MATHEMATICS 2

Exam

03/07/2018, A.Y. 2017/2018

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1) *(10 p.ts)* Solve the following integral

$$\int \sqrt{x}\cos\sqrt{x}dx$$

Applying the substitution $t = \sqrt{x}$, for which $x = t^2$, and $dx = 2tdt$, we get

$$\int \sqrt{x}\cos\sqrt{x}dx = 2\int t^2\cos t dt.$$

Hence we can apply integration by parts to the last integral

$$\int \sqrt{x}\cos\sqrt{x}dx = 2\int t^2\cos t dt = 2\Big(t^2\sin t - 2\int t\sin t dt\Big)$$

$$= 2\Big[t^2\sin t - 2\Big(-t\cos t + \int \cos t dt\Big)\Big] = 2\Big[t^2\sin t - 2\Big(-t\cos t + \sin t\Big)\Big] + c$$

$$= 2x\sin x + 4\sqrt{x}\cos\sqrt{x} - 4\sin\sqrt{x} + c$$

2) *(10 p.ts)* Given the following linear system

$$\begin{cases} 2x + 3y = 1 \\ -kx + y = -1 \\ x - y = k \end{cases}$$

discuss and find solutions as k changes.

To apply Rouché Capelli let us analyze the rank of incomplete and complete matrices

$$A = \begin{pmatrix} 2 & 3 \\ -k & 1 \\ 1 & -1 \end{pmatrix}, \quad A|b = \begin{pmatrix} 2 & 3 & 1 \\ -k & 1 & -1 \\ 1 & -1 & k \end{pmatrix}$$

As $rkA|b = 3$ if $detA|b \neq 0$ and

$$detA|b = 2k - 3 + k - 1 - 2 + 3k^2 = 3k^2 + 3k - 6 = 3(k^2 + k - 2) = 3(k-1)(k+2)$$

we get that $detA|b \neq 0$ if $k \neq 1$ and $k \neq -2$. Moreover $rkA \leq 2$, hence surely if $k \neq 1$ and $k \neq -2$, $rkA|b = 3$ and $rkA \leq 2$, hence the two ranks are different and the system is incompatible.
Let us analyze the two different cases, $k = 1$ and $k = -2$.
If $k = 1$

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A|b = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Notice that the number of variables is $n = 2$ and $rkA = rkA|b = 2$, hence the number of solutions is $\infty^{(2-2)} = \infty^0 = 1$ solution,that is given by solving the following equivalent system

$$\begin{cases} 2x + 3y = 1 \\ -x + y = -1 \end{cases}$$

whose solution is

$$\begin{pmatrix} x = \frac{4}{5} \\ y = -\frac{1}{5} \end{pmatrix},$$

If $k = -2$

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad A|b = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

the system is compatible as $rkA = rkA|b = 2$, and as $n = 2$ and $rkA = rkA|b = 2$, hence the number of solutions is $\infty^{(2-2)} = \infty^0 = 1$ solution. If I consider the following order 2 submatrix

$$M = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

the equivalent system that gives me the solution is

$$\begin{cases} 2x + 3y = 1 \\ 2x + y = -1 \end{cases}$$

whose solution is

$$\begin{pmatrix} x = -1 \\ y = 1 \end{pmatrix},$$

3) *(10 p.ts)* Find max/min of the following function

$$f(x,y) = xy$$

subject to the following constraint

$$x^2 + y^2 = 2$$

The Lagrangian of the problem is

$$L(x,y,\lambda) = xy - \lambda(x^2 + y^2 - 2)$$

Let us calculate the points for which the gradient of the Lagrangian is null

$$\begin{cases} L_x = y - 2\lambda x = 0 \\ L_y = y - 2\lambda y = 0 \\ L_\lambda = -(x^2 + y^2 - 2) = 0 \end{cases}$$

hence all the possible combinations that solve the previous system are $(-1, 1, -\frac{1}{2})$, $(1, -1, -\frac{1}{2})$, $(-1, -1, \frac{1}{2})$, $(1, 1, \frac{1}{2})$. To evaluate the nature of these points let us calculate the Bordered Hessian matrix

$$H(x,y,\lambda) = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 1 \\ 2y & 1 & -2\lambda \end{pmatrix}$$

whose determinant is $|H(x,y,\lambda)| = 8xy + 8\lambda y^2 + 8\lambda x^2$.
Let us analyze each single point:

- $(-1, 1, -\frac{1}{2})$ is such that $|H(-1, 1, -\frac{1}{2})| = -16 < 0$, hence it is a local minimum
- $(1, -1, -\frac{1}{2})$ is such that $|H(1, -1, -\frac{1}{2})| = -16 < 0$, hence it is a local minimum
- $(1, 1, \frac{1}{2})$ is such that $|H(1, 1, \frac{1}{2})| = 16 > 0$, hence it is a local maximum
- $(-1, -1, \frac{1}{2})$ is such that $|H(-1, -1, \frac{1}{2})| = 16 > 0$, hence the point is a local maximum.