Mathematics II

Second Practice: Solutions

1. Let A, B and C be the following matrices:

$$A := \begin{pmatrix} 2 & 3 \\ 4 & 0 \\ 1 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 4 \end{pmatrix}$$

Compute (if possible):

$a)$ $2A + C$ $\qquad$ $b)$ $A + B + C$ $\qquad$ $c)$ $A^T \cdot C$

$d)$ $(A^T \cdot C)^{-1}$ $\qquad$ $e)$ $C^T \cdot A + B$ $\qquad$ $f)$ $A \cdot C^T + B$

$g)$ $\det(A^T \cdot C) + B$ $\qquad$ $h)$ $B + B^{-1}$ $\qquad$ $i)$ $B^{-1} \cdot (A \cdot C^T)$

Solution:

$a)$ $\begin{pmatrix} 5 & 8 \\ 11 & 1 \\ 3 & 2 \end{pmatrix}$ $\qquad$ $b)$ **Nonsense** $\qquad$ $c)$ $\begin{pmatrix} 15 & 12 \\ 2 & 2 \end{pmatrix}$

$d)$ $\begin{pmatrix} 1/3 & -2 \\ -1/3 & 15/6 \end{pmatrix}$ $\qquad$ $e)$ **Nonsense** $\qquad$ $f)$ $\begin{pmatrix} 9 & 10 & 14 \\ 6 & 12 & 3 \\ 0 & 3 & -4 \end{pmatrix}$

$g)$ **Nonsense** $\qquad$ $h)$ $\begin{pmatrix} \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 2 & 1 & -2 \end{pmatrix}$ $\qquad$ $i)$ $\begin{pmatrix} \frac{13}{2} & \frac{19}{2} & \frac{21}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 9 & 7 & 17 \end{pmatrix}$

2. Compute $A \cdot B$ when such an operation is possible:

$$a) \quad A := \begin{pmatrix} 3 & 3 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 6 \\ -2 & -4 \\ 1 & 3 \end{pmatrix}$$

$$b) \quad A := \begin{pmatrix} 3 & -2 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 2 & -2 & -3 & 1 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} -5 & 6 & 17 & 4 \\ 3 & -3 & -8 & -1 \\ -3 & 3 & 1 & -4 \end{pmatrix}$$

$$c) \quad A := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 5 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

$$d) \quad A := \begin{pmatrix} 1 & 1 & -\frac{1}{3} & \frac{1}{6} \\ 1 & -1 & \frac{4}{3} & -\frac{1}{6} \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 7 & -1 \end{pmatrix}$$

3. Calculate the following determinants:

$$a) \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{vmatrix} = -20, \quad b) \begin{vmatrix} -1 & 5 & 3 \\ 4 & 0 & 0 \\ 2 & 7 & 8 \end{vmatrix} = -76, \quad c) \begin{vmatrix} 3 & 1 & 1 \\ 2 & 5 & 5 \\ 8 & 9 & 9 \end{vmatrix} = 0$$

$$d)\begin{vmatrix}1 & 1 & -2 & 4\\0 & 1 & 1 & 3\\0 & -1 & 1 & 0\\3 & 1 & 2 & 5\end{vmatrix} = -32, \qquad e)\begin{vmatrix}1 & 0 & 0 & -1 & 0\\1 & 1 & -2 & 1 & 4\\0 & 1 & 1 & 1 & 3\\0 & -1 & 1 & -1 & 0\\3 & 1 & 2 & 1 & 5\end{vmatrix} = -32$$

4. Given in $\mathbb{R}^3$ the vectors $\vec{v} = (1, 0, -1)$, $\vec{w} = (2, 2, -1)$ and $\vec{z} = (-2, 1, 0)$, calculate:

   a) $\vec{v} + 2\vec{w} - 3\vec{z}$;

   b) $(\vec{v} \cdot \vec{w})\,\vec{z} - \vec{w}$;

   c) $(\vec{v} \times \vec{w}) \cdot \vec{z}$;

   d) the length of each vector.

   Solution: a) $(11, 1, -3)$; b) $(-8, 1, 1)$; c) -5; d) $\sqrt{2}$, 3 , $\sqrt{5}$;

5. Given in $\mathbb{R}^3$ the vectors $\vec{v} = (1, 0, -1)$, $\vec{w} = (2, 2, -1)$ and $\vec{z} = (0, -2, -1)$, calculate the values of the parameters α, β, γ, such that: $\alpha\vec{v} + \beta\vec{w} + \gamma\vec{z} = \vec{0}$.

   Solution: $\alpha = 2t$, $\beta = -t$, $\gamma = -t \quad \forall t \in \mathbb{R}$.

6. Given the vectors $\vec{v} = (k, 1, -1)$ and $\vec{w} = (2, 2k, -1)$ determine values for k such that their inner product is zero.

   Solution: $k = -1/4$.

7. For each of the following vectors find the corresponding versor

   a) $\vec{v} = (1, \alpha)$;

   b) $\vec{v} = (-\alpha, \alpha)$;

   c) $\vec{v} = (1, \alpha, 2\alpha)$.

   Solution: a) $\frac{1}{\sqrt{1+\alpha^2}}(1, \alpha)$; b) $\frac{1}{\sqrt{2}\alpha}(-\alpha, \alpha)$; c) $\frac{1}{+\sqrt{1+5\alpha^2}}(1, \alpha, 2\alpha)$.

8. Determine the parametric equation and the Cartesian equation of the line on the space

   a) passing through the points $A(1, 0, 2)$ and $B(1, 3, 0)$;

   b) passing through the point $P(8, 0, 1)$ and parallel to the vector $\vec{v} = (0, 10, 0)$;

   c) passing through the point $P(-1, 0, 4)$ and parallel to the line of parametric equations $x = 2t$, $y = -t$, $z = 3t$.

Solution:

$$\mathbf{a)} : \begin{cases} x - 1 = 0 \\ 2y + 3z - 6 = 0 \end{cases} \quad ; \quad \begin{cases} x = 1 \\ y = 3t \\ z = 2 - 2t \end{cases}$$

$$\mathbf{b)} : \begin{cases} x - 8 = 0 \\ z - 1 = 0 \end{cases} \quad ; \quad \begin{cases} x = 8 \\ y = t \\ z = 1 \end{cases}$$

$$\mathbf{c)} : \begin{cases} x + 2y + 1 = 0 \\ 3x - 2z + 11 = 0 \end{cases} \quad ; \quad \begin{cases} x = -1 + 2t \\ y = -t \\ z = 4 + 3t \end{cases}$$

9. Determine the reciprocal position of the lines r and s of Cartesian equations

$$\begin{cases} x + y + z = 0 \\ y = 2 \end{cases} \quad ; \quad \begin{cases} x + 2y + z = 2 \\ y = 3 \end{cases}$$

   Solution: They are parallel.

10. Establish if the vectors $\vec{v}_1 = (1, 1, 1)$, $\vec{v}_2 = (1, 0, 0)$, $\vec{v}_1 = (0, 1, 0)$ and $\vec{v}_1 = (2, 1, 1)$ are linearly dependent.

   Solution: Yes, they are.

11. Determine for which value of k the following vectors in $\mathbb{R}^4$ are linearly independent

$$\vec{v} = (2, -1, 2, -1); \quad \vec{w} = (1, 1, 1, 1); \quad \vec{z} = (k, 0, k, 0).$$

   Solution: No value. They are linearly dependent for any $k \in \mathbb{R}$.

12. For any $k \in \mathbb{R}$ compute the rank of the following matrix:

$$\begin{pmatrix} k & 1 & 8 & k \\ k & 1 & 8 & 1 \\ 2k & 2 & 16 & 2 \end{pmatrix}$$

   Solution: For $k = 1$ the rank is equal to 1, otherwise it is equal to 2.