MATHEMATICS - THIRD PRACTICE

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1 Logarithm and exponential.

1. Find the domain of
$$f(x) = \log_{10}\left(x^4 - 4\,x^2 + 1\right).$$
Hint: $\sqrt{2-\sqrt{3}} \approx 0.51 < \sqrt{2+\sqrt{3}} \approx 1.93$.
Solution: We have to find the x such that
$$x^4 - 4\,x^2 + 1 > 0.$$
To solve the inequality, we lower the degree defining $t = x^2$
$$t^2 - 4\,t + 1 > 0.$$
The solutions of $t^2 - 4\,t + 1 = 0$ are
$$t_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}.$$
Whence
$$t^2 - 4\,t + 1 > 0 \Leftrightarrow t < 2 - \sqrt{3} \text{ or } t > 2 + \sqrt{3}$$
Going back to the original x we have
$$t < 2 - \sqrt{3} \Leftrightarrow x^2 < 2 - \sqrt{3} \Leftrightarrow x \in \left(-\sqrt{2-\sqrt{3}}, +\sqrt{2-\sqrt{3}}\right)$$
and
$$t > 2 + \sqrt{3} \Leftrightarrow x^2 > 2 + \sqrt{3} \Leftrightarrow x \in \left(-\infty, -\sqrt{2+\sqrt{3}}\right) \cup \left(+\sqrt{2+\sqrt{3}}, +\infty\right).$$
In summary the domain is
$$D = \left(-\infty, -\sqrt{2+\sqrt{3}}\right) \cup \left(-\sqrt{2-\sqrt{3}}, +\sqrt{2-\sqrt{3}}\right) \cup \left(+\sqrt{2+\sqrt{3}}, +\infty\right)$$

2. Find the domain of the function
$$f(x) = \frac{1}{2^x - 1}.$$
Solution: We have to impose
$$2^x - 1 \neq 0 \Leftrightarrow 2^x \neq 1 \Leftrightarrow 2^x \neq 2^0 \Leftrightarrow x \neq 0.$$
Whence
$$D = \mathbb{R} \setminus \{0\}$$

3. Find the domain and study the sign of the function

$$f(x) = \log_{\frac{1}{3}}\left(1 - x^4\right).$$

Solution: to find the domain we have to impose

$$1 - x^4 > 0 \Leftrightarrow x^4 < 1$$

Again put $t^2 = x$ *to have*

$$t^2 < 1 \Leftrightarrow t \in (-1, 1)$$

which means

$$x^2 \in (-1, 1),$$

but $x^2 > -1$ *is always verified, so the previous condition is equivalent to*

$$x^2 < 1 \Leftrightarrow x \in (-1, 1).$$

Whence the domain is

$$D = (-1, 1).$$

Since the base of the logarithm is < 1 *the function is positive if and only if*

$$1 - x^4 \leq 1 \Leftrightarrow -x^4 \leq 0,$$

so we can conclude that

$$f(x) \geq 0 \forall x \in D$$

and $f(x) = 0$ *if and only if* $x = 0$.

4. Solve the inequality

$$\log_7(x) > 2.$$

Solution. The condition

$$\log_7(x) > 2.$$

is verified if and only if

$$7^{\log_7(x)} > 7^2 = 49$$

which is equivalent to

$$x > 49.$$

2 Using the *Principle of Induction*, show that:

For all $n \in N$

$$\boxed{\textbf{P:}\ 1 + 3 + 3^2 + ... + 3^{(n-1)} = \frac{(3^n - 1)}{2}}$$

For $n = 1$, is verified. Now, let $P(k)$ be true for some positive integer k,

$$1 + 3 + 3^2 + ... + 3^{(k-1)} = \frac{(3^k - 1)}{2},$$

now prove that $P(k+1)$ is true. Consider

$$\begin{aligned}1+3+3^2+\ldots+3^{k-1}+3^k &= \frac{(3^k-1)}{2}+3^k \\ &= \frac{(3^k-1)+(2)(3^k)}{2} \\ &= \frac{(1+2)3^k-1}{2} \\ &= \frac{(3)3^k-1}{2} \\ &= \frac{3^{k+1}-1}{2}\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true using all the natural numbers.

For all $n \in N$

$$\boxed{\textbf{P: } 1^3+2^3+3^3+...+n^3 = \Big(\frac{n(n+1)}{2}\Big)^2}$$

For $n=1$ is verified. Now, let $P(k)$ be true for some positive integer k,

$$1^3+2^3+3^3+...+k^3 = \Big(\frac{k(k+1)}{2}\Big)^2,$$

now prove that $P(k+1)$ is true. Consider

$$\begin{aligned}1^3+2^3+3^3+...+k^3+(k+1)^3 &= \Big(\frac{k(k+1)}{2}\Big)^2+(k+1)^3 \\ &= \Big(\frac{k^2(k+1)^2}{4}\Big)+(k+1)^3 \\ &= \Big(\frac{k^2(k+1)^2+4(k+1)^3}{4}\Big) \\ &= \Big(\frac{(k+1)^2[k^2+4(k+1)]}{4}\Big) \\ &= \Big(\frac{(k+1)^2[k^2+4k+4]}{4}\Big) \\ &= \Big(\frac{(k+1)^2(k+2)^2}{4}\Big) \\ &= \Big(\frac{(k+1)^2(k+1+1)^2}{4}\Big) \\ &= \Big(\frac{(k+1)(k+1+1)}{2}\Big)^2\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true using all the natural numbers.

For all $n \in N$

$$\boxed{\textbf{P: } (1*2)+(2*3)+(3*4)+...+n*(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]}$$

For $n=1$ is verified. Now, let $P(k)$ be true for some positive integer k,

$$(1*2)+(2*3)+(3*4)+...+k*(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right],$$

now prove that $P(k+1)$ is true. Consider

$$\begin{aligned}(1*2)+(2*3)+(3*4)+...+k*(k+1)+(k+1)*(k+2) &= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2)\\ &= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}\\ &= \frac{(k+1)(k+2)(k+3)}{3}\\ &= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true using all the natural numbers.

For all $n \in N$

$$\boxed{\textbf{P: } 1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+...+\frac{1}{(1+2+3+...n)}=\frac{2n}{(n+1)}}$$

For $n=1$ is verified. Now, let $P(k)$ be true for some positive integer k,

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+...+\frac{1}{(1+2+3+...k)}=\frac{2k}{(k+1)}\ ,$$

now prove that $P(k+1)$ is true. Consider

$$\begin{aligned}
&= 1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+...+\frac{1}{(1+2+3+...k)}+\frac{1}{(1+2+3+...k+(k+1))}\\
&= \frac{2k}{(k+1)}+\frac{1}{(1+2+3+...+k+(k+1))}\\
&= \frac{2k}{(k+1)}+\frac{1}{\frac{(k+1)(k+1+1)}{2}}\\
&= \frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}\\
&= \frac{k}{(k+1)}\Big(k+\frac{1}{(k+2)}\Big)\\
&= \frac{2}{(k+1)}\Big(\frac{k(k+2)+1}{(k+2)}\Big)\\
&= \frac{2}{(k+1)}\Big(\frac{k^2+2k+1}{(k+2)}\Big)\\
&= \frac{2(k+1)^2}{(k+1)(k+2)}\\
&= \frac{2(k+1)}{(k+2)}\\
&= \frac{2(k+1)}{((k+1)+1)}
\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true using all the natural numbers.

3 Compute each of the following limits:

$$\boxed{\lim_{n\to\infty}\frac{n^2(2n+1)(3n-2)}{2n^2(5n-8)(n+6)}} = \frac{n^2[6n^2-n-3]}{2n^2[5n^2+22n-48]} = \frac{6n^4-n^3-3}{10n^4+44n^3-96} = \frac{\frac{6n^4}{n^4}-\frac{n^3}{n^4}-\frac{3}{n^4}}{\frac{10n^4}{n^4}+\frac{44n^3}{n^4}-\frac{96}{n^4}}$$
$$= \frac{6-\frac{1}{n}-\frac{3}{n^4}}{10+\frac{44}{n}-\frac{96}{n^4}} = \frac{6}{10} = \frac{3}{5}$$

$$\boxed{\lim_{n\to\infty}\left(\frac{n^3}{2n^2-1}-\frac{n^2}{2n+1}\right)} = \frac{n^3(2n+1)-n^2(2n^2-1)}{(2n^2-1)(2n+1)} = \frac{2n^4+n^3-2n^4+n^2}{4n^3+2n^2-2n-1} = \frac{n^3+n^2}{4n^3+2n^2-2n-1}$$
$$= \frac{\frac{n^3}{n^3}+\frac{n^2}{n^3}}{\frac{4n^3}{n^3}+\frac{2n^2}{n^3}-\frac{2n}{n^3}-\frac{1}{n^3}} = \frac{1+\frac{1}{n}}{4+\frac{2}{n}-\frac{2}{n^2}-\frac{1}{n^3}} = \frac{1}{4}$$

$$\boxed{\lim_{n\to\infty}\sqrt{n^2+n}-n} = \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n} = \frac{(\sqrt{n^2+n})^2+n\sqrt{n^2+n}-n\sqrt{n^2+n}-n^2}{\sqrt{n^2+n}+n}$$
$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2(\frac{n^2}{n^2}+\frac{n}{n^2})}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

$$\boxed{\lim_{n\to\infty}\frac{4n^2+n^6}{1-5n^3}} = \frac{\frac{4n^2}{n^3}+\frac{n^6}{n^3}}{\frac{1}{n^3}-\frac{5n^3}{n^3}} = \frac{\frac{4}{n}+n^3}{\frac{1}{n^3}-5} = -\infty$$

$$\boxed{\lim_{n\to\infty}\frac{2n^4-n^2+8n}{-5n^4+7}} = \frac{\frac{2n^4}{n^4}-\frac{n^2}{n^4}+\frac{8n}{n^4}}{-\frac{5n^4}{n^4}+\frac{7}{n^4}} = \frac{2-\frac{1}{n^2}+\frac{8}{n^3}}{-5+\frac{7}{n^4}} = -\frac{2}{5}$$