1 Eigenvalues and Eigenvectors

1.1 Characteristic Polynomial and Characteristic Equation

Procedure. How to find the eigenvalues?

A vector x is an e.vector if x is nonzero and satisfies $Ax = \lambda x$

$\Rightarrow (A - \lambda I)x = 0$ must have nontrivial solutions

$\Rightarrow (A - \lambda I)$ is not invertible by the theorem on properties of determinants

$\Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow$ Solve $\det(A - \lambda I) = 0$ for λ to find eigenvalues.

Definition. $P(\lambda) = \det(A - \lambda I)$ is called *the characteristic polynomial.* $\det(A - \lambda I) = 0$ is called *a characteristic equation.*

Proposition. A scalar λ is an e.v. of a $n \times n$ matrix if λ satisfies $P(\lambda) = \det(A - \lambda I) = 0$.

Example. Find the e.v. of $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$.

Since $A - \lambda I = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{pmatrix}$,

we have the characteristic equation
$\det(A - \lambda I) = -\lambda(5-\lambda) + 6 = (\lambda - 2)(\lambda - 3) = 0$.
So $\lambda = 2$, $\lambda = 3$ are eigenvalues of A.

<u>Theorem.</u> Let A be a $n \times n$ matrix. Then A is invertible if and only if:
a) $\lambda = 0$ is not an e.v. of A;
or
b) $\det A \neq 0$.

<u>Proof.</u> For b) we have discussed the proof on the determinant section.

For a):
$(\Rightarrow)$: Let A be invertible $\Rightarrow \det A \neq 0 \Rightarrow \det(A - 0I) \neq 0$ $\Rightarrow \lambda = 0$ is not an e.v.

$(\Leftarrow)$. Let 0 be not an e.v of A $\Rightarrow \det(A - 0I) \neq 0$ $\Rightarrow \det A \neq 0 \Rightarrow A$ is invertible.

<u>Theorem.</u> The eigenvalues of a triangular matrix are the entries of the main diagonal.

Proof. Recall that a determinant of a triangular matrix is a product of main diagonal elements. Hence, if
$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$
then the characteristic equation is
$$\det(A - \lambda I) = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{pmatrix}$$
$$= (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$$

$\Rightarrow a_{11}, a_{22}, \dots, a_{nn}$ are the eigenvalues of A.

Example. Find the eigenvalues of $A = \begin{pmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{pmatrix}$

Solution. $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{pmatrix}$.
Thus the characteristic equation is $(3-\lambda)(6-\lambda)(2-\lambda) = 0 \Rightarrow$ eigenvalues are 3, 6, 2.

Example. Suppose λ is e.v. of A. Determine an e.v. of A^2 and A^3. What is an e.v. of A^n?

Solution. Since λ is e.v. of A
$\Rightarrow \exists$ nonzero vector x such that $Ax = \lambda x$
$\Rightarrow AAx = A\lambda x = \lambda Ax = \lambda^2 x$.
Therefore λ^2 is an e.v. of A^2.

Analogously for A^3. We have $Ax = \lambda x$ and $A^2 x = \lambda^2 x$
$\Rightarrow AA^2 x = A^3 x = A\lambda^2 x = \lambda^2 Ax = \lambda^3 x$.
Thus λ^3 is an e.v. of A^3.

In general, λ^n is an e.v. of A^n.

1.2 Similar Matrices

Definition. A $n \times n$ matrix B is called similar to matrix A if there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem. If $n \times n$-matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Proof. If $B = P^{-1}AP$, then $B - \lambda I$
$= P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$.

Using the multiplicative property of determinant, we have $\det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$
$= \det P^{-1} \det(A - \lambda I) \det P = \det(A - \lambda I)$.

Hence, matrices A and B have the same e.v.

Theorem. Hamilton-Caley. (Without proof. Try to prove it as an exercise) If $P(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 = 0$ then $P(A) = A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I = 0$.

1.3 Algebraic and Geometric Multiplicity of Eigenvalues

Definition. The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation $(\mathrm{mult}_a(\lambda))$.

Example. Find the polynomial of $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{pmatrix}$ and find e.v. with the algebraic multiplicity.

Solution. The characteristic equation is $\det(A - \lambda I)$

$$= \det \begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 5 & 3-\lambda & 0 & 0 \\ 9 & 1 & 3-\lambda & 0 \\ 1 & 2 & 5 & -1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(3-\lambda)(3-\lambda)(-1-\lambda) = 0$$

Thus the e.v. are $\lambda_1 = 2$, $\lambda_{2,3} = 3$ and $\lambda_4 = -1$.
The algebraic multiplicity of $\lambda = 3$ is 2, or $\mathrm{mult}_a(3) = 2$.

Definition. The eigenspace E_λ consists of the zero vector and all eigenvectors corresponding to an e. v. λ.

Definition. The geometric multiplicity of an e.v. λ is the dimension of the corresponding eigenspace E_λ ($\text{mult}_g(\lambda)$). Recall that the dimension of a vector space is equal to the number of linearly independent vectors it contains.

Example. Find e.v. and their algebraic and geometric multiplicity for $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Solution. The characteristic equation is $\det(A - \lambda I)$ $= \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + 3\lambda + 2 = -(\lambda - 2)(\lambda + 1)^2$.
So the e.v. are $\lambda_1 = 2$, $\lambda_{2,3} = -1$.

Solving the equation $(A - \lambda_i I)x = 0$ for $i = 1, 2, 3$ we find that

$$E_{\lambda=2} = \text{Span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$$

$$E_{\lambda=-1} = \text{Span}\{\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}\}$$

Thus $\text{mult}_a(2)$=$\text{mult}_g(2) = 1$ and $\text{mult}_a(-1)$=$\text{mult}_g(-1) = 2$

Example. Find e.v. and their algebraic and geometric multiplicity for $A = \begin{pmatrix}0 & 0 & 1\\1 & 0 & -3\\0 & 1 & 3\end{pmatrix}$.

Solution. The characteristic equation is $\det(A - \lambda I)$ $= \det\begin{pmatrix}-\lambda & 0 & 1\\1 & -\lambda & -3\\0 & 1 & 3-\lambda\end{pmatrix} = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$.
So the e.v. are $\lambda_{1,2,3} = 1$.

Solving the equation $(A - \lambda_i I)x = 0$ for $i = 1, 2, 3$ we find that

$$E_{\lambda=1} = \text{Span}\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\}$$

Thus $\text{mult}_a(1) = 3$ and $\text{mult}_g(1) = 1$.

Multiplicity Theorem.

For any eigenvalue $\lambda_i, i = 1, 2, \ldots, n$ of a $n \times n$-matrix A holds

$$\text{mult}_g(\lambda) \leq \text{mult}_a(\lambda).$$

Proof. Let λ_i be an eigenvalue of A.

Let $B_{\lambda_i} = \{v_1, \ldots\ v_m\}$ be a basis of the corresponding eigenspace E_{λ_i} where $\text{mult}_g(\lambda_i) = m$.

Note that each v_j in B_{λ_i} is an eigenvector of A corresponding to λ_i.

Thus

$$Av_j = \lambda_i v_j, \quad j = 1, 2, \ldots, m$$

Extend B_{λ_i} to form a basis $B = \{v_1, \ldots\ v_m, v_{m+1}, \ldots, v_n\}$.

Note that B is now a basis in the whole n-dimensional space (R^n or C^n) while B_{λ_i} is only a basis in the eigenspace corresponding to e.v. λ_i.

Note that eigenspace E_{λ_i} is only subspace of the whole n-dimensional space ($E_{\lambda_i} \in R^n$ or $E_{\lambda_i} \in C^n$).

Let $Q = (v_1|v_2|\dots|v_m|v_{m+1}|\dots|v_n)$ be a matrix which columns are vectors $v_1, \dots v_m, v_{m+1}, \dots, v_n$ of the vector basis B.

Since these vectors are linearly independent, the matrix Q is invertible.

Notice that $v_j = Qe_j$ where $e_j = (0, 0, \dots, 0, \overset{j}{1}, 0 \dots, 0)^T$.

Such vector e_j is called a j-th ort.

Now using the definition of e.v.:

$$Q^{-1}Av_j = Q^{-1}\lambda_i v_j = \lambda_i Q^{-1} v_j = \lambda_i e_j, \quad j = 1, 2, \dots, m.$$

Thus $\tilde{A} = Q^{-1}AQ = Q^{-1}A[v_1, \dots v_m, v_{m+1}, \dots, v_n]$

$$= [\lambda_i e_1|\lambda_i e_2|\dots, |\lambda_i e_m|Q^{-1}Av_{m+1}|\dots|Q^{-1}Av_n]]$$

$$= \begin{pmatrix} \lambda_i I_m & C \\ 0 & D \end{pmatrix},$$

where I_m is the $m \times m$-identity matrix.

The matrix $\tilde{A}$ is similar to the matrix A since $\tilde{A} = Q^{-1}AQ$.

Hence using the property of determinant for the block diagonal matrixes (see the assignment 2)

$$P_A(\lambda) = P_{\tilde{A}} = \mathrm{det}(\tilde{\mathrm{A}} - \lambda \mathrm{I_n})$$

$$= \mathrm{det}((\lambda - \lambda_\mathrm{i})\mathrm{I_m})\mathrm{det}(\mathrm{D} - \lambda\mathrm{I_{n-m}}) = (\lambda - \lambda_\mathrm{i})^\mathrm{m}\mathrm{P_D}(\lambda).$$

Here I_n and I_{n-m} are $n \times n$- and $(n-m) \times (n-m)$-identity matrixes respectively.

Thus a characteristic polynomial $P_A(\lambda)$ has a root of λ_i of at least degree m, where $m = \mathrm{mult_g}(\lambda_\mathrm{i})$.

$\Rightarrow \mathrm{mult}_g(\lambda) \leq \mathrm{mult}_a(\lambda)$.

Example. Find e.v. and eigenspace of $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Define the algebraic and geometric multiplicities of e.v.

Solution. Since the matrix A is upper-triangular, its the only e.v. is $\lambda_{1,2,3,4} = 1$.

Thus $\mathrm{mult}_a(\lambda) = 4$.

To find the eigenspace and geometric multiplicity we need to solve the equation $(A - 1I)x = 0$ and find basis for the null space of $(A - 1I)$.

$$A - 1I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the solution is

x_1, x_2 and x_3 are arbitrary numbers, $x_4 = 0$.

Thus we can choose 3 linearly independent vectors, for example,

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}$$

Therefore, $\mathrm{mult}_g(\lambda) = 3$

Notice that the eigenspace E_1 is a 3-dimensional hyperplane in R^4.

Example. Suggest a 4×4-matrix with e.v. $\lambda = 1$ and $\text{mult}_a(1) = 4$ and $\text{mult}_g(1) \neq 3$.

Solution. From the Multiplicity Theorem we have the following options $\text{mult}_g(\lambda) = 1$, $\text{mult}_g(\lambda) = 2$ and $\text{mult}_g(\lambda) = 4$.

From the previous example it is easy to see that

if $A = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

then $\text{mult}_a(1) = 4$ and $A - 1I = 0_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence the solution of $(A - 1I)x = 0$ is

x_1, x_2 x_3 and x_4 are arbitrary numbers.

Thus we can choose 4 linearly independent vectors, for example,

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$$

Therefore, $\text{mult}_g(\lambda) = 4$. Notice that for this case the eigenspace E_1 coincides with R^4.

To get $\text{mult}_g(\lambda) = 2$, it is possible to think backward and choose such a matrix A such

that the null space of $(A - 1I)$ has only 2 linearly independent vectors.

This would imply that $x_3 = 0$ and $x_4 = 0$ while x_1 and x_2 are arbitrary.

For example, $A - 1I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

hence $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Notice that $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 1 \end{pmatrix}$ also works for arbitrary α and β, $\alpha, \beta \neq 0$.

<u>Exercise.</u> Find a 4×4-matrix with e.v. $\lambda = 1$ and $\text{mult}_a(1) = 4$ and $\text{mult}_g(1) = 3$.

1.4 Trace of a Matrix

Definition. The trace of an $n \times n$-matrix A is defined to be $\mathrm{Tr}(A) = \mathrm{Sp}(A) = \sum_{i=1}^{n} a_{ii}$, i.e., the sum of the diagonal elements. (Tr is English, Sp is German from "Spur".)

Properties.

- $\mathrm{Tr}(\mathrm{A}) = \mathrm{Tr}(\mathrm{A}^{\mathrm{T}})$
- $\mathrm{Tr}(\alpha \mathrm{A}) = \alpha \mathrm{Tr}(\mathrm{A})$
- $\mathrm{Tr}(\mathrm{A} + \mathrm{B}) = \mathrm{Tr}(\mathrm{A}) + \mathrm{Tr}(\mathrm{B})$
- $\mathrm{Tr}(\mathrm{AB}) = \mathrm{Tr}(\mathrm{BA})$

Proof as an exercise.

Theorem. Let A be a $n \times n$-matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its e.v.

Then $\mathrm{Tr}(\mathrm{A}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}$ and $\det(\mathrm{A}) = \prod_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}$.

Proof. Let for simplicity assume that A is similar to a diagonal matrix $D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{n}}\}$. Hence $A = P^{-1}DP$.

From the properties of trace

$\mathrm{tr}(\mathrm{A}) = \mathrm{tr}(\mathrm{P}^{-1}\mathrm{DP}) = \mathrm{tr}(\mathrm{PP}^{-1}\mathrm{D}) = \mathrm{tr}(\mathrm{D}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}$.

From the properties of determinants

$\det(\mathrm{A}) = \det(\mathrm{P}^{-1}\mathrm{DP}) = \det(\mathrm{P})\det(\mathrm{D})\det(\mathrm{P}^{-1})$

$= \det(\mathrm{D}) = \prod_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}$.

Example. Find eigenvalues of $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ without calculation.

Solution. Notice that $\det(\text{A}) = \lambda_1\lambda_2 = 0$

and $\text{Tr}(\text{A}) = \lambda_1 + \lambda_2 = 2\text{a}$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 2a$.

1.5 Diagonalization

Definition. The matrix is *diagonal* if all its entries are only on the main diagonal.

Example. $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

If a matrix is diagonal, it is trivial to compute D^k, $\det D$, etc.

Example 1. Let $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Compute D^2, D^3, and D^k, $\forall k \in N$.

Solution. $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$.

Analogously, $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} == \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix}$.

In general, $D^k = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}$.

Example 2. Let $A = PDP^{-1}$, find a formula for A^k, $\forall k \in N$.

Solution. By induction, $A^2 = (PDP^{-1})(PDP^{-1}) = (PD^2P^{-1})$.

If it is true for $(n-1)$, then $A^n = (PDP^{-1})(PD^{(n-1)}P^{-1}) = PD^nP^{-1}$.

Definition. A $n \times n$-matrix is said to be *diagonalizable* if A is similar to a diagonal matrix, i.e. $\exists\ P$ — invertible such that $A = PDP^{-1}$, where D is diagonal.

Theorem*. A $n \times n$-matrix is diagonalizable iff A has n independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, iff the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are e.v. of A, corresponding to eigenvectors-columns of P.

Proof. ($\Rightarrow$): Given $A = PDP^{-1}$. Notice that if P is a $n \times n$-matrix with columns $v_1, \ldots, v_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$\begin{aligned} AP &= A\left[v_1 \mid v_2 \mid \ldots \mid v_n\right] \\ &= \left[Av_1 \mid Av_2 \mid \ldots \mid Av_n\right], \end{aligned}$$

while

$$\begin{aligned} PD &= P\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \\ &= \left[\lambda_1 v_1 \mid \lambda_2 v_2 \mid \ldots \mid \lambda_n v_n\right]. \end{aligned}$$

If $A = PDP^{-1} \Rightarrow PA = PD \Rightarrow$

$$\begin{aligned} &\left[Av_1 \mid Av_2 \mid \ldots \mid Av_n\right] \\ &= \left[\lambda_1 v_1 \mid \lambda_2 v_2 \mid \ldots \mid \lambda_n v_n\right] \Rightarrow \end{aligned}$$

$$Av_1 = \lambda v_1,\ av_2 = \lambda_2 v_2,\ \ldots,\ Av_n = \lambda_n v_n.$$

Since P is invertible, $v_1, v_2, \ldots, v_n$ are linearly independent non-zero vectors.

Hence by definition $\lambda_1, \lambda_2, \ldots, \lambda_n$ are e.v. of A are $v_1, v_2, \ldots, v_n$ are corresponding eigenvectors of A.

($\Leftarrow$): Given n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, use them to construct P and use n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinctive) to construct a diag D.

Then by definition of e.v. $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, $\ldots$, $Av_n = \lambda_n v_n$.

$$\Rightarrow \begin{bmatrix} Av_1 & | & Av_2 & | & \ldots & | & Av_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 v_1 & | & \lambda_2 v_2 & | & \ldots & | & \lambda_n v_n \end{bmatrix}$$

$\Rightarrow AP = PD$. Since P is invertible (all $v_1, v_2, \ldots, v_n$ are linearly independent) $\Rightarrow A = PDP^{-1}$.

Definition. Linearly independent vectors $v_1, v_2, \ldots, v_n$ form an *eigenvector basis* in R^n.

Example. Diagonalize $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ if possible.

1) Find the eigenvalues of A:

$$
\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix} \\
&= (2-\lambda)^2(1-\lambda) = 0.
\end{aligned}
$$

Thus, $\lambda_1 = 1$, $\lambda_{2,3} = 2$ are e.v.

2) Find three linearly independent eigenvectors if possible.

By solving $(A - \lambda_i I)x = 0$, $i = 1, 2, 3$, we get

$$
v_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},
$$

corresponding to $\lambda_1 = 1$, $\lambda_{2,3} = 2$.

Vectors v_1, v_2, v_3 are clearly linearly independent $\Rightarrow$ for a basis.

3) Construct P from v_1, v_2, v_3:

$$
P = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
$$

4) Construct D from the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

5) Check your results by verifying that $AP = PD$:

$$AP = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

$$PD = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

<u>Example.</u> Diagonalize $A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$ if possible.

<u>Solution.</u>

1) Since A is triangular, it is clear that $\lambda_1 = 4$, $\lambda_{2,3} = 2$ are e.v.

2) Solve $(A - \lambda_i I)x = 0$, $i = 1, 2, 3$, and find the eigenbasis.

Eigenvector for $\lambda_1 = 4$ is $v = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$,

Eigenvector for $\lambda_{2,3} = 2$ is $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Thus dimension of eigenspace corresponding to $\lambda_{2,3} = 1$ is 1 ($\text{mult}_g(2) = 1$ and $\text{mult}_a(2) = 1$)

$\Rightarrow P$ is singular (it is not enough eigenvectors to form a basis for R^3) $\Rightarrow A$ is not diagonalizable.

Example. Diagonalize $A = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ if possible.

Solution. Since it is a triangular matrix, the e.v. are $\lambda_1 = 5$, $\lambda_1 = 5$, $\lambda_3 = -2$.

For each $\lambda_1, \lambda_2, \lambda_3$ we can find corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ v_3 = \begin{pmatrix} -\frac{4}{7} \\ \frac{1}{5} \\ 1 \end{pmatrix}.$$

Clearly v_1, v_2, v_3 are linearly independent

$\Rightarrow$ form an eigenvector basis in R^3

$\Rightarrow P = \begin{bmatrix} v_1 & | & v_2 & | & v_3 \end{bmatrix}$ is invertible

$\Rightarrow A$ is diagonalizable and $D = \mathrm{diag}(5, 3, -2)$.

Why $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ $(\lambda_1 = 1,\ \lambda_{2,3} = 2)$ and
$A = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ $(\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = -2)$
are diagonalizable?

<u>Theorem 1.</u> If $v_1, v_2, \ldots, v_r$ are eigenvectors

corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ for a $n \times n$-matrix A, then $v_1, v_2, \ldots, v_n$ are linearly independent.

Proof. From contradiction. Suppose $v_1, v_2, \ldots, v_r$ are linearly dependent.

Let $(p-1)$ be the last index such that v_p is a linear combinations of the preceding linearly independent vectors $(p \leq n)$.

Then there exist $c_1, c_2, \ldots, c_{p-1}$ such that $\exists i$, $i \in N$, $c_i \neq 0$ and $c_1 v_1 + c_2 v_2 + \ldots + c_{p-1} v_{p-1} = v_p$. $(*)$

Multiply both sides of this equation by $A \Rightarrow$
$c_1 A v_1 + c_2 A v_2 + \ldots + c_{p-1} A v_{p-1} = A v_p$.
Using the fact that $A v_i = \lambda_i v_i$, we get
$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_{p-1} \lambda_{p-1} v_{p-1} = \lambda_p v_p$. $(**)$

Multiply $(*)$ by λ_p and subtract result from $(**)$:
$c_1 \lambda_1 v_1 - c_1 \lambda_p v_1 + c_2 \lambda_2 v_2 - c_2 \lambda_p v_2 + \ldots + c_{p-1} \lambda_{p-1} v_{p-1} - c_{p-1} \lambda_p v_{p-1} = \lambda_p v_p - \lambda_p v_p = 0$

$\Rightarrow c_1(\lambda_1 - \lambda_p) v_1 + c_2(\lambda_2 - \lambda_p) v_2 + \ldots + c_{p-1}(\lambda_{p-1} - \lambda_p) v_{p-1} = 0$.

Notice that by construction $v_1, v_2, \ldots, v_{p-1}$ are linearly independent

$\Rightarrow$ by definition of linear independence

$$c_1(\lambda_1 - \lambda_p) = c_2(\lambda_2 - \lambda_p) = \ldots = c_{p-1}(\lambda_{p-1} - \lambda_p) = 0.$$

But $\exists i,\ i \in N : c_i \neq 0 \Rightarrow \exists i,\ i \in N : \lambda_i - \lambda_p = 0 \Rightarrow$ contradiction since by statement of Theorem $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$.

Corollary. If a $n \times n$-matrix A has n distinct eigenvalues, A is disgonalizable.

Proof. If A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$\Rightarrow$ eigenvectors $v_1, v_2, \ldots, v_n$ are linearly independent and form an eigenvector basis

$\Rightarrow p = \begin{pmatrix} v_1 & | & v_2 & | & \ldots & | & v_n \end{pmatrix}$ is invertible and A is diagonalizable.

Diagonalization theorem 2. A $n \times n$-matrix A is diagonalizable iff $\text{mult}_g(\lambda_i)$ $=\text{mult}_a(\lambda_i)$, $\forall i = 1, 2, \ldots, n$, i.e. eigenvectors of A form a basis in R^n.

Proof. ($\Leftarrow$): Let $\text{mult}_g(\lambda_i)$ $=\text{mult}_a(\lambda_i)$, $\forall i = 1, 2, \ldots, n$

$\Rightarrow$ each eigenspace of dimension n_i has n_i linearly independent vectors but

$$n_i = \mathrm{mult_a(\lambda_i)}$$

and

$$\mathrm{mult_a(\lambda_1) + mult_a(\lambda_2) + \ldots + mult_a(\lambda_n) = n}$$

$\Rightarrow n_1 + n_2 + \ldots + n_n = n \Rightarrow$ there are n linearly independent eigenvectors which form an eigenvector basis in $R^n \Rightarrow A$ is diagonalizable by Theorem $(*)$.

$(\Leftarrow)$: Let A be diagonalizable. Then by the Theorem* $\exists n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ which form the matrix P such that $A = P^{-1}DP$ and D is diagonal. Let B be a set of $\{v_1, v_2, \ldots, v_n\}$.

If all e.v. $\lambda_1, \lambda_2, \ldots, \lambda_n$ have algebraic multiplicity 1 ($\mathrm{mult_a\lambda_i = 1}$, $i = 1, 2, \ldots, n$) then clearly

$$\mathrm{mult_a\lambda_i = mult_g\lambda_i = 1, i = 1, 2, \ldots, n}$$

since dimension of any vector space cannot be less than 1 (trivial case).

(General case.) Assume for simplicity that $\exists$ only one eigenvalue λ_k^* such that $\mathrm{mult_a\lambda_k^* = p}$ and all other e.v. $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, $i \neq k$ have single algebraic multiplicity ($\mathrm{mult_a\lambda_i = 1}$).

Since clearly eigenspaces do not intersect, i.e. eigenvector v_j corresponding to the e.v. λ_j **do not** belong to the eigenspace E_{λ_i} corresponding to the e.v. λ_i ($\lambda_j \neq \lambda_j$) (verify it at home)

$\Rightarrow$ the eigenvectors $v_1, v_2, \ldots, v_{n-p}$, $i \neq k$, corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, $i \neq k$, do not belong to $E_{\lambda_k^*}$.

$\Rightarrow$ vectors which belong to the set $B/\{v_1, v_2, \ldots, v_{n-p}\}_{i\neq k}$ belong to the eigenspace $E_{\lambda_k^*}$. Note that these vectors are linearly independent by statement of the Theorem and $\mathrm{dimB/\{v_1, v_2, \ldots, v_{n-p}\}_{i\neq k} = dimE_{\lambda_k^*} = p = mult_a\lambda_k^*}$.

Example. Why is $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ diagonalizable?

Solution. A has 3 disticnt e.v. $\lambda_1 = 2$, $\lambda_2 = 6$, $\lambda_3 = 1$ $\Rightarrow$ A is diagonalizable.

Example. Diagonalize if possible $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 24 & -12 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Solution. A has two e.v. of $\mathrm{mult}_a = 2$: $\lambda_{1,2} = -2$, $\lambda_{3,4} = 2$. Solve $(A - \lambda_i E)x = 0$ and find eigenvectors.

Basis for $\lambda_{1,2} = -2$: $v_1 = \begin{pmatrix} 1 \\ 0 \\ -6 \\ 6 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}$. Thus $\text{mult}_g(-2) = 2$.

Basis for $\lambda_{3,4} = 2$: $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Thus $\text{mult}_g(2) = 2$.

Hence by The Diagonalization Theorem, since

$$\text{mult}_{\text{a}}(-2) = \text{mult}_{\text{g}}(-2) = 2$$

and

$$\text{mult}_{\text{a}}(2) = \text{mult}_{\text{g}}(2) = 2,$$

the matrix A is diagonalizable.