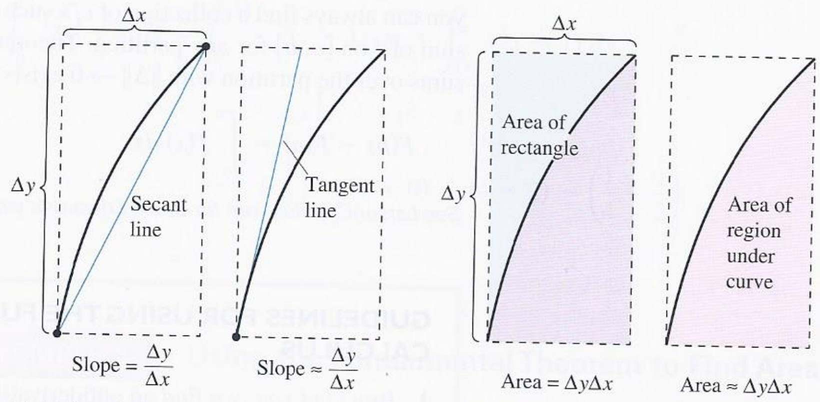
4.4 The Fundamental Theorem of Calculus

- **Evaluate a definite integral using the Fundamental Theorem of Calculus.**
- **Understand and use the Mean Value Theorem for Integrals.**
- **Find the average value of a function over a closed interval.**
- **Understand and use the Second Fundamental Theorem of Calculus.**
- **Understand and use the Net Change Theorem.**

The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus (introduced with the tangent line problem) and integral calculus (introduced with the area problem). So far, these two problems might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in the **Fundamental Theorem of Calculus.**

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.27. The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line). Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle). So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations. The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.



(a) Differentiation (b) Definite integration

Differentiation and definite integration have an "inverse" relationship.

Figure 4.27

ANTIDIFFERENTIATION AND DEFINITE INTEGRATION

Throughout this chapter, you have been using the integral sign to denote an antiderivative (a family of functions) and a definite integral (a number).

Antidifferentiation: $\int f(x)\,dx$ Definite integration: $\int_a^b f(x)\,dx$

The use of the same symbol for both operations makes it appear that they are related. In the early work with calculus, however, it was not known that the two operations were related. The symbol $\int$ was first applied to the definite integral by Leibniz and was derived from the letter S. (Leibniz calculated area as an infinite sum, thus, the letter S.)

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof The key to the proof is writing the difference $F(b) - F(a)$ in a convenient form. Let Δ be any partition of $[a, b]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

By pairwise subtraction and addition of like terms, you can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

By the Mean Value Theorem, you know that there exists a number c_i in the ith subinterval such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(c_i) = f(c_i)$, you can let $\Delta x_i = x_i - x_{i-1}$ and obtain

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

This important equation tells you that by repeatedly applying the Mean Value Theorem, you can always find a collection of c_i's such that the *constant* $F(b) - F(a)$ is a Riemann sum of f on $[a, b]$ for any partition. Theorem 4.4 guarantees that the limit of Riemann sums over the partition with $\|\Delta\| \to 0$ exists. So, taking the limit (as $\|\Delta\| \to 0$) produces

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative.

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

EXAMPLE 1 **Evaluating a Definite Integral**

See LarsonCalculus.com for an interactive version of this type of example.

Evaluate each definite integral.

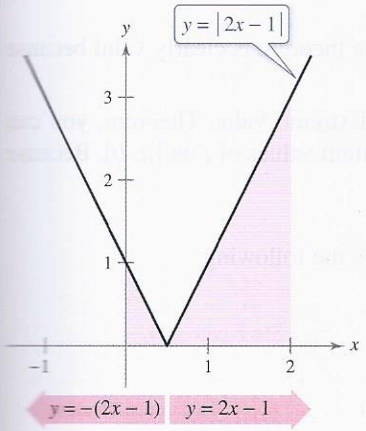
a. $\int_1^2 (x^2 - 3)\, dx$ **b.** $\int_1^4 3\sqrt{x}\, dx$ **c.** $\int_0^{\pi/4} \sec^2 x\, dx$

Solution

a. $\int_1^2 (x^2 - 3)\, dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$

b. $\int_1^4 3\sqrt{x}\, dx = 3\int_1^4 x^{1/2}\, dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$

c. $\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.
Figure 4.28

EXAMPLE 2 **A Definite Integral Involving Absolute Value**

Evaluate $\int_0^2 |2x - 1|\, dx$.

Solution Using Figure 4.28 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x - 1|\, dx &= \int_0^{1/2} -(2x - 1)\, dx + \int_{1/2}^2 (2x - 1)\, dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$

EXAMPLE 3 **Using the Fundamental Theorem to Find Area**

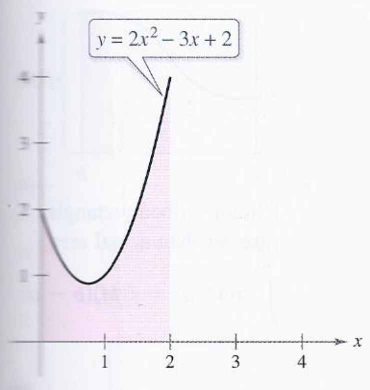
Find the area of the region bounded by the graph of

$$y = 2x^2 - 3x + 2$$

the x-axis, and the vertical lines $x = 0$ and $x = 2$, as shown in Figure 4.29.

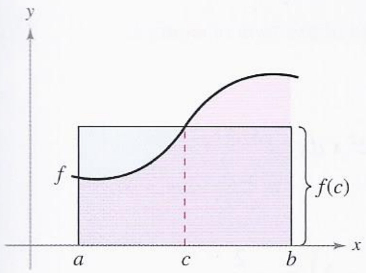
Solution Note that $y > 0$ on the interval $[0, 2]$.

$$\begin{aligned}\text{Area} &= \int_0^2 (2x^2 - 3x + 2)\, dx && \text{Integrate between } x = 0 \text{ and } x = 2. \\ &= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^2 && \text{Find antiderivative.} \\ &= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) && \text{Apply Fundamental Theorem.} \\ &= \frac{10}{3} && \text{Simplify.}\end{aligned}$$



The area of the region bounded by the graph of y, the x-axis, $x = 0$, and $x = 2$ is $\frac{10}{3}$.
Figure 4.29

The Mean Value Theorem for Integrals



Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

Figure 4.30

In Section 4.2, you saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle. The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles, there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.30.

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof

Case 1: If f is constant on the interval $[a, b]$, then the theorem is clearly valid because c can be any point in $[a, b]$.

Case 2: If f is not constant on $[a, b]$, then, by the Extreme Value Theorem, you can choose $f(m)$ and $f(M)$ to be the minimum and maximum values of f on $[a, b]$. Because

$$f(m) \le f(x) \le f(M)$$

for all x in $[a, b]$, you can apply Theorem 4.8 to write the following.

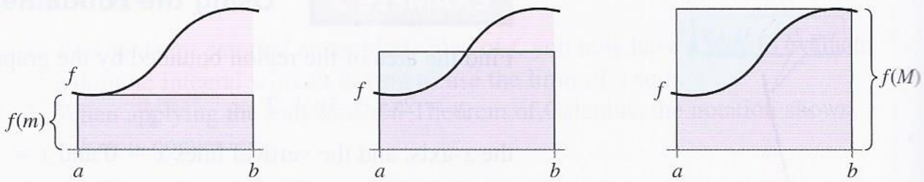
$$\int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx \qquad \text{See Figure 4.31.}$$

$$f(m)(b-a) \le \int_a^b f(x)\,dx \le f(M)(b-a) \qquad \text{Apply Fundamental Theorem.}$$

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M) \qquad \text{Divide by } b-a.$$

From the third inequality, you can apply the Intermediate Value Theorem to conclude that there exists some c in $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or} \quad f(c)(b-a) = \int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area) $\int_a^b f(m)\,dx = f(m)(b-a)$

Mean value rectangle (equal to actual area) $\int_a^b f(x)\,dx$

Circumscribed rectangle (greater than actual area) $\int_a^b f(M)\,dx = f(M)(b-a)$

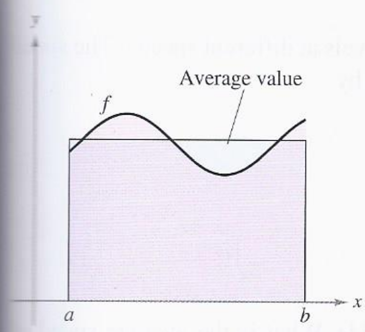
Figure 4.31

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Notice that Theorem 4.10 does not specify how to determine c. It merely guarantees the existence of at least one number c in the interval.

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.



Average value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Figure 4.32

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

See Figure 4.32.

To see why the average value of f is defined in this way, partition $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

If c_i is any point in the ith subinterval, then the arithmetic average (or mean) of the function values at the c_i's is

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdot\cdot\cdot + f(c_n)].$$

Average of $f(c_1), \ldots, f(c_n)$

By multiplying and dividing by $(b-a)$, you can write the average as

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \end{aligned}$$

Finally, taking the limit as $n \to \infty$ produces the average value of f on the interval $[a, b]$, as given in the definition above. In Figure 4.32, notice that the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value.

This development of the average value of a function on an interval is only one of many practical uses of definite integrals to represent summation processes. In Chapter 7, you will study other applications, such as volume, arc length, centers of mass, and work.

EXAMPLE 4 **Finding the Average Value of a Function**

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution The average value is

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx \\ &= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4 \\ &= \frac{1}{3}[64 - 16 - (1-1)] \\ &= \frac{48}{3} \\ &= 16. \end{aligned}$$

See Figure 4.33.

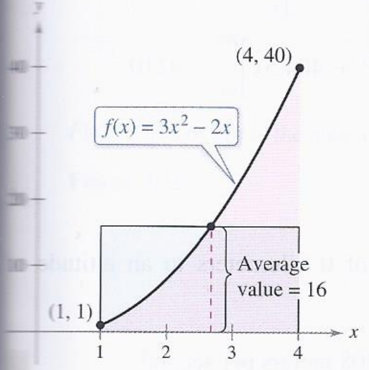


Figure 4.33

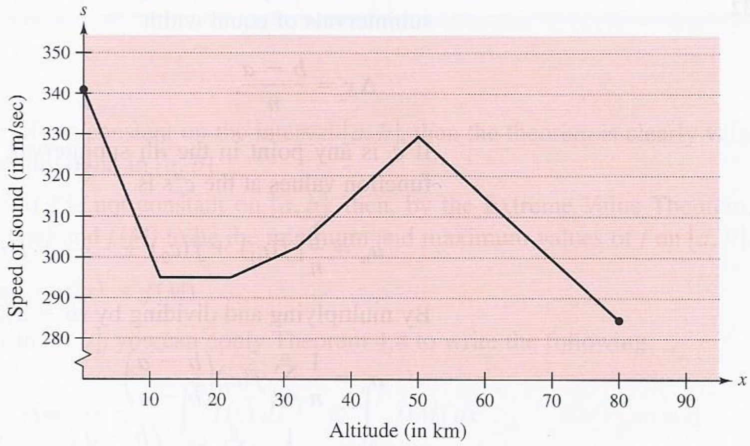
The first person to fly at a speed greater than the speed of sound was Charles Yeager. On October 14, 1947, Yeager was clocked at 295.9 meters per second at an altitude of 12.2 kilometers. If Yeager had been flying at an altitude below 11.275 kilometers, this speed would not have "broken the sound barrier." The photo shows an F/A-18F Super Hornet, a supersonic twin-engine strike fighter. A "green Hornet" using a 50/50 mixture of biofuel made from camelina oil became the first U.S. naval tactical aircraft to exceed 1 mach.

EXAMPLE 5 **The Speed of Sound**

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

where x is the altitude in kilometers (see Figure 4.34). What is the average speed of sound over the interval $[0, 80]$?



Speed of sound depends on altitude.
Figure 4.34

Solution Begin by integrating $s(x)$ over the interval $[0, 80]$. To do this, you can break the integral into five parts.

$$\int_0^{11.5} s(x)\,dx = \int_0^{11.5} (-4x + 341)\,dx = \Big[-2x^2 + 341x\Big]_0^{11.5} = 3657$$

$$\int_{11.5}^{22} s(x)\,dx = \int_{11.5}^{22} 295\,dx = \Big[295x\Big]_{11.5}^{22} = 3097.5$$

$$\int_{22}^{32} s(x)\,dx = \int_{22}^{32} \left(\tfrac{3}{4}x + 278.5\right)dx = \Big[\tfrac{3}{8}x^2 + 278.5x\Big]_{22}^{32} = 2987.5$$

$$\int_{32}^{50} s(x)\,dx = \int_{32}^{50} \left(\tfrac{3}{2}x + 254.5\right)dx = \Big[\tfrac{3}{4}x^2 + 254.5x\Big]_{32}^{50} = 5688$$

$$\int_{50}^{80} s(x)\,dx = \int_{50}^{80} \left(-\tfrac{3}{2}x + 404.5\right)dx = \Big[-\tfrac{3}{4}x^2 + 404.5x\Big]_{50}^{80} = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)\,dx = 24{,}640.$$

So, the average speed of sound from an altitude of 0 kilometers to an altitude of 80 kilometers is

$$\text{Average speed} = \frac{1}{80}\int_0^{80} s(x)\,dx = \frac{24{,}640}{80} = 308 \text{ meters per second.}$$

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration. However, a slightly different situation may arise in which the variable x is used in the upper limit of integration. To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of x.

The Definite Integral as a Function of x

F is a function of x.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

f is a function of t.

Exploration

Use a graphing utility to graph the function

$$F(x) = \int_0^x \cos t\,dt$$

for $0 \le x \le \pi$. Do you recognize this graph? Explain.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

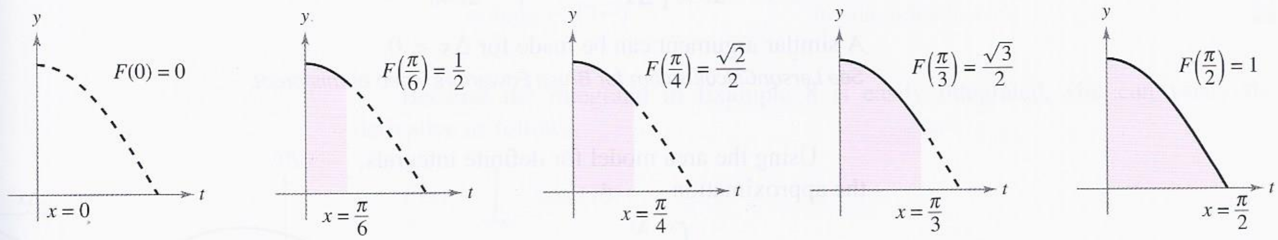
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}$, and $\dfrac{\pi}{2}$.

Solution You could evaluate five different definite integrals, one for each of the given upper limits. However, it is much simpler to fix x (as a constant) temporarily to obtain

$$\begin{aligned}\int_0^x \cos t\,dt &= \sin t\Big]_0^x \\ &= \sin x - \sin 0 \\ &= \sin x.\end{aligned}$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.35.



$F(x) = \displaystyle\int_0^x \cos t\,dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 4.35

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$. For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$. This interpretation of an integral as an **accumulation function** is used often in applications of integration.

In Example 6, note that the derivative of F is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t\, dt\right] = \cos x.$$

This result is generalized in the next theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x).$$

Proof Begin by defining F as

$$F(x) = \int_a^x f(t)\, dt.$$

Then, by the definition of the derivative, you can write

$$\begin{aligned} F'(x) &= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt - \int_a^x f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)\, dt + \int_x^a f(t)\, dt\right] \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_x^{x+\Delta x} f(t)\, dt\right]. \end{aligned}$$

From the Mean Value Theorem for Integrals (assuming $\Delta x > 0$), you know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the expression above is equal to $f(c)\, \Delta x$. Moreover, because $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. So, you obtain

$$F'(x) = \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x} f(c)\, \Delta x\right] = \lim_{\Delta x \to 0} f(c) = f(x).$$

A similar argument can be made for $\Delta x < 0$.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

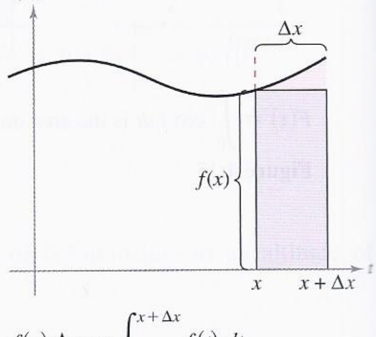
Using the area model for definite integrals, the approximation

$$f(x)\, \Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

can be viewed as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval

$$[x, x + \Delta x]$$

as shown in the figure at the right.



$$f(x)\, \Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

Note that the Second Fundamental Theorem of Calculus tells you that when a function is continuous, you can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section P.3.)

EXAMPLE 7 **The Second Fundamental Theorem of Calculus**

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real number line. So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$

The differentiation shown in Example 7 is a straightforward application of the Second Fundamental Theorem of Calculus. The next example shows how this theorem can be combined with the Chain Rule to find the derivative of a function.

EXAMPLE 8 **The Second Fundamental Theorem of Calculus**

Find the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t\,dt$.

Solution Using $u = x^3$, you can apply the Second Fundamental Theorem of Calculus with the Chain Rule as shown.

$$F'(x) = \frac{dF}{du}\frac{du}{dx} \qquad \text{Chain Rule}$$

$$= \frac{d}{du}[F(x)]\frac{du}{dx} \qquad \text{Definition of } \frac{dF}{du}$$

$$= \frac{d}{du}\left[\int_{\pi/2}^{x^3} \cos t\,dt\right]\frac{du}{dx} \qquad \text{Substitute } \int_{\pi/2}^{x^3} \cos t\,dt \text{ for } F(x).$$

$$= \frac{d}{du}\left[\int_{\pi/2}^{u} \cos t\,dt\right]\frac{du}{dx} \qquad \text{Substitute } u \text{ for } x^3.$$

$$= (\cos u)(3x^2) \qquad \text{Apply Second Fundamental Theorem of Calculus.}$$

$$= (\cos x^3)(3x^2) \qquad \text{Rewrite as function of } x.$$

Because the integrand in Example 8 is easily integrated, you can verify the derivative as follows.

$$F(x) = \int_{\pi/2}^{x^3} \cos t\,dt$$

$$= \sin t\,\Big]_{\pi/2}^{x^3}$$

$$= \sin x^3 - \sin\frac{\pi}{2}$$

$$= \sin x^3 - 1$$

In this form, you can apply the Power Rule to verify that the derivative of F is the same as that obtained in Example 8.

$$\frac{d}{dx}[\sin x^3 - 1] = (\cos x^3)(3x^2) \qquad \text{Derivative of } F$$