution gives a new "shape" to the problem which can be easier to be solved

In the case of a definite integral we can also transform the integration interval or work with the indefinite integral and if $g(t)$ is invertible at the end when we find the solution to the original (indefinite) integral solve the definite one.

Integration by substitution: Example

Example:

$$\int_0^1 \frac{e^{2x}}{e^x + 1}\, dx$$

set $1 + e^x = t$, $x = \ln(t-1)$ hence (we consider the indefinite integral)

$$\int \frac{e^{2x}}{e^x + 1}\, dx = \int \frac{(t-1)^2}{t} \frac{1}{t-1} dt = \int \frac{t-1}{t}\, dt = t - \ln|t|$$

replacing t with $1 + e^x$ we get

$$\int_0^1 \frac{e^{2x}}{e^x + 1}\, dx = 1 + e^x - \ln(1 + e^x)\,\Big|_0^1 = -2 - \ln(2)$$

Exercises

Solve the following integrals

1. $\int_1^3 \frac{1}{\sqrt{x}+1} \; dx$

2. $\int \ln\left(\frac{x-x^2}{x^3}\right) \; dx$

3. $\int_1^e \sqrt{x}\ln(x) \; dx$

4. $\int_2^3 \sqrt{2x+1} \; dx$

Improper Integral

In statistics and in economics it is common to encounter integrals over an infinite interval.

An improper integral is the limit (if it exists) of a definite integral as an endpoint of the integration interval goes to a value for which the function is not specified or goes to ∞ or $-\infty$. We write

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

Improper Integral: example

The exponential distribution in statistics is defined by the density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and λ a positive constant
We show that $\int_0^\infty f(x)\, dx = 1$. Set $b > 0$ and compute

$$\int_0^b \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^b = -e^{-\lambda b} + 1$$

the limit of $-e^{-\lambda b} + 1$ as $b \to \infty$ is 1.

Exercises

Prove the following integrals converge and find their values

1. For $c > 0$,

$$\int_{-\infty}^{\infty} xe^{-cx^2}\ dx$$

2. For $a > 1$

$$\int_{1}^{\infty} \frac{1}{x^a}\ dx$$

Consumer Surplus

Present value

An example of improper integral in economics

Denoting by $c(t)$ consumption at time t, by U the instantaneous utility function and r a discount rate. An integral

$$\int_{t_0}^{\infty} U(c(t))e^{-\alpha t}dt$$

represents the present value of future cumulative utility coming from consumption.