Mathematics 1 A   Fall 2019
Second Practice

1. Consider the function equation $f(x) = x^2 - 2x - 1$.

   (a) Determine the maximal domain for this equation, that is, the maximal subset $D \subseteq \mathbb{R}$ such that

   $$f : D \to \mathbb{R}, \quad x \mapsto f(x) = x^2 - 2x - 1$$

   is a well-defined function.

   (b) Determine the image $I_f = f(D)$ of f.

   (c) Determine whether f is injective, surjective, bijective and/or invertible.

   (d) Let $A = \{0, 11, 101\}$. Determine the image $I_{f|A} = f(A)$ of the restriction $f|_A : A \to \mathbb{R}, \quad x \mapsto f(x) = x^2 - 2x - 1$.

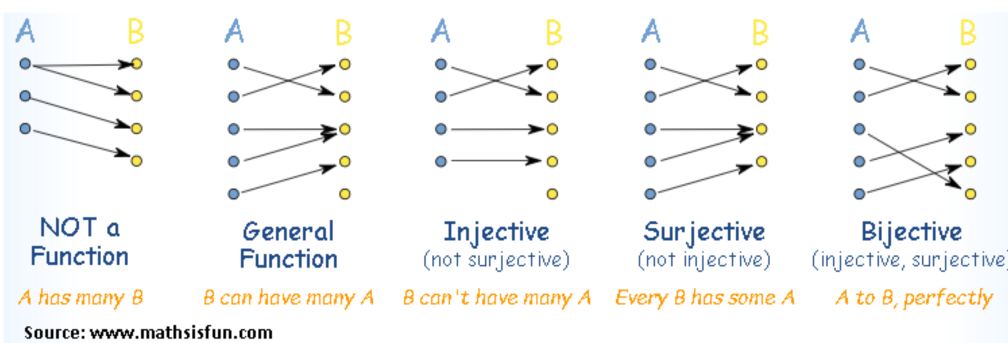
   (e) If f is not injective, determine a suitable subset $E \subseteq D$ such that the *restriction* $f|_E : E \to \mathbb{R}, \quad x \mapsto f(x) = x^2 - 2x - 1$ is injective. Note that then $f|_E : E \to I_{f|E} = f(E)$ is invertible.

   (f) Determine the inverse function $(f|_E)^{(-1)}$ of $f|_E : E \to f(E)$ (including its domain).

Solution:

*Remark***:** Let $f : D \to \mathbb{R}$. For $x \in \mathbb{R}$, we call $y = f(x) \in \mathbb{R}$ the *image* of x (under f). For $y \in \mathbb{R}$, we call each $x \in \mathbb{R}$ with $f(x) = y$ a *preimage* of x (under f). Then:

-f is injective, if and only if, every $y \in \mathbb{R}$ has at most one preimage under f.
-f is surjective, if and only if, every $y \in \mathbb{R}$ has at least one preimage under f.
-f is bijective, if and only if, every $y \in \mathbb{R}$ has exactly one preimage under f.



1. (a) $f(x) = x^2 - 2x - 1$ is well-defined (in $\mathbb{R}$) for every $x \in \mathbb{R}$, so $D = \mathbb{R}$

   (b) For every $x \in \mathbb{R}$, $x^2 - 2x + 1 = (x-1)^2 \geq 0$, so $y = f(x) = (x-1)^2 - 2 \geq -2$, thus $I_f = [-2, +\infty)$.

(c) For $x_1 = 0, x_2 = 2$,

$$f(x_1) = -1 = 2^2 - 2 \cdot 2 - 1 = f(x_2)$$

or in more general for $x_1 = a, x_2 = 2 - a$,

$$f(x_1) = (a-1)^2 - 2 = (1-a)^2 - 2 = f(x_2)$$

so $\exists x_1, x_2 \in \mathbb{R}, x_1 \neq x_2 : f(x_1) = f(x_2)$, thus f is not injective.
f is not surjective as it has not any preimage for every $y \in \mathbb{R}$
Since it is both not injective and not surjective, f is not bijective whence it is not invertible in $\mathbb{R}$.

(d) $f(0) = -1$, $f(11) = (11-1)^2 - 2 = 98$, $f(101) = (101-1)^2 - 2 = 9998$, so $I_{f|A} = f(A) = \{-1, 98, 9998\}$

(e) $y = -2$ has exactly one preimage 1 ($f(1) = -2$). Each $y > -2$ has two possible preimages: $x = \sqrt{y+2} + 1 > 0$ and $x = -\sqrt{y+2} + 1 < 0$, so we have to choose one of those preimages in every case and "throw out" the other. For example, we can choose the preimages $x = \sqrt{y+2} + 1$ for which $E = [1, +\infty)$.
Then $f|_{\mathbb{E}} : [1, +\infty) \to [-2, +\infty)$ is bijective and therefore invertible.
(Alternatively, we could also choose the other preimages $x = -\sqrt{y+2} + 1$ for which $E = (-\infty, 1]$ Then $f|_{\mathbb{E}} : (-\infty, 1] \to [-2, +\infty)$ is bijective and therefore invertible.)

(f) The inverse $(f|_E)^{(-1)} : [-2, +\infty) \to [1, +\infty)$ must map each $y \in [-2, +\infty)$ to its (unique) preimage, so we need to solve $y = f(x) = x^2 - 2x - 1$ for $x \in E = [1, +\infty)$:

$$\begin{aligned} y &= f(x) = (x-1)^2 - 2 \\ y + 2 &= (x-1)^2 \\ x &= \pm\sqrt{y+2} + 1 \end{aligned}$$

Since we assume $x \in E = [1, +\infty)$, we get $x = \sqrt{y+2} + 1$, so the inverse function is

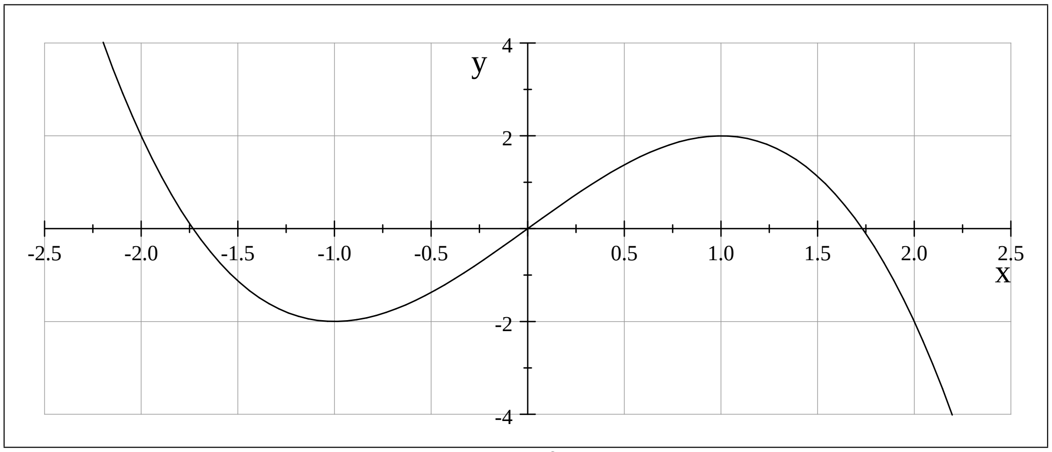
$$(f|_E)^{(-1)}(x) = \sqrt{x+2} + 1$$

and its domain is $[-2, +\infty)$ and codomain is $[1, +\infty)$

$$(f|_E)^{(-1)} : [-2, +\infty) \to [1, +\infty),\ x \mapsto \sqrt{x+2} + 1$$

2. Solution of the problem 1 using the following functions instead of $f(x) = x^2 - 2x - 1$:

f(x)	**D**	$\mathbf{I}_f$	**inj.**	**surj.**	**bij.**	f($\{0, 11, 101\}$)	**E**	$(\mathbf{f}\|_E)^{(-1)}(\mathbf{x})$
2x-8	$\mathbb{R}$	$\mathbb{R}$	✓	✓	✓	$\{-8, 14, 198\}$	$\mathbb{R}$	$\frac{x+8}{2}$
$\sqrt[4]{x}$	$\mathbb{R}^+$	$\mathbb{R}^+$	✓	X	X	$\{0, \sqrt[4]{11}, \sqrt[4]{101}\}$	$\mathbb{R}^+$	x^4
$\frac{1}{x^2-2x+1}$	$\mathbb{R} \setminus \{1\}$	$\mathbb{R}^+ \setminus \{0\}$	X	X	X	$\{0, 1/100, 1/10000\}$	$(1, +\infty)$ or $(-\infty, 1)$	$1/\sqrt{x}+1$ or $-1/\sqrt{x}+1$
$x^2 - 1$	$\mathbb{R}$	$[-1, +\infty)$	X	X	X	$\{-1, 120, 10200\}$	$\mathbb{R}^+$ or $\mathbb{R}^-$	$\sqrt{x+1}$ or -$\sqrt{x+1}$
$-x^3 + 3x$	$\mathbb{R}$	$\mathbb{R}$	X	✓	X	$\{0, -1298, -1029998\}$	$(-\infty, -2) \cup (1, +\infty)$ or $(-\infty, -1) \cup (2, +\infty)$	



$$f(x) = -x^3 + 3x \tag{1}$$

3. Line L_1 passes from the points A=(1,3) and B=(3,7), line L_2 passes from the points C=(2,1) and D=(-2,3).

   (a) Find the ratio of the distances $\frac{d(A,B)}{d(C,D)}$
   Solution:

$$\frac{d(A,B)}{d(C,D)} = \frac{\sqrt{(y_B - y_A)^2 + (x_B - x_A)^2}}{\sqrt{(y_D - y_C)^2 + (x_D - x_C)^2}} = \frac{\sqrt{(7-3)^2 + (3-1)^2}}{\sqrt{(3-1)^2 + (-2-2)^2}} = 1$$

   (b) Are L_1 and L_2 parallel?
   Solution: Let the slope of L_1=m_1 and the slope of L_2=m_2

$$\begin{aligned} m_1 &= \frac{y_B - y_A}{x_B - x_A} = \frac{7-3}{3-1} = 2 \\ m_2 &= \frac{y_D - y_C}{x_D - x_C} = \frac{3-1}{-2-2} = -1/2 \end{aligned}$$

   Since their slopes are not the same, they are not parallel.

   (c) Are L_1 and L_2 perpendicular?
   Solution: Since $m_1 m_2 = -1$, L_1 and L_2 perpendicular to each other.

   (d) Do they intersect? If yes, what is their intersection point?
   Solution: As they are not paralel, certainly they intersect.
   We can write the lines as $y = m_i x + n_i$ linear function form where m_i being the slope and n_i is the y intercept of line i.
   L_1: y=$2x + n_1$ →inserting point (1,3) → 3=2+n_1 → $n_1 = 1$ → $L_1 : y = 2x + 1$
   L_2: y=$\frac{-1}{2}x + n_2$ →inserting point (2,1)→ 1=-1+n_2 → $n_2 = 2$ → L_2: $y = \frac{-1}{2}x + 2$
   The intersection point is the point of both of the lines.

Hence $y = 2x + 1 = \frac{-1}{2}x + 2 \rightarrow x = \frac{2}{5}$
Plugging $x = \frac{2}{5}$ in one of the functions $y = 2\frac{2}{5} + 1 = \frac{9}{5}$
So the intersection point of the lines L_1 and L_2 is $(\frac{2}{5}, \frac{9}{5})$

4. For the functions $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x+1}{x+2}$, find the equations $fof(x)$, $gog(x)$, $gof(x)$, and $fog(x)$, and determine their corresponding domains.

Solution: $D_f = \mathbb{R}\setminus\{0\}$, $D_g = \mathbb{R}\setminus\{-2\}$

$$fof(x) = f(f(x)) = f(x + \frac{1}{x}) = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}} = \frac{x^2+1}{x} + \frac{x}{x^2+1} = \frac{x^4+3x^2+1}{x(x^2+1)}$$

$f(x)$ is defined for $x \neq 0$, so$f(f(x))$ is only defined if $x \neq 0$ and $f(x) \in D_f$, whence

$$f(x) = x + \frac{1}{x} \neq 0 \rightarrow \frac{x^2+1}{x} \neq 0 \text{ since this inequality always holds:}$$

$$D_{fof} = D_f \cap \mathbb{R} = \mathbb{R}\setminus\{0\}$$

$$gog(x) = g(g(x)) = g(\frac{x+1}{x+2}) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{2x+3}{3x+5}$$

$g(x)$ is defined for $x \neq -2$, so $g(g(x))$ is only defined if $x \neq -2$ and $g(x) \in D_g$, whence

$$g(x) = \frac{x+1}{x+2} \neq -2 \rightarrow x + 1 \neq -2x - 4 \rightarrow x \neq -5/3$$

Therefore

$$D_{gog} = D_g \cap \mathbb{R}\setminus\{-5/3\} = \mathbb{R}\setminus\{-2, -5/3\}$$

$$gof(x) = g(f(x)) = g(x + \frac{1}{x}) = \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{x^2+x+1}{x^2+2x+1}$$

$f(x)$ is defined for $x \neq 0$, so $g(f(x))$ is only defined if $x \neq 0$ and $f(x) \in D_g$, whence

$$f(x) = x + \frac{1}{x} \neq -2 \rightarrow \frac{x^2+2x+1}{x} \neq 0 \rightarrow x \neq -1$$

So

$$D_{gof} = D_f \cap \mathbb{R}\setminus\{-1\} = \mathbb{R}\setminus\{-1, 0\}$$

$$fog(x) = f(g(x)) = f(\frac{x+1}{x+2}) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{(x+1)^2 + (x+2)^2}{(x+2)(x+1)} = \frac{2x^2+6x+5}{x^2+3x+2}$$

$g(x)$ is defined for $x \neq -2$, so $f(g(x))$ is only defined if $x \neq -2$ and $g(x) \in D_f$, whence

$$g(x) = \frac{x+1}{x+2} \neq 0 \rightarrow\rightarrow x \neq -1$$

Thus

$$D_{fog} = D_g \cap \mathbb{R}\setminus\{-1\} = \mathbb{R}\setminus\{-1, -2\}$$

5. Prove that the functions

$$\begin{aligned} f &: \mathbb{R}\backslash\{2\} \rightarrow \mathbb{R}\backslash\{0\}, \ x \rightarrow f(x) = \frac{1}{x-2} \\ g &: \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}\backslash\{2\}, x \rightarrow g(x) = \frac{1}{x} + 2 \end{aligned}$$

are each other's inverses.

Note: This shows that f and g are invertible, hence we also know both are bijective. Finding an inverse function $[g(x) = f^{-1}(x)]$ and proving $fog(x) = gof(x) = x$ $=\mathcal{I}(x)$ (an identitiy function) is an alternative way of showing bijectivity.

Solution: For all $x \in \mathbb{R}\backslash\{0\}$, we have

$$fog(x) = f(g(x) = \frac{1}{\frac{1}{x} + 2 - 2} = x$$

Hence $fog(x) = \mathcal{I}_{\mathbb{R}\backslash\{0\}}$,where $\mathcal{I}$ is the identity function.

For all $x \in \mathbb{R}\backslash\{2\}$, we have

$$gof(x) = g(f(x) = \frac{1}{\frac{1}{x-2}} + 2 = x$$

So $gof(x) = \mathcal{I}_{\mathbb{R}\backslash\{2\}}$.Thus,$f$ and g are each other's inverses.

6. Given the set $A = \{0, 1, 2\}$, find the power set $\mathcal{P}(A)$, the cardinality of A and $\mathcal{P}(A)$.

Solution: The power set of A is the the set whose elements are all the subsets of A. Therefore

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{1,2,3\}\}$$

The cardinality of a set is the number of elements in the set. Whence, the cardinality of the set A is

$$card(A) = 3$$

The cardinality of the power set is found either by counting all the subsets of $\mathcal{P}(A)$ that is:

$$card(\mathcal{P}(A)) = 8$$

Or without using the explicit demonstration of $\mathcal{P}(A)$,its cardinality is found as:

$$card(\mathcal{P}(A)) = 2^{card(A)} = 2^3 = 8$$