Matrices

A. Fabretti

Mathematics 2
A.Y. 2015/2016

Table of contents

Introduction

A **matrix** is a rectangular array of numbers. The size of a matrix is indicated by the number of rows and the number of its columns. A matrix with k rows and n columns is called a $k \times n$ (" k by n") matrix. The number in row i and column j is called the (i, j)th entry and it is denoted by a_{ij}.
The matrix A with entries a_{ij} with $i = 1, ..., k$ and $j = 1, ...n$ is a $k \times n$ matrix is representing as follows

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Matrices can be added, subtracted, multiplied and even divided if their sizes respect some conditions.

Addition

Two matrices can be added only if they have the same size, i.e. the same number of rows and columns. Their sum is a new matrix of the same size as the two matrices being added.
Let A and B be two $k \times n$ matrices with entries a_{ij} and b_{ij} for $i = 1, ..., k$ and $j = 1, ..., n$, respectively. Their sum $A + B$ is a $k \times n$ matrix with entries $a_{ij} + b_{ij}$ for $i = 1, ..., k$ and $j = 1, ..., n$.

Addition (2)

Explicitly

$$A + B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij} + b_{ij} & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}$$

Addition: Example

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 & 5 \\ 10 & 7 & -3 \end{pmatrix}$ two 2×3 matrices.

Their sum is a 2×3 matrix

$$A+B = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 5 \\ 10 & 7 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 5 \\ 13 & 8 & -6 \end{pmatrix}$$

Addition

The matrix $\mathbf{0}$ is a special matrix whose entries are all zero. It is an additive identity since $A + \mathbf{0} = A$ for all matrices A:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

Subtraction

Subtraction, as addition, is possible only if the two matrices have the same size. Writing $A - B$ is a shorthand for $A + (-B)$ where $-B$ is what one adds to B to get $\mathbf{0}$ (as opposite for numbers).

$$A - B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} - \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & a_{ij} - b_{ij} & \vdots \\ a_{k1} - b_{k1} & \cdots & a_{kn} - b_{kn} \end{pmatrix}$$

Scalar multiplication

Matrices can be multiplied by real numbers called **scalar**, the result is a matrix with the same size with all the entries multiplied by the scalar. Explicitly, let A be a $k \times n$ matrices and γ a scalar ($\gamma \in \mathbb{R}$)

$$\gamma A = \gamma \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} \gamma a_{11} & \cdots & \gamma a_{1n} \\ \vdots & \gamma a_{ij} & \vdots \\ \gamma a_{k1} & \cdots & \gamma a_{kn} \end{pmatrix}$$

Exercises

Given $A = \begin{pmatrix} -3 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \\ -2 & -4 & 3 \end{pmatrix}$, $B \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 3 & 1/2 \\ -2 & -1/2 & 2 \end{pmatrix}$, $\alpha = -2$ and $\beta = 3$.

Solve the following expression

1. $\beta A + \alpha(B - A)$
2. $B - \alpha B + A$
3. $(\alpha + \beta)(A + B)$

Matrix Multiplication

In matrix multiplication sizes and order in which matrices are multiplied matter!
We can define a matrix product of A and B, AB, only if **the number of columns of A equals the number of rows of B.**
Let A be a $k \times m$ matrix and B a $m \times n$ matrix, their product AB exists and it is a matrix $k \times n$ whose (i, j) th entry is given by the inner product of the ith row of A and jth column of B

$$(AB)_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} \; a_{i2}b_{2j} \; \cdots \; a_{im}b_{mj}.$$

Matrix Multiplication (2)

In other words, the (i, j) th entry is given by

$$\sum_{h=1}^{m} a_{ih} b_{hj}.$$

Note that if AB exists, BA is not equal (this operation is not commutative) and even does not exist! Indeed, if $k \neq n$, BA cannot be calculated.

Matrix Multiplication: Examples

1. Calculate AB and say if BA exists.
$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 0 \\ 2 & -3 \\ 0 & 1 \end{pmatrix}$$

2. Calculate CD and DC and verify that $CD \neq DC$.
$$C = \begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} -2 & 0 & 5 \\ 1 & 4 & 1 \end{pmatrix}$$

Identity Matrix

The $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

is called the **identity matrix** and is such that $a_{ii} = 1$ for all i and $a_{ij} = 0$ for all $i \neq j$. This matrix has the property that for any $m \times n$ matrix A

$$AI = A$$

and for any $n \times l$ matrix B

$$IB = B.$$

Law of Matrix Algebra

- *Associative law*

$$(A+B)+C = A+(B+C)$$
$$(AB)C = A(BC)$$

- *Commutative law for addition*

$$A+B = B+A$$

- *Distributive law*

$$A(B+C) = AB+AC$$
$$(A+B)C = AC+BC$$

Recall that matrix multiplication is not commutative!

Transpose

The **transpose** of a $k \times n$ matrix is a $n \times k$ matrix obtained by interchanging the rows and the columns of A. This matrix is denoted by A^T. Thus the (i, j)th entry of A becomes the (j, i)th entry of A^T.

For example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}$$

Laws for transpose matrices

Let A and B be $k \times n$ matrices and γ a scalar, the following rules hold

- $(A+B)^T = A^T + B^T$,
- $(A-B)^T = A^T - B^T$,
- $(A^T)^T = A$,
- $(\gamma A)^T = \gamma A^T$

Transpose of matrices product

Let A be $k \times m$ matrix and B be $m \times n$ matrix. Then, $(AB)^T = B^T A^T$.
This can be easily proved

$$
\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} = \sum_h A_{jh} B_{hi} \\
&= \sum_h (A^T)_{hj} (B^T)_{ih} = \sum_h (B^T)_{ih} (A^T)_{hj} \\
&= (B^T A^T)_{ij}
\end{aligned}
$$

Exercises

Using matrices

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix},$$

calculate (where possible)

1. $A + B - 2D$
2. $(2A - 3B)^T D$
3. $(AD)^T C + B$

Special Matrices

Here a list of special matrices:

- A matrix is called **square** if it has the same number of columns and rows ($k = n$)
- A matrix is a **column** matrix if it has only one column. For example

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- A matrix is a **row** matrix if it has only one row. For example

$$\begin{pmatrix} e & f & d \end{pmatrix}$$

Special Matrices (2)

Special matrices list follows:

- A **diagonal** matrix is a square matrix in which all nondiagonal entries are zeros. For example

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- A **upper-triangular** matrix is a (square) matrix in which all entries below the diagonal are zeros. For example

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

Special Matrix (3)

list follows:

- A **lower-triangular** matrix is a (square) matrix in which all entries above the diagonal are zeros. For example

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

- A square matrix B is **idempotent** if $BB = B$.

Symmetric Matrix

A square matrix is **symmetric** if $A = A^T$ or similarly if $a_{ij} = a_{ji}$ for all i, j. For example

$$\begin{pmatrix} 1 & 0.1 & -2 \\ 0.1 & 1 & 0 \\ -2 & 0 & 6 \end{pmatrix}.$$

Example: The variance covariance matrix is a symmetric matrix.

Exercises

1. Find real numbers a, b and x such that
$$\begin{pmatrix} a & b \\ x & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ x & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}.$$
2. A square matrix B is **anti-symmetric** (or skew-symmetric) is $B = -B^T$. Show that if A is any square matrix, $A_1 = \frac{1}{2}(A + A^T)$ is symmetric and $A_2 = \frac{1}{2}(A - A^T)$ is anti-symmetric. Verify that $A = A_1 + A_2$.
3. If P and Q are $n \times n$ matrices with $PQ - QP = P$, prove that $P^2Q - QP^2 = 2P^2$ and $P^3Q - QP^3 = 3P^3$. Then use induction to prove that $P^kQ - QP^k = kP^k$ for $k = 1, 2, \ldots$.

Square Matrices

Within the class of $n \times n$ matrices, denoted by $\mathcal{M}_n$, all the arithmetic operations are possibile and the result is a $n \times n$ matrix.

The identity matrix I is such that $AI = IA = A$.

Since we have the product, can we divide square matrices? We must introduce the idea of inverse as for numbers.

Definition

Let A be a matrix in $\mathcal{M}_n$. The matrix $B \in \mathcal{M}_n$ is an inverse for A if $AB = BA = I$.

If the matrix B exists we say that A is **invertible**.

Inverse matrix is unique

Theorem

An $n \times n$ matrix can have at most one inverse

Proof.

Suppose that B and C are both inverse of A. Then we have

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

If a matrix $A \in \mathcal{M}_n$ is invertible we denote its unique inverse A^{-1}.

Exercises

- Verify that $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is invertible and $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$
- Verify that $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ is invertible and $B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$

Inverse matrix of a 2×2 matrix

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the inverse exists if $ad - bc \neq 0$ and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Theorem

Let A and B be square invertible matrices. Then

- $(A^{-1})^{-1} = A$;
- $(A^T)^{-1} = (A^{-1})^T$;
- AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem

If A is invertible:

- (A^m) *is invertible for any integer m and* $(A^m)^{-1} = (A^{-1})^m$;
- *for any integer r and s* $A^r A^s = A^{r+s}$;
- *for any scalar* γ, γA *is invertible and* $(\gamma A)^{-1} = \gamma^{-1}A^{-1}$.

Exercises

1. Given $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ find (if it exists) A^{-1}.
2. Show that the inverse of a 2×2 symmetric matrix is symmetric.
3. What is the inverse of an $n \times n$ diagonal matrix? When does it exist?
4. Show that $(A+B)^{-1}$, if it exists, is generally not $A^{-1} + B^{-1}$.

Defining the determinant

The determinant of a matrix can be defined inductively.

- ($n = 1$) A 1×1 matrix is a scalar a. The inverse of such a matrix exists if $a \neq 0$ hence we define

$$det(a) = a$$

- ($n = 2$) A matrix 2×2 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible if $a_{11}a_{22} - a_{12}a_{21} \neq 0$ hence we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

which equals

$$\det(A) = a_{11}det(a_{22}) - a_{12}det(a_{21}).$$

Definition

Let A be an $n \times n$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then the scalar

$$M_{ij} = \det(A_{ij})$$

is called the (i,j)th **minor** of A and the scalar

$$C_{ij} = (-1)^{i+j} M_{ij}$$

is called the (i,j)th **cofactor**.

The determinant of a 2×2 matrix

The determinant of a 2×2 matrix

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

can be seen

$$\begin{aligned} \det(A) &= a_{11}det(a_{22}) - a_{12}det(a_{21}) \\ &= a_{11}M_{11} - a_{12}M_{12} \\ &= a_{11}C_{11} + a_{12}C_{12} \end{aligned}$$

The determinant of a 3×3 matrix

The determinant of a 3×3 matrix

$$det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}\,\mathrm{det}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\,\mathrm{det}\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\,\mathrm{det}\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant of a $n \times n$ matrix

Definition

The determinant of a $n \times n$ matrix

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}$$

$$= a_{11}M_{11} - a_{12}M_{12} + ... + (-1)^{n+1}a_{1n}M_{1n}$$

Notation In referring to determinant of matrix A we write $det(A)$, $|A|$ or

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \quad \text{or} \quad \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Exercises

Compute the determinants of the following matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 0 & -1 & 2 \\ 2 & 3 & 0 & 0 \\ -2 & 3 & -4 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & -4 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}$$

Theorem

The determinant of a lower-triangular, an upper-triangular or a diagonal matrix is simply the product of its diagonal entries.

Proof.

Easy to see when 2×2.

For a upper-triangular 3×3:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix} = a_{11}a_{22}a_{33}.$$

Theorem

Let A be a square matrix. Then

- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(\gamma A) = \gamma^n \det(A)$ *for* γ *scalar*

Adjoint matrix

For $n \times n$ matrix A we denote by C_{ij} the (i, j)th cofactor of A, that is the $(-1)^{i+j}$ times the determinant of the submatrix obtained deleting row i and columns j from A.

Definition

The $n \times n$ matrix whose (i, j)th entry is C_{ji} is called the **adjoint** of A and is denoted by **adj**A.

Note that we can alternatively say that the adjoint matrix is the transpose of the matrix having as entries the cofactors of A.

Inverse matrix

Theorem

Let A be an $n \times n$ invertible matrix. Then,

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Note that a matrix is invertible if and only if its determinant is different from zero.
Note that $AA^{-1} = I$ thus $\det(AA^{-1}) = \det(I) = 1$, since $\det(AA^{-1}) = \det(A)\det(A^{-1})$ we derive that $\det(A^{-1}) = \frac{1}{\det(A)}$.

Exercises

Find the inverse (if it exists) of the following matrices

1. $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$
3. $C = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 1 \\ 4 & -16 & 8 \end{pmatrix}$

Exercises

Exericise 1 Let $A_t = \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

1. For what values of t does A_t have an inverse?
2. Find a matrix X such that $B + XA_1^{-1} = A_1^{-1}$.

Exericise 2 Solve the equation

$$\begin{vmatrix} 1-x & 2 & 2 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix} = 0$$

Trace

Definition

The trace of a square matrix is the sum of its diagonal entries:

$$trace(A) = a_{11} + a_{22} + \ldots + a_{nn}.$$