EX. CLASS MAY 8, 2017 (1)

EX. 3 Find the local max and min values and saddle points of the following functions

$$f(x,y,z) = -x^2 - y^2 - z^2 + xy - yz + x + y$$

1st step: Find critical points, means, find points $(x,y,z) \in D_f$ such that $\nabla f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} f_x(x,y,z) = -2x + y + 1 = 0 \Rightarrow y = 2x - 1 \\ f_y(x,y,z) = -2y + x - z + 1 = 0 \\ f_z(x,y,z) = -2z - y = 0 \Rightarrow y = 2z \end{cases}$$

$2x - 1 = 2z \Rightarrow x = \frac{2z+1}{2}$

substitute $x = \frac{2z+1}{2}$ and $y = 2z$ in the 2nd equation

$$-2 \cdot 2z + \frac{2z+1}{2} - z + 1 = 0 \iff -8z + 2z + 1 - 2z + 2 = 0$$

$$\Rightarrow z = \frac{3}{8},\ x = \frac{2 \cdot \frac{3}{8} + 1}{2} = \frac{7}{8},\ y = 2 \cdot \frac{3}{8} = \frac{3}{4}$$

2nd step: to test the nature of the critical point $(x_0, y_0, z_0) = \left(\frac{7}{8}, \frac{3}{4}, \frac{3}{8}\right)$ let us calculate the Hessian in correspondence of $(x_0, y_0, z_0) \Rightarrow$

$$H(x,y,z) = \begin{pmatrix} f_{xx}(x,y,z) & f_{xy}(x,y,z) & f_{xz}(x,y,z) \\ f_{yx}(x,y,z) & f_{yy}(x,y,z) & f_{yz}(x,y,z) \\ f_{zx}(x,y,z) & f_{zy}(x,y,z) & f_{zz}(x,y,z) \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

if $H(x_0, y_0, z_0)$ defines a POSITIVE DEFINITE QUADRATIC FORM $\Rightarrow (x_0, y_0, z_0)$ is a MINIMUM, if $H(x_0, y_0, z_0)$ defines a NEGATIVE DEFINITE QUADRATIC FORM $\Rightarrow (x_0, y_0, z_0)$ is a MAXIMUM. We obtain this information by analyzing the signs of the PRINCIPAL MINORS $|f_{xx}| = -2 < 0$, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0$

NOT ABSOLUTE VALUE but Determinant

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix} = 2$$

$= -8 + 2 + 2 = -4 < 0$

as these minors are ALTERNATE in signs $\Rightarrow$ $H(x_0, y_0, z_0)$ defines a NEGATIVE DEFINITE QUADRATIC FORM $\Rightarrow$ $(x_0, y_0, z_0) = \left(\frac{7}{8}, \frac{3}{4}, \frac{3}{8}\right)$ is a LOCAL MAXIMUM

EX. 4 $\quad f(x,y,z) = \ln(x^2+y^2+z^2+1)$

following the steps of EX. 3 $\Rightarrow$:

$$\begin{cases} f_x = \dfrac{2x}{x^2+y^2+z^2+1} = 0 \\[2ex] f_y = \dfrac{2y}{x^2+y^2+z^2+1} = 0 \\[2ex] f_z = \dfrac{2z}{x^2+y^2+z^2+1} = 0 \end{cases}$$

the only solution to this system is $(x_0, y_0, z_0) = (0,0,0)$

Let us calculate the Hessian:

$$f_{xx} = \frac{2(x^2+y^2+z^2+1) - 2x \cdot 2x}{(x^2+y^2+z^2+1)^2} = \frac{2x^2+2y^2+2z^2+2-4x^2}{(x^2+y^2+z^2+1)^2}$$

$$= \frac{2(-x^2+y^2+z^2+1)}{(x^2+y^2+z^2+1)^2}$$

$$f_{xy} = -\frac{4xy}{(x^2+y^2+z^2+1)^2}, \quad f_{xz} = -\frac{4xz}{(x^2+y^2+z^2+1)^2}$$

$$f_{yy} = \frac{2(x^2-y^2+z^2+1)}{(x^2+y^2+z^2+1)^2}, \quad f_{yx} = f_{xy}, \quad f_{yz} = -\frac{4yz}{(x^2+y^2+z^2+1)}$$

$$f_{zz} = \frac{2(x^2+y^2-z^2+1)}{(x^2+y^2+z^2+1)^2}, \quad f_{zx} = f_{zx}, \quad f_{zy} = f_{yz}$$

$$H(x,y,z) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{2(-x^2+y^2+z^2+1)}{(x^2+y^2+z^2+1)^2} & -\dfrac{4xy}{(x^2+y^2+z^2+1)^2} & -\dfrac{4xz}{(x^2+y^2+z^2+1)^2} \\[2ex] -\dfrac{4xy}{(x^2+y^2+z^2+1)^2} & \dfrac{2(x^2+y^2+z^2+1)}{(x^2+y^2+z^2+1)^2} & -\dfrac{4yz}{(\ldots)^2} \\[2ex] -\dfrac{4yz}{(x^2+y^2+z^2+1)^2} & -\dfrac{4yz}{(x^2+y^2+z^2+1)^2} & \dfrac{2(x^2+y^2-z^2+1)}{(x^2+y^2+z^2+1)^2} \end{pmatrix}$$

$$H(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Analyzing the principal minors:

$$|f_{xx}| = 2 > 0 \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \text{ and}$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0$$

we can declare that $(0,0,0)$ is a LOCAL MINIMUM as $H(0,0,0)$ defines a DEFINITE POSITIVE QUADRATIC FORM.

Find the extreme values of the following functions subject to the given constraints

Ex. 6 $f(x,y)=x^2y$, $x^2+2y^2=6 \Leftrightarrow \underbrace{x^2+2y^2-6}_{g(x)}=0$

Identify the LAGRANGIAN function:

$$\mathcal{L}(x,y,\lambda)=x^2y-\lambda(x^2+2y^2-6)$$

Find critical points for $\mathcal{L}(x,y,\lambda)$:

$\mathcal{L}_x=2xy-2\lambda x=0 \Rightarrow 2x(y-\lambda)=0$ — $x=0$ or $y=\lambda$

$\mathcal{L}_y=x^2-2\lambda y=0$

$\mathcal{L}_\lambda=-(x^2+2y^2-6)=0$

if $x=0 \Rightarrow 2\lambda y=0$ from $x^2-2\lambda y=0$ and hence

— $y=0$ IMPOSSIBLE for 3rd equation as we would have $-6=0$!!

— $\lambda=0 \Rightarrow 2y^2-6=0 \Rightarrow y^2=3 \Rightarrow y=\pm\sqrt{3}$

$\Rightarrow$ the first critical points are:

$$\begin{array}{cc} x\ \ y\ \ \lambda & x\ \ \ y\ \ \ \lambda \\ (0,\sqrt{3},0), & (0,-\sqrt{3},0) \end{array}$$

if instead $y=\lambda \Rightarrow x^2-2y^2=0 \Rightarrow x^2=2y^2 \Rightarrow$

$2y^2+2y^2-6=0 \Rightarrow 4y^2-6=0 \Rightarrow y^2=\frac{3}{2} \Rightarrow y=\pm\frac{\sqrt{3}}{\sqrt{2}}=\pm\frac{\sqrt{6}}{2}$

$\Rightarrow x=\pm\sqrt{2}y=\pm\sqrt{3}$ and as $y=\lambda$, we have the following critical points

$$\left(\sqrt{3},\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right),\ \left(\sqrt{3},-\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{2}\right)$$

$$\left(-\sqrt{3},\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right),\ \left(-\sqrt{3},-\frac{\sqrt{6}}{2},-\frac{\sqrt{6}}{2}\right)$$

Let us calculate now the BORDERED HESSIAN

$$H(x,y,\lambda)=\begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{pmatrix}=\begin{pmatrix} 0 & 2x & 4y \\ 2x & 2y-2\lambda & 2x \\ 4y & 2x & -2\lambda \end{pmatrix}$$

$$|H(x,y,\lambda)|=16x^2y+16x^2y-32y^2(y-\lambda)+8\lambda x^2=$$
$$=32x^2y+8\lambda x^2-32y^2(y-\lambda)$$

Recall that if $|H(x_0, y_0, \lambda_0)| > 0 \rightarrow (x_0, y_0, \lambda_0)$
determinant
is a MAX, if inverse $|H(x_0, y_0, \lambda_0)| < 0 \Rightarrow$ (x_0, y_0, λ_0) is a MIN. If $|H(x_0, y_0, \lambda_0)| = 0$ this method is inconclusive. Let us analyze point by point:

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- $(0, \sqrt{3}, 0)$ $|H(0, \sqrt{3}, 0)| = -32(\sqrt{3})^2 \cdot \sqrt{3} = -96\sqrt{3} < 0$
  $\rightarrow (0, \sqrt{3}, 0)$ is a MIN
- $(0, -\sqrt{3}, 0)$ $|H(0, -\sqrt{3}, 0)| = -32 \cdot (-\sqrt{3})^2 \cdot (-\sqrt{3}) = 96\sqrt{3} > 0$
  $\rightarrow (0, -\sqrt{3}, 0)$ is a MAX
- $|H\left(\sqrt{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)| = 32(\sqrt{3})^2 \frac{\sqrt{6}}{2} + 8\frac{\sqrt{6}}{2}(\sqrt{3})^2 - 0$
  $= 60\sqrt{6} > 0 \rightarrow \left(\sqrt{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)$ is a MAX
- $|H\left(\sqrt{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\right)| = -60\sqrt{6} < 0 \rightarrow \left(\sqrt{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\right)$
  is a MIN
- $|H\left(-\sqrt{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)| = 60\sqrt{6} > 0 \rightarrow \left(-\sqrt{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right)$
  is a MAX
- $H\left(-\sqrt{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\right) = -60\sqrt{6} < 0$ is a MIN.

EX 8 $f(x,y,z) = x^2+2y^2+3z^2$, $x+y+z=1$, $x-y+2z=2$

1st METHOD: write the variables x and y in terms of z by solving the system

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$$\begin{cases} x+y = 1-z \\ x-y = 2-2z \end{cases}$$ (consider z as a number/parameter) I solve the system with CRAMER

$$x = \frac{\begin{vmatrix} 1-z & 1 \\ 2-2z & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{z-1-2+2z}{-1-1} = \frac{3z-3}{-2} = \frac{3}{2}(1-z)$$

$$y = \frac{\begin{vmatrix} 1 & 1-z \\ 1 & 2-2z \end{vmatrix}}{-2} = \frac{2-2z-1+z}{-2} = \frac{1-z}{-2} = \frac{z-1}{2}$$

substitute x and y, written in terms of z, in $f(x,y,z) \Rightarrow$ $h(z) =$

$$f(x(z), y(z), z) = \frac{9}{4}(1-z)^2 + 2\cdot\frac{1}{4}(z-1)^2 + 3z^2$$

$$= \frac{11}{4}(1-z)^2 + 3z^2 = \frac{11}{4}(1-2z+z^2) + 3z^2$$

$$= \frac{23}{4}z^2 - \frac{11}{2}z + \frac{11}{4}$$ this is now a function of ONE variable to which I can apply optimization in one variable $\Rightarrow$:

$$h'(z) = \frac{23}{2}z - \frac{11}{2} = 0 \Leftrightarrow 23z = 11 \Rightarrow z = \frac{11}{23}$$

and $h''(z) = \frac{23}{2} > 0 \Rightarrow z = \frac{11}{23}$ is a LOC. MIN

and $x = \frac{3}{2}\left(1 - \frac{11}{23}\right) = \frac{3}{\cancel{2}} \cdot \frac{\cancel{12}^{6}}{23} = \frac{18}{23}$

$y = \frac{1}{2}\left(\frac{11}{23} - 1\right) = \frac{1}{\cancel{2}}\left(-\frac{\cancel{12}^{6}}{23}\right) = -\frac{6}{23}$

2nd METHOD: LAGRANGIAN METHOD

$$\mathcal{L}(x,y,z,\lambda,\mu) = x^2+2y^2+3z^2 - \lambda(x+y+z-1) - \mu(x-y+2z-2)$$

hence:

$$\begin{cases} \mathcal{L}_x = 2x - \lambda - \mu = 0 \\ \mathcal{L}_y = 4y - \lambda + \mu = 0 \\ \mathcal{L}_z = 6z - \lambda - 2\mu = 0 \\ \mathcal{L}_\lambda = -(x+y+z-1) = 0 \\ \mathcal{L}_\mu = -(x-y+2z-2) = 0 \end{cases}$$

I will write λ and μ in terms of x and y

$$\begin{cases} \lambda + \mu = 2x \\ \lambda - \mu = 4y \end{cases} \rightarrow \lambda = \frac{\begin{vmatrix} 2x & 1 \\ 4y & -1 \end{vmatrix}}{-2} = \frac{-2x-4y}{-2} = x + 2y$$

$$\mu = \frac{\begin{vmatrix} 1 & 2x \\ 1 & 4y \end{vmatrix}}{-2} = \frac{4y-2x}{-2} = x - 2y$$

substitute λ and μ in the third equation

$6z - x - 2y - 2x + 4y = 0 \Rightarrow$ joined with the last two equations, we obtain:

$$\begin{cases} -3x + 2y + 6z = 0 \\ x + y + z = 1 \\ x - y + 2z = 2 \end{cases}$$

I will solve this system through CRAMER

$$x = \frac{\begin{vmatrix} 0 & 2 & 6 \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} -3 & 2 & 6 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}} = \frac{18}{23}$$

$$\lambda = x + 2y = \frac{18}{23} - \frac{12}{23} = \frac{6}{23}$$

$$\mu = x - 2y = \frac{18}{23} + \frac{12}{23} = \frac{30}{23}$$

$$y = \frac{\begin{vmatrix} -3 & 0 & 6 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}}{-23} = -\frac{6}{23}$$

$$z = \frac{\begin{vmatrix} -3 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}}{-23} = \frac{11}{23}$$

We will test the nature of this critical point

$$\left(\overset{x}{\frac{18}{23}}, \overset{y}{-\frac{6}{23}}, \overset{z}{\frac{11}{23}}, \overset{\lambda}{\frac{6}{23}}, \overset{\mu}{\frac{30}{23}}\right)$$ through the analysis

of the BORDERED HESSIAN. Let us briefly

summarize some results for testing nature of CRITICAL POINTS.

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Consider the general problem of maximizing or minimizing $f: \mathbb{R}^n \to \mathbb{R}$ over the set

$$\tilde{D} = U \cap \{x \mid g(x) = 0\} \to g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_k(x) \end{pmatrix}$$

where $g: \mathbb{R}^n \to \mathbb{R}^k$ (these are the constraints, if you have 1 constraint $\Rightarrow g: \mathbb{R}^n \to \mathbb{R}$, if you have two constraints $g: \mathbb{R}^n \to \mathbb{R}^2$, ...)

We will assume that f and g are both C^2 functions (continuous functions with continuous first and second derivative). We form the Lagrangian

$$\mathcal{L}(x, \underline{\lambda}) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x)$$

$$\lambda \in \mathbb{R}^k \Rightarrow \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}$$

Let us consider the Hessian matrix of $\mathcal{L}(x, \lambda)$:

$$\bar{H} = D^2\mathcal{L}(x,\lambda) = \begin{pmatrix}
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1^2} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial \lambda_k \partial \lambda_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n \partial \lambda_1} \\
\vdots & & & & & & \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial \lambda_k} & \frac{\partial^2 \mathcal{L}}{\partial \lambda_2 \partial \lambda_k} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial \lambda_k^2} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda_k} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n \partial \lambda_k} \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x_1} & \cdots & & \frac{\partial^2 \mathcal{L}}{\partial \lambda_k \partial x_n} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n \partial x_1} \\
\vdots & & & & \vdots & & \\
\frac{\partial^2 \mathcal{L}}{\partial \lambda_1 \partial x_n} & \cdots & & \frac{\partial^2 \mathcal{L}}{\partial \lambda_k \partial x_n} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n^2}
\end{pmatrix}$$

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$$\underbrace{\left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_k}{\partial x_1} & \cdots & \frac{\partial g_k}{\partial x_n} \\ \hline \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_k}{\partial x_1} & \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n \partial x_1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_k}{\partial x_n} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n^2} \end{array}\right)} = D^2 \mathcal{L}_{(\lambda, x)}(x, \lambda) = \overline{H}$$

called BORDERED HESSIAN: this matrix plays the role of normal Hessian matrix in constraing problems for second order conditions

Hence, suppose there exists a point $x^* \in \widetilde{D}$ (recall this is the set of points that satisfy the constraints) and $\lambda^* \in \mathbb{R}^k$ such that $\operatorname{rank} Dg(x^*) = k$ (this is the JACOBIAN of the constraint functions, i.e. the matrix of the first partial derivatives of the constraints

$$Dg(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_k}{\partial x_1} \\ \vdots & & \\ \frac{\partial g_k}{\partial x_n} & \cdots & \frac{\partial g_k}{\partial x_k} \end{pmatrix}$$

and $Df(x^*) + \sum_{i=1}^{k} \lambda_i^* Dg_i(x^*) = 0$ (FIRST ORDER CONDITIONS, this is what we do when we calculate the partial first derivatives of the Lagrangian, and we set them to 0). Consider the bordered Hessian given above $\overline{H}$. Let $\overline{H}_r$ denote the r-th order leading principal submatrix of $\overline{H}$ $\Rightarrow$;

1) if $(-1)^{r-k} |\overline{H}_r| > 0$ for all $r = 2k+1, \dots, n+k$ then x^* is a STRICT LOCAL MAXIMIZER of f on $\widetilde{D}$

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2) if $(-1)^k |\overline{H}_r| > 0$ for all $r = 2k+1, \dots, n+k$ then x^* is a STRICT LOCAL MINIMIZER of f on $\widetilde{D}$

3) if either of the above conditions is violated by nonzero leading minors, then x^* is neither a local maximizer nor a local minimizer

Let us go back to our exercise. We have to test the nature of $\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}, \frac{6}{23}, \frac{30}{23}\right)$ and the Bordered Hessian is

$$\overline{H} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 & 6 \end{pmatrix}$$

in our case:

$f: \mathbb{R}^3 \xrightarrow{(x,y,z)} \mathbb{R} \rightarrow n = 3$

$$g(x,y,z) = \begin{pmatrix} g_1(x,y,z) \\ g_2(x,y,z) \end{pmatrix} = \begin{pmatrix} x+y+z-1 \\ x-y+2z-2 \end{pmatrix}$$

hence $g(x,y,z): \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \longrightarrow k = 2$

hence we have to analyze the sign of $|\overline{H}_r|$ for $r = 2k+1, \dots, n+k = \underset{k}{2} \cdot 2 + 1, \dots, \underset{n+k}{3+2} = \underset{\text{only case}}{5}$

this means we have to see the sign of $|\overline{H}|$,

1) if $(-1)^{r-k} |\overline{H}_r| > 0$ $r = 2k+1, \dots, n+k \Rightarrow x^*$ is a Max, and in this case $r = 5, k = 2 \Rightarrow 5 - 2 = 3$

$\Rightarrow (-1)^3 = -1 \rightarrow |\overline{H}_5| = |\overline{H}| < 0$ if this condition has to verify $\rightarrow$ if $|\overline{H}| < 0 \Rightarrow x^*$ is Max

2) if instead $(-1)^k |\bar{H}_r| > 0 \quad r = 2k+1, \dots, n+k$

$\rightarrow x^*$ is a MIN, $\Rightarrow$ here $k = 2$, $r = 5 \Rightarrow (-1)^2 = 1$

$\Rightarrow$ for this condition to verify I want $|\bar{H}_5| = |\bar{H}| > 0 \Rightarrow$ here if $|\bar{H}| > 0 \Rightarrow x^*$ is a MIN

Hence let us check the sign of $|\bar{H}|$:

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 & 6 \end{vmatrix} = (-1)^{5+1} \cdot 1 \cdot \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 4 & 0 \end{vmatrix}$$

Laplace according to the last row

$$+ (-1)^{5+2} \cdot 2 \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 4 & 0 \end{vmatrix} + (-1)^{5+5} \cdot 6 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 4 \end{vmatrix}$$

$$= \underbrace{(-1)^{3+1} \cdot 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 4 & 0 \end{vmatrix} + (-1)^{4+1} \cdot (-1) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 0 \end{vmatrix}}_{\text{determinant through Laplace according to the first column}} - 2 \left[(-1)^{3+1} \cdot 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 4 & 0 \end{vmatrix} \right.$$

$$\left. + (-1)^{4+1} \cdot 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 0 \end{vmatrix} \right] + 6 \left[(-1)^{3+1} \cdot 1 \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 4 \end{vmatrix} + (-1)^{4+1} \cdot 1 \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{vmatrix} \right]$$

$$= -4 + 6 - 2(-4 - 6) + 6(2 + 2) = +2 - 2(-10) + 24$$

$= 46 > 0 \Rightarrow$ the point $\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right)$ with

LAGRANGE MULTIPLIERS $(\lambda, \mu) = \left(\frac{6}{23}, \frac{30}{23}\right)$

is a CONSTRAINED LOCAL

MINIMIZER.