Mathematics II

# Fifth Practice: Solutions

1. Determine for which values of the parameter $k \in \mathbb{R}$ the following matrix is diagonalizable:
$$\begin{pmatrix} k & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & k \end{pmatrix}$$
**Sol**: $k = 5$.

2. Describe the domain $D_f \subset \mathbb{R}^2$ of each of the following functions:

$a)$ $f(x,y) = \sqrt{4 - x^2 - y^2}$ $\quad [D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}]$

$b)$ $f(x,y) = \sqrt{\sin^2(xy) + 2e^{x+y}}$ $\quad [D_f \equiv \mathbb{R}^2]$

$c)$ $f(x,y) = \dfrac{cos(x+y) + e^x}{x - y}$ $\quad [D_f = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\}$

$d)$ $f(x,y) = \dfrac{x+y}{\sqrt{x^2 + y^2}}$ $\quad [D_f = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}]$

$e)$ $f(x,y) = \dfrac{\log y}{e^x}$ $\quad [D_f = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$f)$ $f(x,y) = \dfrac{\log(x-y) + \log(x+y)}{\sqrt{x}}$ $\quad [D_f = \{(x,y) \in \mathbb{R}^2 \mid x > 0, -x < y < x\}$

3. Describe the contour curves of the following functions:

$a)$ $f(x,y) = \sqrt{x^2 + y^2}$ $\quad$ [**Sol** : **circles centered at** $(0,0)$]

$b)$ $f(x,y) = \sqrt{x - y}$ $\quad$ [**Sol** : **lines** $y = x + k$ **with** $k \leq 0$]

$c)$ $f(x,y) = \dfrac{y}{x}$ $\quad$ [**Sol** : **lines** $y = kx$ $(k \in \mathbb{R})$ **except for the point** $(0,0)$]

4. Find the following multivariable limits:

$a)$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sin xy}$ $\quad$ [**Sol**: the limit doesn't exist]

$b)$ $\displaystyle\lim_{(x,y)\to(2,0)} \frac{1 - \cos y}{xy^2}$ $\quad$ [**Sol**: $\frac{1}{4}$]

$c)$ $\displaystyle\lim_{(x,y)\to(0,0)} xy\, e^{\frac{xy}{x^2+y^2}}$ $\quad$ [**Sol**: 0]

5. Consider the function

$$f(x,y) = \frac{2xy^2 \sin^2(y)}{(x^2+y^2)^2}.$$

Does the limit exist when $(x,y)$ tends to $(0,0)$?

**Sol**: Yes.

6. Calculate the partial derivatives, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, of the following functions:

$$a) f(x,y) = x^2 + y - e^{xy} \qquad [\textbf{Sol}: \frac{\partial f}{\partial x} = 2x - ye^{xy}\,, \ \frac{\partial f}{\partial y} = 1 - xe^{xy}]$$

$$b) f(x,y) = \frac{1}{1+xy} \qquad [\textbf{Sol}: \frac{\partial f}{\partial x} = -\frac{y}{(1+xy)^2}\,, \ \frac{\partial f}{\partial y} = -\frac{x}{(1+xy)^2}$$

$$c) f(x,y) = \sqrt{x^2+y^2} \qquad [\textbf{Sol}: \frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\,, \ \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}]$$

7. Study the continuity and the differentiability of the following function

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$$

**Sol**: $f$ is not continuous at the origin.

8. Study the differentiability of

$$f(x,y) = 2xy + \frac{x}{y}$$

at $(1,1)$.

**Sol**: It is differentiable.

9. Study the continuity and the differentiability of the following function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$$

**Sol**: It is continuous and differentiable.

10. Find all the stationary points of the following functions and classify them:

$a) \quad x^6 + y^2$

$$\textit{Solution}: \quad \begin{array}{c} (0,0) \\ min \end{array}$$

$b) \quad 3x^2y - xy$

$$\textit{Solution}: \quad \begin{array}{cc} (0,0) & (1/3,0) \\ saddle & saddle \\ point & point \end{array}$$

$c) \quad x^2 + 2y^2 - 3y^3$

$$\textit{Solution}: \quad \begin{array}{cc} (0,0) & (0,4/9) \\ min & saddle \\ & point \end{array}$$

$d) \quad y^3 + y + 7xy + 7x^2 + 4$

$$\textit{Solution}: \quad \begin{array}{cc} (-\frac{1}{3},\frac{2}{3}) & (-\frac{1}{4},\frac{1}{2}) \\ min & saddle \\ & point \end{array}$$

$e) \quad x - x^3 - 4xy^2$

$$\textit{Solution}: \quad \begin{array}{cccc} (0,-\frac{1}{2}) & (0,\frac{1}{2}) & (-\frac{\sqrt{3}}{3},0) & (\frac{\sqrt{3}}{3},0) \\ saddle & saddle & min & max \\ point & point & & \end{array}$$