MATHEMATICS II

Thursday, March 15 2018

Third Exercise Class

1. Given in $\mathbb{R}^3$ the vectors $\vec{v} = (0, 3, 2)$, $\vec{w} = (1, 0, -2)$ and $\vec{z} = (0, 1, 1)$, calculate:
   a) the combination $2\vec{v} + 3\vec{w} - \vec{z}$;
   b) the inner product $\vec{v} \cdot \vec{w}$;
   c) the combination $(\vec{v} + \vec{z}) \times \vec{w}$;
   d) the lenght of each vector.
   Solution
   a) $(3, 5, -3)$;
   b) -4;
   c) $(-8, 3, -4)$,
   d) $||\vec{v}|| = \sqrt{13}$, $||\vec{w}|| = \sqrt{5}$, $||\vec{z}|| = \sqrt{2}$

2. Given in $\mathbb{R}^3$ the vectors $\vec{v} = (1, 0, 1)$, $\vec{w} = (1, 1, 1)$ and $\vec{z} = (2, 1, 2)$, calculate the values of the parameters α, β, γ, such that : $\alpha\vec{v} + \beta\vec{w} + \gamma\vec{z} = 0$.
   Solution
   $$\begin{cases} \alpha = -\gamma \\ \beta = -\gamma \\ \gamma = \gamma \end{cases}$$

3. Given the vectors $\vec{v} = (3, 0, 2)$ and $\vec{w} = (k, 3, k + 2)$ determine values for k such that their inner product is zero.
   Solution
   $k = -\frac{4}{5}$

4. For each of the following vectors, find the correspondent versor
   a) $\vec{v} = (\alpha, 3\alpha)$;
   b) $\vec{v} = (\alpha - 3, 2\alpha, 1)$;
   c) $\vec{v} = (3, 2, -\alpha)$;
   d) $\vec{v} = (1, \alpha, 4\alpha)$.
   Solution
   a) $\left(\frac{\alpha}{|\alpha|\sqrt{10}}, \frac{3\alpha}{|\alpha|\sqrt{10}}\right)$;
   b) $\left(\frac{\alpha-3}{\sqrt{5\alpha^2+6\alpha+1}}, \frac{2\alpha}{\sqrt{5\alpha^2+6\alpha+1}}, \frac{1}{\sqrt{5\alpha^2+6\alpha+1}}\right)$;

c) $\left(\frac{3}{\sqrt{13+\alpha^2}}, \frac{2}{\sqrt{13+\alpha^2}}, \frac{-\alpha}{\sqrt{13+\alpha^2}}\right)$;

d) $\left(\frac{1}{\sqrt{1+17\alpha^2}}, \frac{\alpha}{\sqrt{1+17\alpha^2}}, \frac{4\alpha}{\sqrt{1+17\alpha^2}}\right)$

5. Determine the parametric equation and the Cartesian equation of the line on the space
a) passing through the points A(2,-1,2) and B(4,3,6);
b) passing through the point P(1,-1,0) and parallel to the vector $\vec{v} = (2, 0, 1)$;
c) of Cartesian equation $\begin{cases} y = 2x + 3 \\ x + 2y - z = 0 \end{cases}$.

Solution

a) $\begin{cases} x = 2 + 2t \\ y = -1 + 4t \\ z = 2 + 4t \end{cases}$; $\begin{cases} x = \frac{z-3}{2} \\ y = -3 - 4z \end{cases}$

b) $\begin{cases} x = 1 + 2t \\ y = -1 \\ z = t \end{cases}$; $\begin{cases} x = 1 + 2z \\ y = -1 \end{cases}$

c) $\begin{cases} x = 2 + \frac{4}{3}t \\ y = -1 - \frac{2}{3}t \\ z = t \end{cases}$

6. Determine the reciprocal position of the lines r and s of Cartesian equations

$$r : \begin{cases} x + 3y = 4 \\ 2y + z = 0 \end{cases} \quad ; \quad s : \begin{cases} 3x + 2y = 5 \\ x - 2z = 5 \end{cases} .$$

Solution

The lines are incident.

7. Consider the lines of parametric equations

$$r : \begin{cases} x = 1 + 4t \\ y = -1 + 6t \\ z = 2 - 2t \end{cases} \quad ; \quad s : \begin{cases} x = 3 + 2t \\ y = 1 + 3t \\ z = -2 - t \end{cases} .$$

a) Verify that the lines are parallel;
b) Calculate the distance between r and s;
c) Find the plane π to which both the lines belong (Does the point C(2,0,2) belong to the plane π?).

Solution

a) The lines are on the same plane because the derminant of the matrix defined by a vector that join the two lines and the two direction vectors is zero. Moreover, the cross product of the direction vectors is zero and there is a point on r that not belongs to s. The lines are parallel, then.

b) $\sqrt{10}$
c) $\pi' = ax + by + cz + d = 0$, where $(a, b, c, d) = (a, -\frac{3}{5}a, \frac{a}{5}, 2a)$

8. Given the lines

$$r : \begin{cases} x + y - 2 = 0 \\ x + y - 2z = 0 \end{cases} \quad ; \qquad s : \begin{cases} x - 2y + 1 = 0 \\ 2x - 2z - 2 = 0 \end{cases} \quad ;$$

find the equation of the family of planes parallel to both r and s. Moreover find a plane to which the line r belongs and a plane to which both r and s belong.
Solution
The planes parallel to both r and s are the family $\pi = 2x + 2y - 3z + d = 0$. For $d = -1$ we have a plane to which only the line r belongs. There is not a plane parallel to both lines and to which r and s belong.

9. Establish if the vectors $\vec{v} = (3, 0, 4)$ and $\vec{w} = (6, 0, 8)$in $\mathbb{R}^3$ are linearly dependent.
Solution
Yes, because $\vec{w} = 2\vec{v}$.

10. Detemine for which value of k the following vectors in $\mathbb{R}^4$ are linearly independet

$$\vec{v} = (1, 2, k, 3); \qquad \vec{w} = (k, 2, 3, 1); \qquad \vec{z} = (k, 6, 5, 5).$$

Solution
For each $k \neq -1$

11. Calculate the rank of the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ k & 2k & k \\ 2 & k+2 & -2 \end{pmatrix}$$

Solution
The rank is 3 for each $k \neq 0$.