

Linear Transformations



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6.1 Introduction to Linear Transformations

- A function T that maps a vector space V into a vector space W :

$$T : V \xrightarrow{\text{mapping}} W, \quad V, W : \text{vector spaces}$$

V : the domain of T W : the codomain of T

- Image of \mathbf{v} under T :

If \mathbf{v} is a vector in V and \mathbf{w} is a vector in W such that

$$T(\mathbf{v}) = \mathbf{w},$$

then \mathbf{w} is called the image of \mathbf{v} under T

(For each \mathbf{v} , there is only one \mathbf{w})

- The range of T :

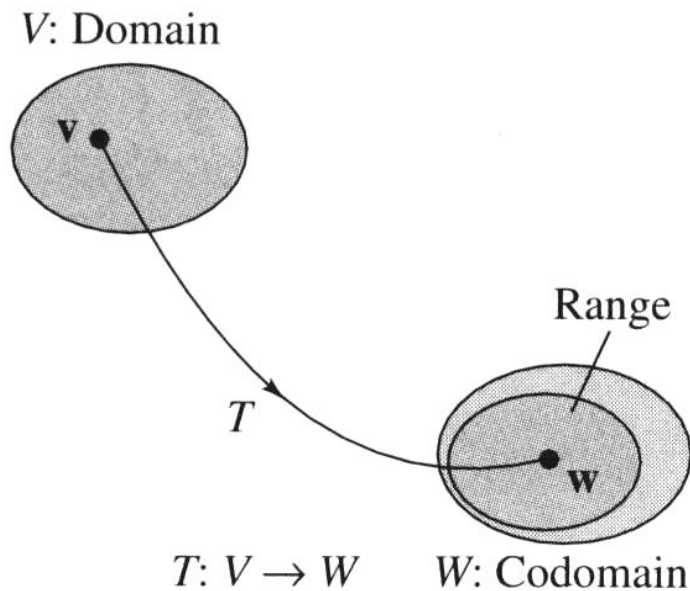
The set of all images of vectors in V (see the figure on the next slide)

- The preimage of \mathbf{w} :

The set of all \mathbf{v} in V such that $T(\mathbf{v})=\mathbf{w}$

(For each \mathbf{w} , \mathbf{v} may not be unique)

- The graphical representations of the domain, codomain, and range



⌘ For example, V is R^3 , W is R^3 , and T is the orthogonal projection of any vector (x, y, z) onto the xy -plane, i.e. $T(x, y, z) = (x, y, 0)$

(we will use the above example many times to explain abstract notions)

⌘ Then the domain is R^3 , the codomain is R^3 , and the range is xy -plane (a subspace of the codomain R^3)

⌘ $(2, 1, 0)$ is the image of $(2, 1, 3)$

⌘ The preimage of $(2, 1, 0)$ is $(2, 1, s)$, where s is any real number

■ **Ex 1: A function from R^2 into R^2**

$$T : R^2 \rightarrow R^2 \quad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\mathbf{v}=(-1,2)$ (b) Find the preimage of $\mathbf{w}=(-1,11)$

Sol:

(a) $\mathbf{v} = (-1, 2)$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3)$$

(b) $T(\mathbf{v}) = \mathbf{w} = (-1, 11)$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$$\Rightarrow v_1 = 3, v_2 = 4 \quad \text{Thus } \{(3, 4)\} \text{ is the preimage of } \mathbf{w}=(-1, 11)$$

- **Linear Transformation :**

V, W : vector spaces

$T : V \rightarrow W$: A linear transformation of V into W if the following two properties are true

$$(1) \ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2) \ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Notes:

- (1) A linear transformation is said to be operation preserving
(because the same result occurs whether the operations of addition
and scalar multiplication are performed before or after the linear
transformation is applied)

$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$		$T(c\mathbf{u}) = cT(\mathbf{u})$	
Addition in V	Addition in W	Scalar multiplication in V	Scalar multiplication in W

- (2) A linear transformation $T : V \rightarrow V$ from a vector space into
itself is called a **linear operator**

■ Ex 2: Verifying a linear transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$: vector in R^2 , c : any real number

(1) Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation

■ **Ex 3: Functions that are not linear transformations**

(a) $f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) \neq \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{3}\right)$$

$(f(x) = \sin x \text{ is not a linear transformation})$

(b) $f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(1+2)^2 \neq 1^2 + 2^2$$

$(f(x) = x^2 \text{ is not a linear transformation})$

(c) $f(x) = x+1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

$(f(x) = x+1 \text{ is not a linear transformation, although it is a linear function})$

In fact, $f(cx) \neq cf(x)$

- Notes: Two uses of the term “linear”.

(1) $f(x) = x + 1$ is called a linear function because its graph is a line

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

- Zero transformation:

$$T: V \rightarrow W \quad T(\mathbf{v}) = \mathbf{0}, \quad \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \rightarrow V \quad T(\mathbf{v}) = \mathbf{v}, \quad \forall \mathbf{v} \in V$$

- Theorem 6.1: Properties of linear transformations

$$T: V \rightarrow W, \quad \mathbf{u}, \mathbf{v} \in V$$

$$(1) T(\mathbf{0}) = \mathbf{0} \quad (T(c\mathbf{v}) = cT(\mathbf{v}) \text{ for } c=0)$$

$$(2) T(-\mathbf{v}) = -T(\mathbf{v}) \quad (T(c\mathbf{v}) = cT(\mathbf{v}) \text{ for } c=-1)$$

$$(3) T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) \quad (T(\mathbf{u} + (-\mathbf{v})) = T(\mathbf{u}) + T(-\mathbf{v}) \text{ and property (2)})$$

$$(4) \text{ If } \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

$$\text{ then } T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n)$$

$$= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n)$$

$$(\text{Iteratively using } T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \text{ and } T(c\mathbf{v}) = cT(\mathbf{v}))$$

■ Ex 4: Linear transformations and bases

Let $T : R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$

$$T(0,1,0) = (1,5,-2)$$

$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\left(\begin{array}{l} \text{According to the fourth property on the previous slide that} \\ T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n) \end{array} \right)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1) \\ &= (7,7,0) \end{aligned}$$

■ **Ex 5: A linear transformation defined by a matrix**

The function $T : R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation from R^2 into R^3

Sol:

(a) $\mathbf{v} = (2, -1)$ R^2 vector R^3 vector

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$$\therefore T(2, -1) = (6, 3, 0)$$

(b) $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$

(scalar multiplication)

- **Theorem 6.2: The linear transformation defined by a matrix**

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m

- **Note:**

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

R^n vector R^m vector

$$T(\mathbf{v}) = A\mathbf{v} \quad T : R^n \longrightarrow R^m$$

- ⊗ If $T(\mathbf{v})$ can be represented by $A\mathbf{v}$, then T is a linear transformation
- ⊗ If the size of A is $m \times n$, then the domain of T is R^n and the codomain of T is R^m

■ Ex 7: Rotation in the plane

Show that the L.T. $T : R^2 \rightarrow R^2$ given by the matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

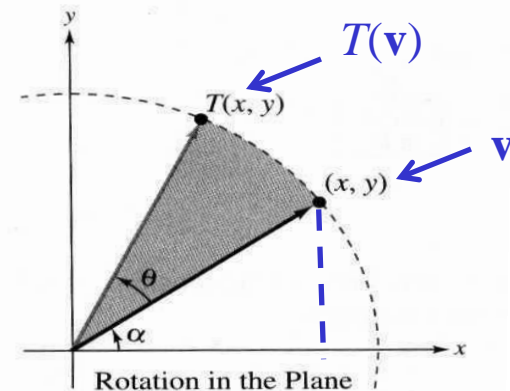
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ

Sol:

$\mathbf{v} = (x, y) = (r \cos \alpha, r \sin \alpha)$ (Polar coordinates: for every point on the xy -plane, it can be represented by a set of (r, α))

r : the length of \mathbf{v} ($= \sqrt{x^2 + y^2}$)

α : the angle from the positive x -axis counterclockwise to the vector \mathbf{v}



$$\begin{aligned}
 T(\mathbf{v}) = A\mathbf{v} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} \\
 &= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix} \\
 &= \begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix}
 \end{aligned}$$

according to the addition
formula of trigonometric
identities

r : remain the same, that means the length of $T(\mathbf{v})$ equals the length of \mathbf{v}

$\theta + \alpha$: the angle from the positive x -axis counterclockwise to the vector $T(\mathbf{v})$

Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector \mathbf{v} counterclockwise through the angle θ

■ Ex 8: A projection in R^3

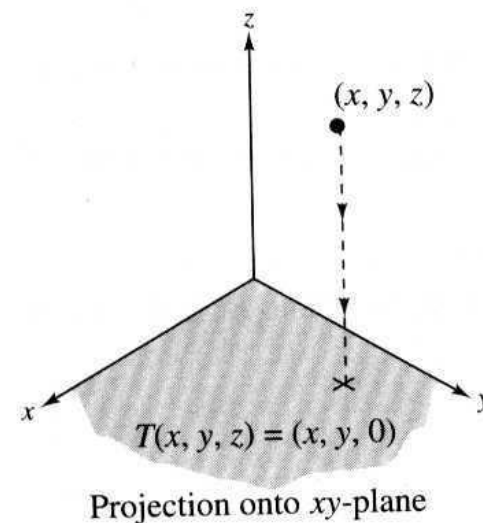
The linear transformation $T : R^3 \rightarrow R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3

$$\text{If } \mathbf{v} \text{ is } (x, y, z), A\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

※ In other words, T maps every vector in R^3 to its orthogonal projection in the xy -plane, as shown in the right figure



Keywords in Section 6.1:

- function
- domain
- codomain
- image of v under T
- range of T
- preimage of w
- linear transformation
- linear operator
- zero transformation
- identity transformation

6.2 The Kernel and Range of a Linear Transformation

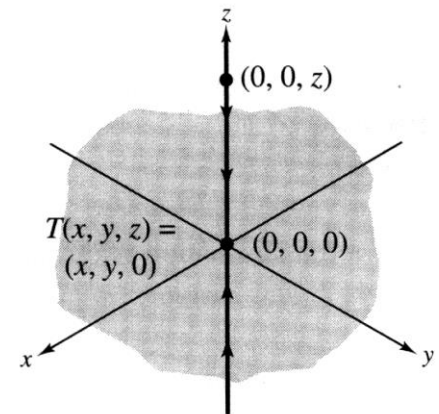
- **Kernel of a linear transformation T :**

Let $T : V \rightarrow W$ be a linear transformation. Then the set of all vectors \mathbf{v} in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the kernel of T and is denoted by $\ker(T)$

$$\ker(T) = \{ \mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V \}$$

- ⌘ For example, V is R^3 , W is R^3 , and T is the orthogonal projection of any vector (x, y, z) onto the xy -plane, i.e. $T(x, y, z) = (x, y, 0)$
- ⌘ Then the kernel of T is the set consisting of $(0, 0, s)$, where s is a real number, i.e.

$$\ker(T) = \{ (0, 0, s) \mid s \text{ is a real number} \}$$



The kernel of T is the set of all vectors on the z -axis.

- Ex 2: The kernel of the zero and identity transformations

(a) If $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T : V \rightarrow W$), then

$$\ker(T) = V$$

(b) If $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T : V \rightarrow V$), then

$$\ker(T) = \{\mathbf{0}\}$$

- Ex 5: Finding the kernel of a linear transformation

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (T : R^3 \rightarrow R^2)$$
$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0, 0), \text{ and } (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0, 0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \ker(T) &= \{t(1, -1, 1) \mid t \text{ is a real number}\} \\ &= \text{span}\{(1, -1, 1)\} \end{aligned}$$

- **Theorem 6.3: The kernel is a subspace of V**

The kernel of a linear transformation $T : V \rightarrow W$ is a subspace of the domain V

Pf:

$\because T(\mathbf{0}) = \mathbf{0}$ (by Theorem 6.1) $\therefore \ker(T)$ is a nonempty subset of V

Let \mathbf{u} and \mathbf{v} be vectors in the kernel of T . Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0} && (\mathbf{u} \in \ker(T), \mathbf{v} \in \ker(T) \Rightarrow \mathbf{u} + \mathbf{v} \in \ker(T)) \\ &\quad \uparrow \text{ } \longrightarrow T \text{ is a linear transformation} \\ T(c\mathbf{u}) &= cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0} && (\mathbf{u} \in \ker(T) \Rightarrow c\mathbf{u} \in \ker(T)) \end{aligned}$$

Thus, $\ker(T)$ is a subspace of V (according to Theorem 4.5 that a nonempty subset of V is a subspace of V if it is closed under vector addition and scalar multiplication)

- Ex 6: Finding a basis for the kernel

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where \mathbf{x} is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for $\ker(T)$ as a subspace of R^5

Sol:

To find $\ker(T)$ means to find all \mathbf{x} satisfying $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$.

Thus we need to form the augmented matrix $[A \mid \mathbf{0}]$ first

$$[A \mid \mathbf{0}] =$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

s

t

$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$: one basis for the kernel of T

- Corollary to Theorem 6.3:

Let $T : R^n \rightarrow R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

Then the kernel of T is equal to the solution space of $A\mathbf{x} = \mathbf{0}$

$$T(\mathbf{x}) = A\mathbf{x} \quad (\text{a linear transformation } T : R^n \rightarrow R^m)$$

$$\Rightarrow \ker(T) = NS(A) = \left\{ \mathbf{x} \mid A\mathbf{x} = \mathbf{0}, \forall \mathbf{x} \in R^n \right\} \quad (\text{subspace of } R^n)$$

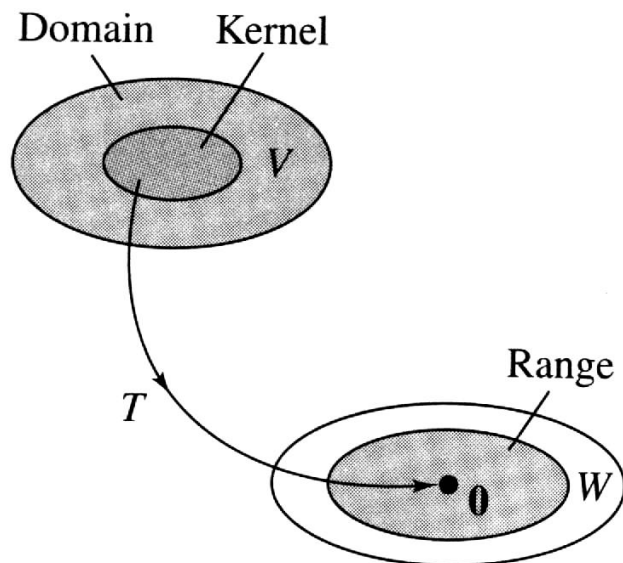
※ The kernel of T equals the nullspace of A (which is defined in Theorem 4.16 on p.239) and these two are both subspaces of R^n)

※ So, the kernel of T is sometimes called the nullspace of T

■ Range of a linear transformation T :

Let $T : V \rightarrow W$ be a linear transformation. Then the set of all vectors \mathbf{w} in W that are images of any vectors in V is called the range of T and is denoted by $\text{range}(T)$

$$\text{range}(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$



- ⌘ For the orthogonal projection of any vector (x, y, z) onto the xy -plane, i.e. $T(x, y, z) = (x, y, 0)$
- ⌘ The domain is $V = \mathbb{R}^3$, the codomain is $W = \mathbb{R}^3$, and the range is xy -plane (a subspace of the codomain \mathbb{R}^3)
- ⌘ Since $T(0, 0, s) = (0, 0, 0) = \mathbf{0}$, the kernel of T is the set consisting of $(0, 0, s)$, where s is a real number

- Theorem 6.4: The range of T is a subspace of W

The range of a linear transformation $T : V \rightarrow W$ is a subspace of W

Pf:

$$\because T(\mathbf{0}) = \mathbf{0} \quad (\text{Theorem 6.1})$$

$$\therefore \text{range}(T) \text{ is a nonempty subset of } W$$

Since $T(\mathbf{u})$ and $T(\mathbf{v})$ are vectors in $\text{range}(T)$, and we have

$$\begin{array}{lcl}
 & \text{because } \mathbf{u} + \mathbf{v} \in V & \\
 T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}) \in \text{range}(T) & \uparrow & \left(\begin{array}{l} \text{Range of } T \text{ is closed under vector addition} \\ \text{because } T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{u} + \mathbf{v}) \in \text{range}(T) \end{array} \right) \\
 \text{ } & \text{ } & \\
 \text{ } & \text{ } & \\
 cT(\mathbf{u}) = T(c\mathbf{u}) \in \text{range}(T) & \downarrow & \left(\begin{array}{l} \text{Range of } T \text{ is closed under scalar multi-} \\ \text{plication because } T(\mathbf{u}) \text{ and } T(c\mathbf{u}) \in \text{range}(T) \end{array} \right) \\
 \text{ } & \text{because } c\mathbf{u} \in V &
 \end{array}$$

Thus, $\text{range}(T)$ is a subspace of W (according to Theorem 4.5 that a nonempty subset of W is a subspace of W if it is closed under vector addition and scalar multiplication)

- Notes:

$T : V \rightarrow W$ is a linear transformation

(1) $\ker(T)$ is subspace of V (Theorem 6.3)

(2) $\text{range}(T)$ is subspace of W (Theorem 6.4)

- **Corollary to Theorem 6.4:**

Let $T : R^n \rightarrow R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

The range of T is equal to the column space of A , i.e. $\text{range}(T) = CS(A)$

- (1) According to the definition of the range of $T(\mathbf{x}) = A\mathbf{x}$, we know that the range of T consists of all vectors \mathbf{b} satisfying $A\mathbf{x}=\mathbf{b}$, which is equivalent to find all vectors \mathbf{b} such that the system $A\mathbf{x}=\mathbf{b}$ is consistent
- (2) $A\mathbf{x}=\mathbf{b}$ can be rewritten as

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}$$

Therefore, the system $A\mathbf{x}=\mathbf{b}$ is consistent iff we can find (x_1, x_2, \dots, x_n) such that \mathbf{b} is a linear combination of the column vectors of A , i.e. $\mathbf{b} \in CS(A)$

Thus, we can conclude that the range consists of all vectors \mathbf{b} , which is a linear combination of the column vectors of A or said $\mathbf{b} \in CS(A)$. So, the column space of the matrix A is the same as the range of T , i.e. $\text{range}(T) = CS(A)$

■ Use our example to illustrate the corollary to Theorem 6.4:

- ⊗ For the orthogonal projection of any vector (x, y, z) onto the xy -plane, i.e. $T(x, y, z) = (x, y, 0)$
- ⊗ According to the above analysis, we already knew that the range of T is the xy -plane, i.e. $\text{range}(T) = \{(x, y, 0) \mid x \text{ and } y \text{ are real numbers}\}$
- ⊗ T can be defined by a matrix A as follows

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ such that } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

- ⊗ The column space of A is as follows, which is just the xy -plane

$$CS(A) = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \text{ where } x_1, x_2 \in R$$

- Ex 7: Finding a basis for the range of a linear transformation

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where \mathbf{x} is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T

Sol:

Since $\text{range}(T) = CS(A)$, finding a basis for the range of T is equivalent to finding a basis for the column space of A

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} \textcircled{1} & 0 & 2 & 0 & -1 \\ 0 & \textcircled{1} & -1 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5$
 $w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5$

$\Rightarrow w_1, w_2,$ and w_4 are independent, so $\{w_1, w_2, w_4\}$ can form a basis for $CS(B)$

\therefore Row operations will not affect the dependency among columns

$\therefore c_1, c_2,$ and c_4 are independent, and thus $\{c_1, c_2, c_4\}$ is a basis for $CS(A)$

That is, $\{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T

- Rank of a linear transformation $T:V \rightarrow W$:

$\text{rank}(T) = \text{the dimension of the range of } T = \dim(\text{range}(T))$

According to the corollary to Thm. 6.4, $\text{range}(T) = \text{CS}(A)$, so $\dim(\text{range}(T)) = \dim(\text{CS}(A))$

- Nullity of a linear transformation $T:V \rightarrow W$:

$\text{nullity}(T) = \text{the dimension of the kernel of } T = \dim(\ker(T))$

According to the corollary to Thm. 6.3, $\ker(T) = \text{NS}(A)$, so $\dim(\ker(T)) = \dim(\text{NS}(A))$

- Note:

If $T : R^n \rightarrow R^m$ is a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$\text{rank}(T) = \dim(\text{range}(T)) = \dim(\text{CS}(A)) = \text{rank}(A)$$

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{NS}(A)) = \text{nullity}(A)$$

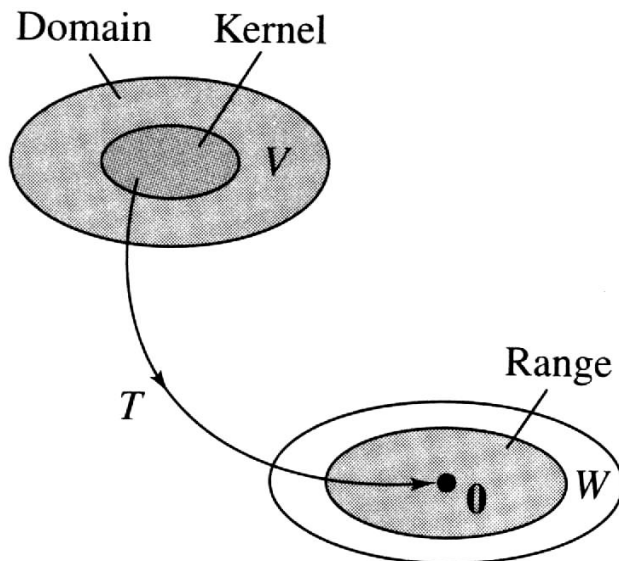
- ⊗ The dimension of the row (or column) space of a matrix A is called the rank of A
- ⊗ The dimension of the nullspace of A ($\text{NS}(A) = \{\mathbf{x} \mid A\mathbf{x} = 0\}$) is called the nullity of A

■ Theorem 6.5: Sum of rank and nullity

Let $T: V \rightarrow W$ be a linear transformation from an n -dimensional vector space V (i.e. the $\dim(\text{domain of } T)$ is n) into a vector space W . Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

(i.e. $\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$)



- ※ You can image that the $\dim(\text{domain of } T)$ should equals the $\dim(\text{range of } T)$ originally
- ※ But some dimensions of the domain of T is absorbed by the zero vector in W
- ※ So the $\dim(\text{range of } T)$ is smaller than the $\dim(\text{domain of } T)$ by the number of how many dimensions of the domain of T are absorbed by the zero vector, which is exactly the $\dim(\text{kernel of } T)$

Pf:

Let T be represented by an $m \times n$ matrix A , and assume $\text{rank}(A) = r$

$$(1) \text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space of } A) = \text{rank}(A) = r$$

$$(2) \text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{null space of } A) = n - r$$

$$\Rightarrow \text{rank}(T) + \text{nullity}(T) = r + (n - r) = n$$

↑
according to Thm. 4.17 where
 $\text{rank}(A) + \text{nullity}(A) = n$

※ Here we only consider that T is represented by an $m \times n$ matrix A . In the next section, we will prove that any linear transformation from an n -dimensional space to an m -dimensional space can be represented by $m \times n$ matrix

- Ex 8: Finding the rank and nullity of a linear transformation

Find the rank and nullity of the linear transformation $T : R^3 \rightarrow R^3$ define by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

✂ The rank is determined by the number of leading 1's, and the nullity by the number of free variables (columns without leading 1's)

■ Ex 9: Finding the rank and nullity of a linear transformation

Let $T : R^5 \rightarrow R^7$ be a linear transformation

- (a) Find the dimension of the kernel of T if the dimension of the range of T is 2
- (b) Find the rank of T if the nullity of T is 4
- (c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Sol:

(a) $\dim(\text{domain of } T) = n = 5$

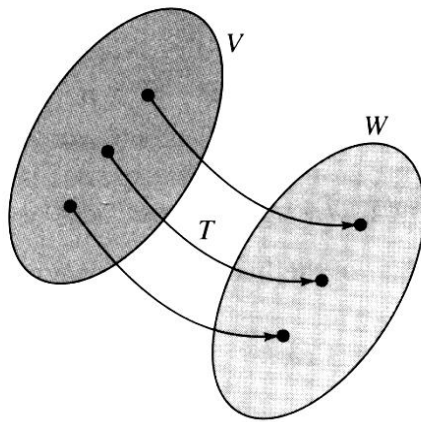
$$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$$

(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

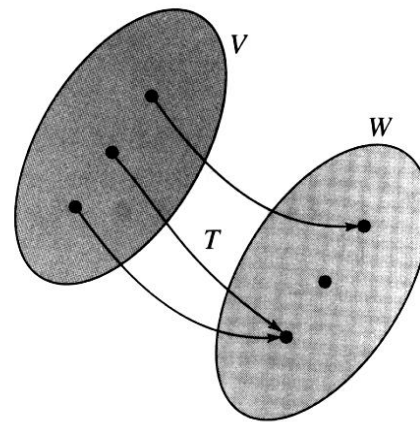
(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

- **One-to-one:**

A function $T : V \rightarrow W$ is called one-to-one if the preimage of every \mathbf{w} in the range consists of a single vector. This is equivalent to saying that T is one-to-one iff for all \mathbf{u} and \mathbf{v} in V , $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$



one-to-one



not one-to-one

- Theorem 6.6: One-to-one linear transformation

Let $T : V \rightarrow W$ be a linear transformation. Then

T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

Pf:

(\Rightarrow) Suppose T is one-to-one

Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution : $\mathbf{v} = \mathbf{0}$

i.e. $\ker(T) = \{\mathbf{0}\}$

Due to the fact
that $T(\mathbf{0}) = \mathbf{0}$ in
Thm. 6.1

(\Leftarrow) Suppose $\ker(T) = \{\mathbf{0}\}$ and $T(\mathbf{u}) = T(\mathbf{v})$

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$$

↑ T is a linear transformation, see Property 3 in Thm. 6.1

$$\therefore \mathbf{u} - \mathbf{v} \in \ker(T) \Rightarrow \mathbf{u} - \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{v}$$

$\Rightarrow T$ is one-to-one (because $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$) 6.40

- Ex 10: One-to-one and not one-to-one linear transformation

(a) The linear transformation $T : M_{m \times n} \rightarrow M_{n \times m}$ given by $T(A) = A^T$ is one-to-one

because its kernel consists of only the $m \times n$ zero matrix

(b) The zero transformation $T : R^3 \rightarrow R^3$ is not one-to-one

because its kernel is all of R^3

- **Onto:**

A function $T : V \rightarrow W$ is said to be onto if every element in W has a preimage in V

(T is onto W when W is equal to the range of T)

- **Theorem 6.7: Onto linear transformations**

Let $T: V \rightarrow W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the rank of T is equal to the dimension of W

$$\text{rank}(T) = \dim(\text{range of } T) = \dim(W)$$



The definition of
the rank of a linear
transformation



The definition
of onto linear
transformations

- Theorem 6.8: One-to-one and onto linear transformations

Let $T : V \rightarrow W$ be a linear transformation with vector space V and W both of dimension n . Then T is one-to-one if and only if it is onto

Pf:

(\Rightarrow) If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$ and $\dim(\ker(T)) = 0$

$$\dim(\text{range}(T)) \stackrel{\text{Thm.6.5}}{=} n - \dim(\ker(T)) = n = \dim(W)$$

Consequently, T is onto

According to the definition of dimension (on p.227) that if a vector space V consists of the zero vector alone, the dimension of V is defined as zero

(\Leftarrow) If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$$\dim(\ker(T)) \stackrel{\text{Thm.6.5}}{=} n - \dim(\text{range of } T) = n - n = 0 \Rightarrow \ker(T) = \{\mathbf{0}\}$$

Therefore, T is one-to-one

■ **Ex 11:**

The linear transformation $T : R^n \rightarrow R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither

$$(a) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$(d) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

	$= \dim(R^n)$ $= n$	$= \dim(\text{range of } T)$ $= \# \text{ of leading 1's}$	$= (1) - (2) = \dim(\ker(T))$	If nullity(T) $= \dim(\ker(T)) = 0$	If rank(T) $= \dim(R^m) = m$
$T: R^n \rightarrow R^m$	\downarrow dim(domain of T) (1)	\downarrow rank(T) (2)	\downarrow nullity(T)	\downarrow 1-1	\downarrow onto
(a) $T: R^3 \rightarrow R^3$	3	3	0	Yes	Yes
(b) $T: R^2 \rightarrow R^3$	2	2	0	Yes	No
(c) $T: R^3 \rightarrow R^2$	3	2	1	No	Yes
(d) $T: R^3 \rightarrow R^3$	3	2	1	No	No

- **Isomorphism :**

A linear transformation $T : V \rightarrow W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W , then V and W are said to be isomorphic to each other

- **Theorem 6.9: Isomorphic spaces and dimension**

Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension

Pf:

(\Rightarrow) Assume that V is isomorphic to W , where V has dimension n

\Rightarrow There exists a L.T. $T : V \rightarrow W$ that is one to one and onto

$\because T$ is one-to-one

$$\Rightarrow \dim(\ker(T)) = 0 \qquad \dim(V) = n$$

||

$$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(\ker(T)) = n - 0 = n_{6.45}$$

$\because T$ is onto

$$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$$

$$\text{Thus } \dim(V) = \dim(W) = n$$

(\Leftarrow) Assume that V and W both have dimension n

Let $B = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \}$ be a basis of V and

$B' = \{ \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \}$ be a basis of W

Then an arbitrary vector in V can be represented as

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n$$

and you can define a L.T. $T : V \rightarrow W$ as follows

$$\mathbf{w} = T(\mathbf{v}) = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + \dots + c_n \mathbf{w}_n \text{ (by defining } T(\mathbf{v}_i) = \mathbf{w}_i \text{)}$$

Since B' is a basis for V , $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is linearly independent, and the only solution for $\mathbf{w}=\mathbf{0}$ is $c_1=c_2=\dots=c_n=0$. So with $\mathbf{w}=\mathbf{0}$, the corresponding \mathbf{v} is $\mathbf{0}$, i.e., $\ker(T) = \{\mathbf{0}\}$.

$\Rightarrow T$ is one - to - one

By Theorem 6.5, we can derive that $\dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(\ker(T)) = n - 0 = n = \dim(W)$.

$\Rightarrow T$ is onto

Since this linear transformation is both one-to-one and onto, then V and W are isomorphic.

- Note

Theorem 6.9 tells us that every vector space with dimension n is isomorphic to R^n

- Ex 12: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other

(a) $R^4 = 4$ -space

(b) $M_{4 \times 1} =$ space of all 4×1 matrices

(c) $M_{2 \times 2} =$ space of all 2×2 matrices

(d) $P_3(x) =$ space of all polynomials of degree 3 or less

(e) $V = \{(x_1, x_2, x_3, x_4, 0), x_i \text{ are real numbers}\}$ (a subspace of R^5)

Keywords in Section 6.2:

- kernel of a linear transformation T
- range of a linear transformation T
- rank of a linear transformation T
- nullity of a linear transformation T
- one-to-one
- onto
- Isomorphism (one-to-one and onto)
- isomorphic space

6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T:R^3\rightarrow R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
 - It is simpler to write
 - It is simpler to read
 - It is more easily adapted for computer use

- **Theorem 6.10: Standard matrix for a linear transformation**

Let $T : R^n \rightarrow R^m$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a standard basis for R^n . Then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$,

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every \mathbf{v} in R^n , A is called the Standard matrix for T

Pf:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n$$

$$\begin{aligned} T \text{ is a linear transformation} &\Rightarrow T(\mathbf{v}) = T(v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n) \\ &= T(v_1 \mathbf{e}_1) + T(v_2 \mathbf{e}_2) + \cdots + T(v_n \mathbf{e}_n) \\ &= v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + \cdots + v_n T(\mathbf{e}_n) \end{aligned}$$

If $A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \cdots \ T(\mathbf{e}_n)]$, then

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$\begin{aligned}
&= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\
&= v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + \cdots + v_n T(\mathbf{e}_n)
\end{aligned}$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each \mathbf{v} in R^n

- **Note**

Theorem 6.10 tells us that once we know the image of every vector in the standard basis (that is $T(\mathbf{e}_i)$), you can use the properties of linear transformations to determine $T(\mathbf{v})$ for any \mathbf{v} in V

- **Ex 1: Finding the standard matrix of a linear transformation**

Find the standard matrix for the L.T. $T : R^3 \rightarrow R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation

$$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 2)$$

$$T(\mathbf{e}_2) = T(0, 1, 0) = (-2, 1)$$

$$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0)$$

Matrix Notation

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

■ Check:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix}$$

i.e., $T(x, y, z) = (x - 2y, 2x + y)$

■ Note: a more direct way to construct the standard matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} \leftarrow 1x - 2y + 0z \\ \leftarrow 2x + 1y + 0z \end{array}$$

※ The first (second) row actually represents the linear transformation function to generate the first (second) component of the target vector

- **Ex 2: Finding the standard matrix of a linear transformation**

The linear transformation $T : R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x - axis. Find the standard matrix for T

Sol:

$$T(x, y) = (x, 0)$$

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2)] = [T(1, 0) \ T(0, 1)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

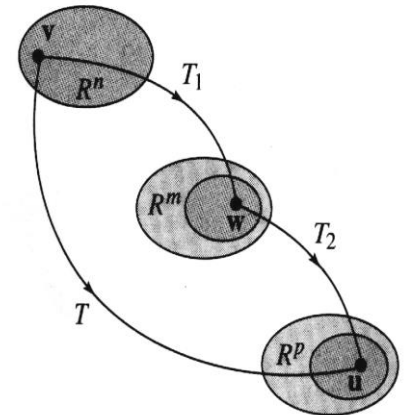
- **Notes:**

- (1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix
- (2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n

- Composition of $T_1: R^n \rightarrow R^m$ with $T_2: R^m \rightarrow R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

This composition is denoted by $T = T_2 \circ T_1$



Composition of Transformations

- Theorem 6.11: Composition of linear transformations

Let $T_1 : R^n \rightarrow R^m$ and $T_2 : R^m \rightarrow R^p$ be linear transformations with standard matrices A_1 and A_2 , then

- (1) The composition $T : R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is still a linear transformation
- (2) The standard matrix A for T is given by the matrix product

$$A = A_2 A_1$$

Pf:

(1) (T is a linear transformation)

Let \mathbf{u} and \mathbf{v} be vectors in R^n and let c be any scalar. Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T_2(T_1(\mathbf{u} + \mathbf{v})) = T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2) (A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

■ **Note:**

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

■ **Ex 3: The standard matrix of a composition**

Let T_1 and T_2 be linear transformations from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$
and $T' = T_1 \circ T_2$

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (\text{standard matrix for } T_1)$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{standard matrix for } T_2)$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

If $T_1 : R^n \rightarrow R^n$ and $T_2 : R^n \rightarrow R^n$ are L.T. s.t. for every \mathbf{v} in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}

- **Theorem 6.12: Existence of an inverse transformation**

Let $T : R^n \rightarrow R^n$ be a linear transformation with standard matrix A ,
Then the following condition are equivalent

(1) T is invertible

(2) T is an isomorphism

(3) A is invertible

※ For (2) \Rightarrow (1), you can imagine that since T is one-to-one and onto, for every \mathbf{w} in the codomain of T , there is only one preimage \mathbf{v} , which implies that $T^{-1}(\mathbf{w}) = \mathbf{v}$ can be a L.T. and well-defined, so we can infer that T is invertible

※ On the contrary, if there are many preimages for each \mathbf{w} , it is impossible to find a L.T to represent T^{-1} (because for a L.T, there is always one input and one output), so T cannot be invertible

- **Note:**

If T is invertible with standard matrix A , then the standard matrix for T^{-1} is A^{-1}

- Ex 4: Finding the inverse of a linear transformation

The linear transformation $T : R^3 \rightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{array}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array} \right] = [I \mid A^{-1}]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words,

$$T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$$

✂ Check $T^{-1}(T(2, 3, 4)) = T^{-1}(17, 19, 20) = (2, 3, 4)$

- The matrix of T relative to the bases B and B' :

$T: V \rightarrow W$ (a linear transformation)

$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ (a nonstandard basis for V)

The coordinate matrix of any \mathbf{v} relative to B is denoted by $[\mathbf{v}]_B$

$$\left(\begin{array}{l} \text{if } \mathbf{v} \text{ can be represented as } c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n, \text{ then } [\mathbf{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \end{array} \right)$$

A matrix A can represent T if the result of A multiplied by a coordinate matrix of \mathbf{v} relative to B is a coordinate matrix of $T(\mathbf{v})$ relative to B' , where B' is a basis for W . That is,

$$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B,$$

where A is called the matrix of T relative to the bases B and B'

- Transformation matrix for nonstandard bases (the generalization of Theorem 6.10, in which standard bases are considered) :

Let V and W be finite - dimensional vector spaces with bases B and B' , respectively, where $B = \{ \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \}$

If $T : V \rightarrow W$ is a linear transformation s.t.

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \quad [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{B'} & [T(\mathbf{v}_2)]_{B'} & \cdots & [T(\mathbf{v}_n)]_{B'} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for every \mathbf{v} in V

- ✂ The above result state that the coordinate of $T(\mathbf{v})$ relative to the basis B' equals the multiplication of A defined above and the coordinate of \mathbf{v} relative to the basis B .
- ✂ Comparing to the result in Thm. 6.10 ($T(\mathbf{v}) = A\mathbf{v}$), it can infer that the linear transformation and the basis change can be achieved in one step through multiplying the matrix A defined above (see the figure on 6.74 for illustration)

- Ex 5: Finding a matrix relative to nonstandard bases

Let $T : R^2 \rightarrow R^2$ be a linear transformation defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis $B = \{(1, 2), (-1, 1)\}$
and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$[T(1, 2)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad [T(-1, 1)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = [[T(1, 2)]_{B'}, [T(-1, 1)]_{B'}] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

■ **Ex 6:**

For the L.T. $T : R^2 \rightarrow R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1)$$

$$B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3)$$

$$B' = \{(1, 0), (0, 1)\}$$

■ **Check:**

$$T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$$

- Notes:

(1) In the special case where $V = W$ (i.e., $T: V \rightarrow V$) and $B = B'$
The matrix A is called the matrix of T relative to the basis B

(2) If $T: V \rightarrow V$ is the identity transformation

$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$: a basis for V

\Rightarrow the matrix of T relative to the basis B

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_B & [T(\mathbf{v}_2)]_B & \cdots & [T(\mathbf{v}_n)]_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

Keywords in Section 6.3:

- standard matrix for T
- composition of linear transformations
- inverse linear transformation
- matrix of T relative to the bases B and B'
- matrix of T relative to the basis B :

6.4 Transition Matrices and Similarity

$T : V \rightarrow V$ (a linear transformation)

$B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ (a basis of V)

$B' = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ (a basis of V)

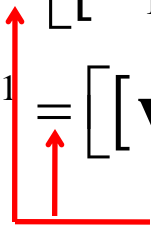
$A = \begin{bmatrix} [T(\mathbf{v}_1)]_B & [T(\mathbf{v}_2)]_B & \cdots & [T(\mathbf{v}_n)]_B \end{bmatrix}$ (matrix of T relative to B)

$A' = \begin{bmatrix} [T(\mathbf{w}_1)]_{B'} & [T(\mathbf{w}_2)]_{B'} & \cdots & [T(\mathbf{w}_n)]_{B'} \end{bmatrix}$ (matrix of T relative to B')

$$\therefore [T(\mathbf{v})]_B = A[\mathbf{v}]_B, \text{ and } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$

$P = \begin{bmatrix} [\mathbf{w}_1]_B & [\mathbf{w}_2]_B & \cdots & [\mathbf{w}_n]_B \end{bmatrix}$ (transition matrix from B' to B)

$P^{-1} = \begin{bmatrix} [\mathbf{v}_1]_{B'} & [\mathbf{v}_2]_{B'} & \cdots & [\mathbf{v}_n]_{B'} \end{bmatrix}$ (transition matrix from B to B')



from the definition of the transition matrix on p.254 in the text book or on Slide 4.108 and 4.109 in the lecture notes

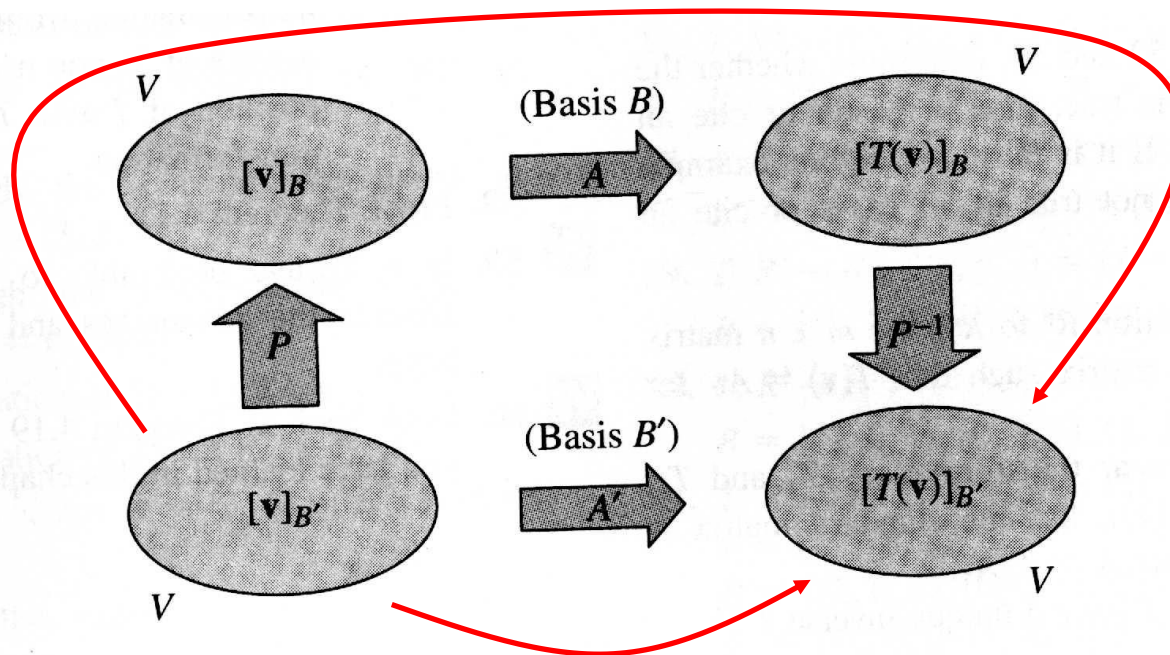
$$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \text{ and } [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

(1) (direct): $A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$

(2) (indirect): $P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'} \Rightarrow A' = P^{-1}AP$

indirect



direct

■ Ex 1: (Finding a matrix for a linear transformation)

Find the matrix A' for $T: R^2 \rightarrow R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

relative to the basis $B' = \{(1, 0), (1, 1)\}$

Sol:

$$(1) A' = \left[[T(1, 0)]_{B'} \quad [T(1, 1)]_{B'} \right]$$

$$T(1, 0) = (2, -1) = 3(1, 0) - 1(1, 1) \Rightarrow [T(1, 0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1, 1) = (0, 2) = -2(1, 0) + 2(1, 1) \Rightarrow [T(1, 1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1, 0)]_{B'} \quad [T(1, 1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(2) standard matrix for T (matrix of T relative to $B = \{(1, 0), (0, 1)\}$)

$$A = [T(1, 0) \quad T(0, 1)] = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$P = \left[\begin{bmatrix} (1, 0) \end{bmatrix}_B \quad \begin{bmatrix} (1, 1) \end{bmatrix}_B \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

※ Solve $a(1, 0) + b(0, 1) = (1, 0) \Rightarrow (a, b) = (1, 0)$
※ Solve $c(1, 0) + d(0, 1) = (1, 1) \Rightarrow (c, d) = (1, 1)$

transition matrix from B to B'

$$P^{-1} = \left[\begin{bmatrix} (1, 0) \end{bmatrix}_{B'} \quad \begin{bmatrix} (0, 1) \end{bmatrix}_{B'} \right] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

※ Solve $a(1, 0) + b(1, 1) = (1, 0) \Rightarrow (a, b) = (1, 0)$
※ Solve $c(1, 0) + d(1, 1) = (0, 1) \Rightarrow (c, d) = (-1, 1)$

matrix of T relative B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- **Ex 2: (Finding a matrix for a linear transformation)**

Let $B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ be basis for

\mathbb{R}^2 , and let $A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ be the matrix for $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ relative to B .

Find the matrix of T relative to B'

Sol:

Because the specific function is unknown, it is difficult to apply the direct method to derive A' , so we resort to the indirect method where $A' = P^{-1}AP$

transition matrix from B' to B : $P = \begin{bmatrix} [(-1, 2)]_B & [(2, -2)]_B \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

transition matrix from B to B' : $P^{-1} = \begin{bmatrix} [(-3, 2)]_{B'} & [(4, -2)]_{B'} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

matrix of T relative to B' :

$$A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

■ **Ex 3: (Finding a matrix for a linear transformation)**

For the linear transformation $T : R^2 \rightarrow R^2$ given in Ex.2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$, for the vector \mathbf{v} whose coordinate matrix is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$\text{or } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

-
- **Similar matrix :**
 - For square matrices A and A' of order n , A' is said to be similar to A if there exist an invertible matrix P s.t. $A' = P^{-1}AP$

- **Theorem 6.13: (Properties of similar matrices)**

Let A , B , and C be square matrices of order n .

Then the following properties are true.

(1) A is similar to A

(2) If A is similar to B , then B is similar to A

(3) If A is similar to B and B is similar to C , then A is similar to C

Pf for (1) and (2) (the proof of (3) is left in Exercise 23):

(1) $A = I_n A I_n$ (the transition matrix from A to A is the I_n)

(2) $A = P^{-1}BP \Rightarrow PAP^{-1} = P(P^{-1}BP)P^{-1} \Rightarrow PAP^{-1} = B$

$\Rightarrow Q^{-1}AQ = B$ (by defining $Q = P^{-1}$), thus B is similar to A

■ Ex 4: (Similar matrices)

(a) $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$ and $A' = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$ are similar

because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ and $A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ are similar

because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

■ Ex 5: (A comparison of two matrices for a linear transformation)

Suppose $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is the matrix for $T : R^3 \rightarrow R^3$ relative

to the standard basis. Find the matrix for T relative to the basis

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

Sol:

The transition matrix from B' to the standard matrix

$$P = \left[\begin{bmatrix} (1, 1, 0) \end{bmatrix}_B \quad \begin{bmatrix} (1, -1, 0) \end{bmatrix}_B \quad \begin{bmatrix} (0, 0, 1) \end{bmatrix}_B \right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix of T relative to B' :

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad (A' \text{ is a diagonal matrix, which is simple and with some computational advantages})$$

✂ You have seen that the standard matrix for a linear transformation $T: V \rightarrow V$ depends on the basis used for V . What choice of basis will make the standard matrix for T as simple as possible? This case shows that it is not always the standard basis.

- Notes: Diagonal matrices have many computational advantages over nondiagonal ones (it will be used in the next chapter)

$$\text{for } D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(1) D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D^T = D$$

$$(3) D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}, \quad d_i \neq 0$$

Keywords in Section 6.4:

- matrix of T relative to B
- matrix of T relative to B'
- transition matrix from B' to B
- transition matrix from B to B'
- similar matrix

6.5 Applications of Linear Transformation

- The geometry of linear transformation in the plane (p.407-p.110)
 - Reflection in x -axis, y -axis, and $y=x$
 - Horizontal and vertical expansion and contraction
 - Horizontal and vertical shear
- Computer graphics (to produce any desired angle of view of a 3-D figure)