

MATHEMATICS II
Thursday, March 29 2018
Fifth Exercise Class

1. Given the set of vectors $F = \{(1, 3, 0), (1, -1, 2), (0, 0, 1)\}$, verify that F is a base for \mathbb{R}^3 . Moreover, given the vector v whose coordinates in F are $(2, -1, 0)$, write its coordinates in the natural base.

Solution The set F is a base for \mathbb{R}^3 . The vector \vec{v} in the natural base is $\vec{v} = (1, 7, -2)$

2. Determine the value for the parameter h such that the following matrix A admits eigenvalue $\lambda = 1$. For this value of h determine the eigenvalues of A :

$$D = \begin{pmatrix} h & 1 & 0 \\ 1-h & 0 & 2 \\ 1 & 1 & h \end{pmatrix}$$

Solution $h = -2$. The eigenvalues are $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 1$.

3. Calculate the eigenvalues and the eigenvectors of the following matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 & 1 \\ -3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & -5 \end{pmatrix}$$

Solution A). $\lambda_1 = 4, \lambda_2 = -1, \vec{v}_1 = t(-1, 1), \vec{v}_2 = t(-2/3, 1)$

B). $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1, \vec{v}_1 = t(-1/3, -1/3, 1), \vec{v}_2 = t(-1, 1, 0), \vec{v}_3 = t(-1, 2, 1)$

C). $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 3, \vec{v}_1 = t(-3, 1, 2), \vec{v}_2 = t(1, 1, 0), \vec{v}_3 = t(1, 1, 0)$

D). $\lambda_1 = -5, \lambda_2 = -2, \lambda_3 = 1, \vec{v}_1 = t(0, 0, 1), \vec{v}_2 = t(1, -1, 1), \vec{v}_3 = t(-2, 1, 0)$

4. Determine the algebraic and the geometric multiplicity of the eigenvalues of the following matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ -3 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} -7 & -3 & -16 \\ -2 & 0 & -2 \\ 6 & 3 & 15 \end{pmatrix}$$

Solution A). $\lambda = 1 \rightarrow m_a(1) = 2, m_g(1) = 2; \lambda = -1 \rightarrow m_a(-1) = 1, m_g(-1) = 1;$

B). $\lambda = 2 \rightarrow m_a(2) = 2, m_g(2) = 2; \lambda = -1 \rightarrow m_a(-1) = 1, m_g(-1) = 1;$

C). $\lambda = 2 \rightarrow m_a(2) = 1, m_g(1) = 1; \lambda = -1 \rightarrow m_a(-1) = 1, m_g(-1) = 1, \lambda = 1 \rightarrow m_a(1) = 1, m_g(1) = 1;$

D). $\lambda = 0 \rightarrow m_a(0) = 1, m_g(0) = 1; \lambda = -1 \rightarrow m_a(-1) = 1, m_g(-1) = 1, \lambda = 9 \rightarrow m_a(9) = 1, m_g(9) = 1;$

5. Verify that the following matrices are similar and determine the invertible matrix M such that $B = M \cdot A \cdot M^{-1}$

$$A = \begin{pmatrix} 4 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution The matrices A and B are similar since they have the same eigenvalues. The matrix M is given by

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

where the coefficients are given by the following system

$$\begin{cases} b = a \\ c = 2a \\ f = 4a \\ d = 2a \\ h = -2g \\ i = i \end{cases}$$

6. Determine whether the following matrices are diagonalizable. If yes, diagonalize it.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{pmatrix}$$

Solution A). The matrix is diagonalizable and

$$D_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1 + i\sqrt{3})/2 & 0 \\ 0 & 0 & (-1 - i\sqrt{3})/2 \end{pmatrix}$$

B). The matrix is diagonalizable and

$$D_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- C). The matrix is not diagonalizable because the eigenvalue $\lambda = -1$ has $m_a(-1) \neq m_g(-1)$.
D). The matrix is diagonalizable and

$$D_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$