Week 6

Prof. C. Lhotka
(slides by Prof. K. Colaneri)

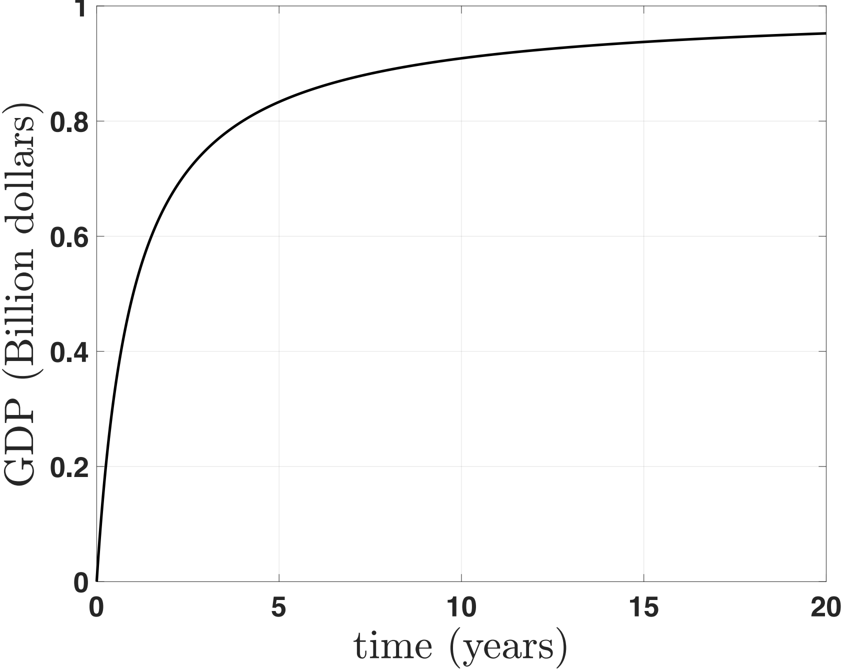
Mathematics I

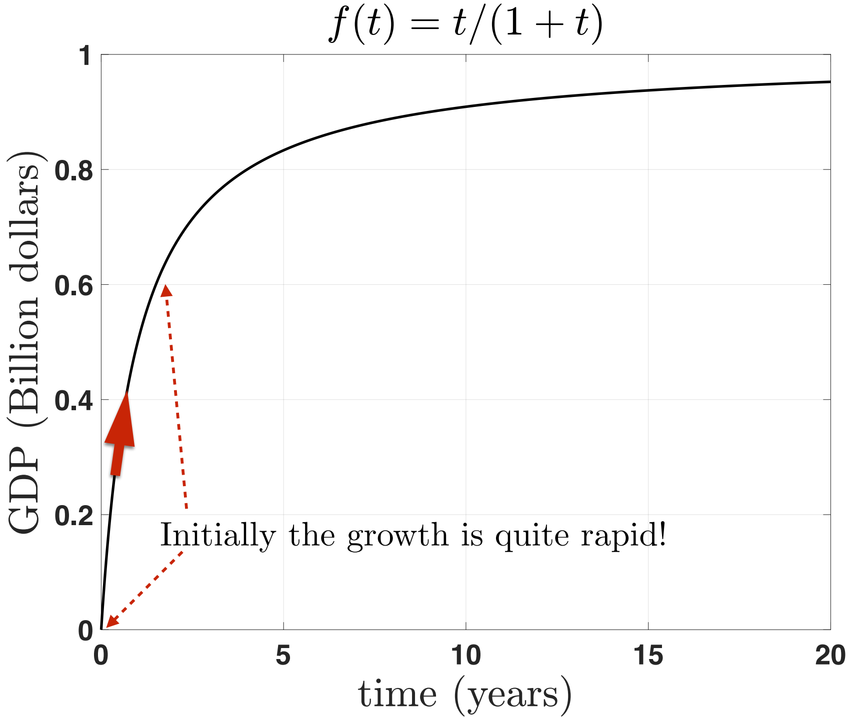
University of Rome Tor Vergata

20-26 October, 2024

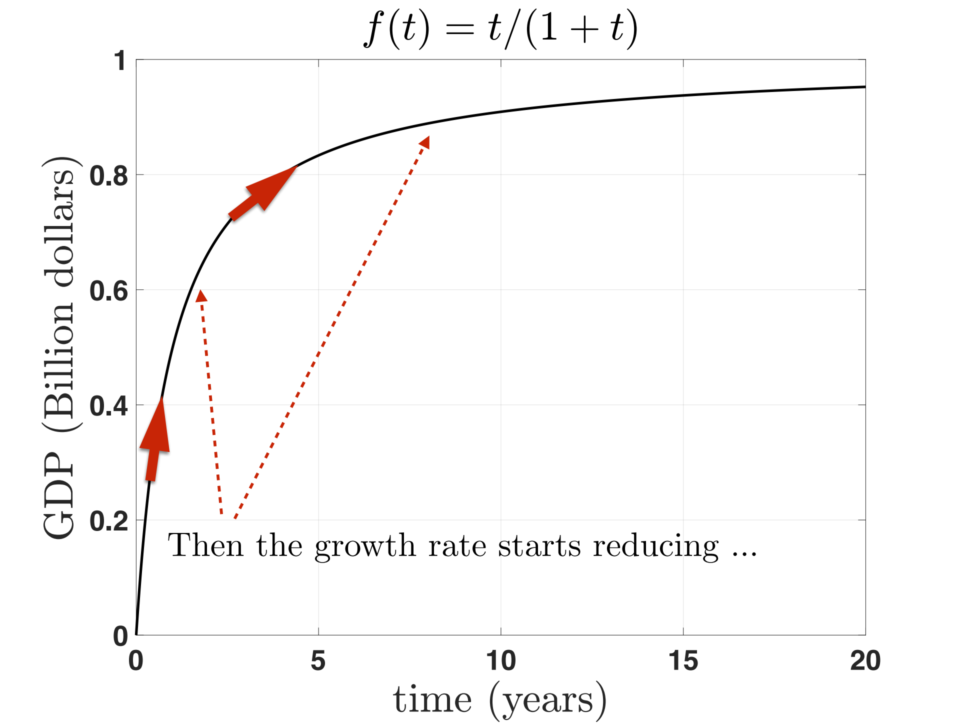
Outline

$f(t) = t/(1+t)$

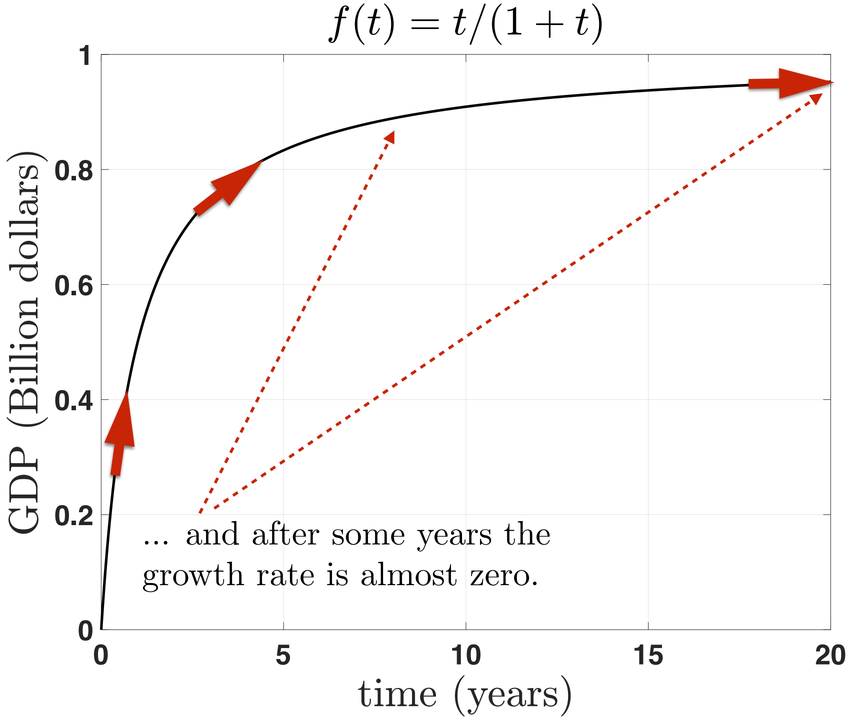




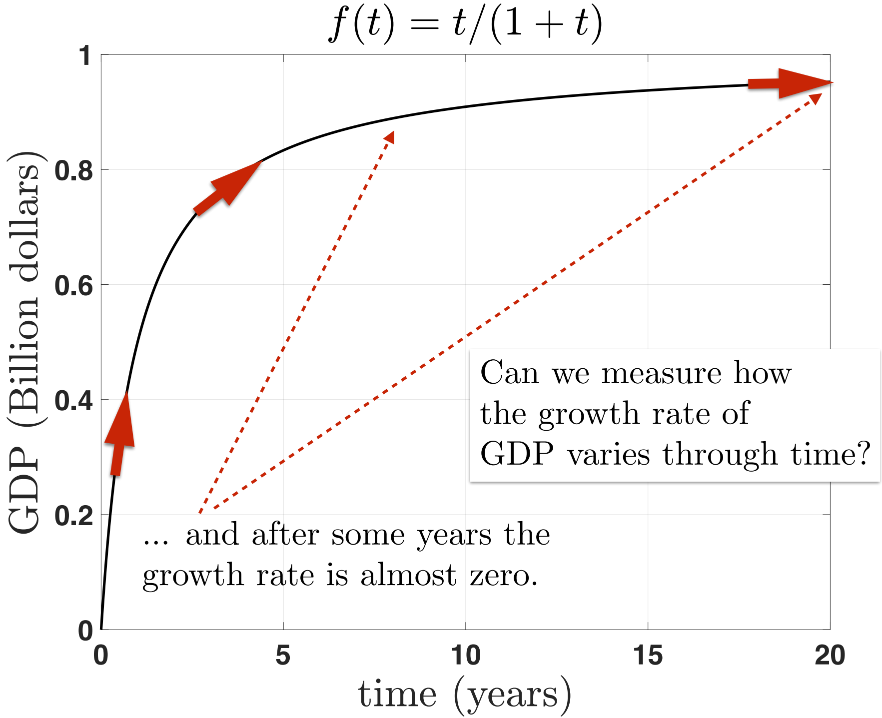
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
Initially the growth is quite rapid!



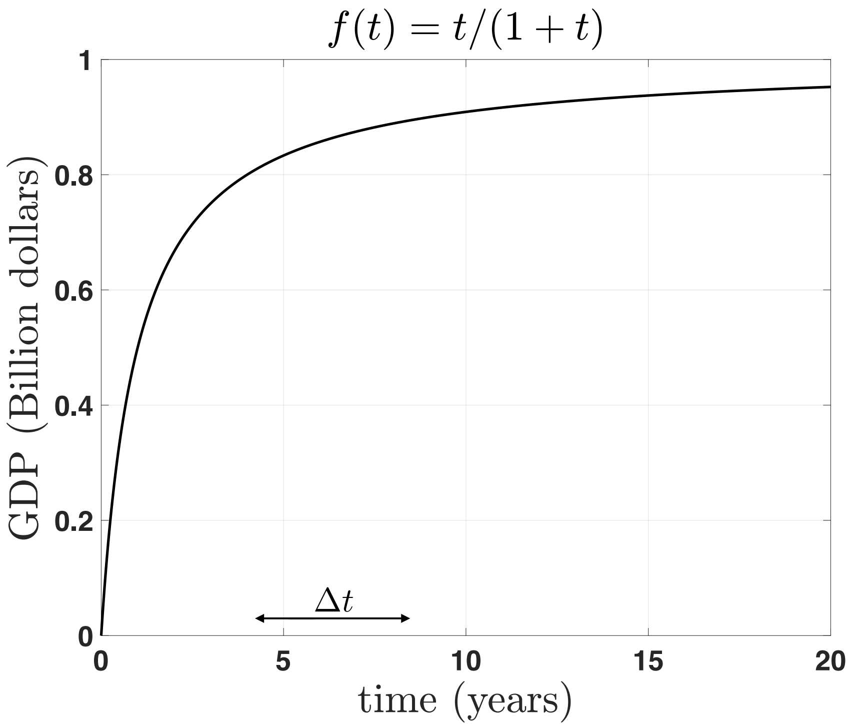
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Then the growth rate starts reducing ...
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20



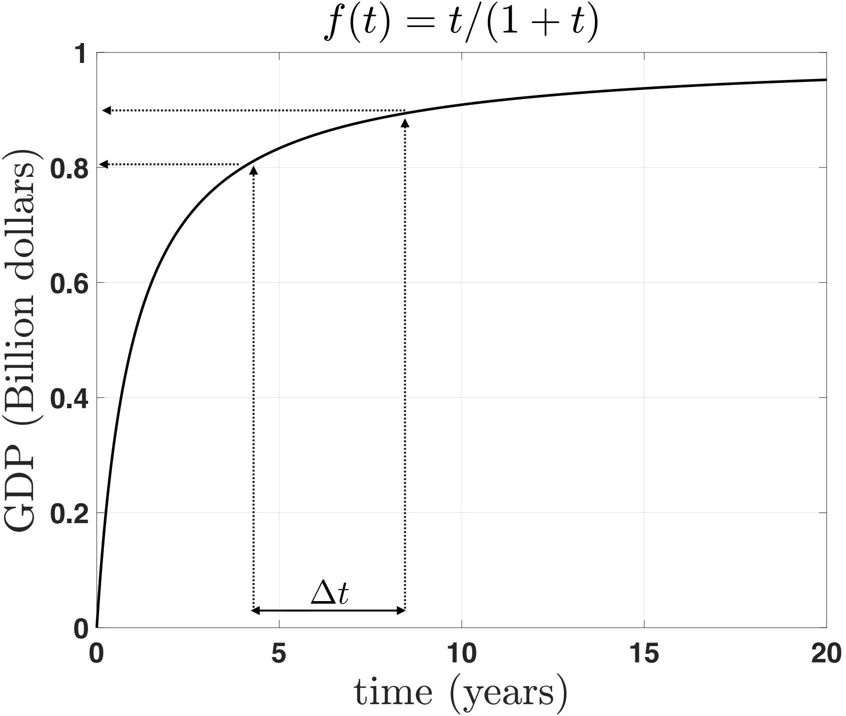
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
... and after some years the
growth rate is almost zero.
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20



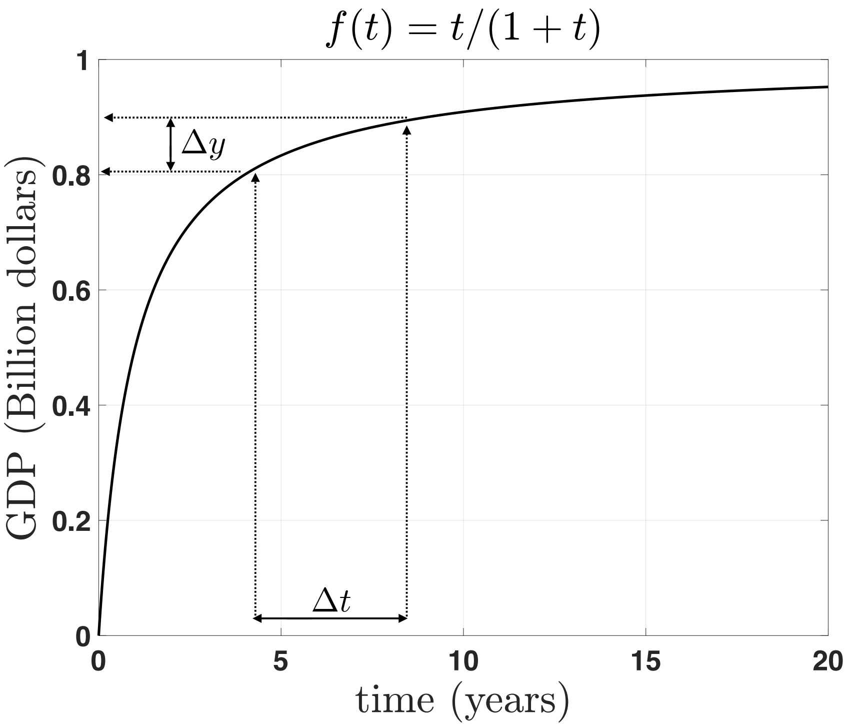
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
Can we measure how the growth rate of GDP varies through time?
... and after some years the growth rate is almost zero.



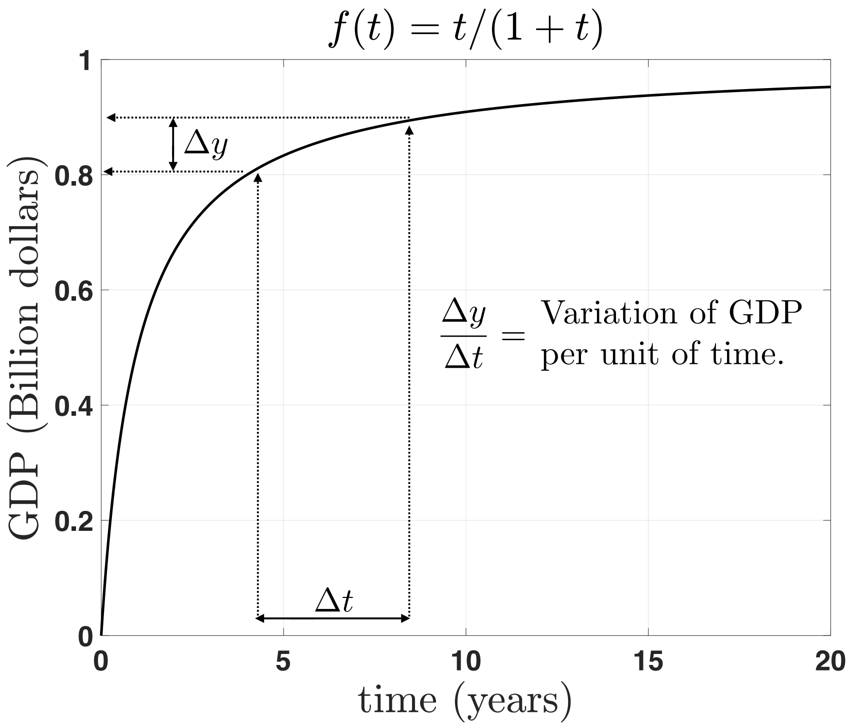
$f(t) = t/(1+t)$
GDP (Billion dollars)
1
0.8
0.6
0.4
0.2
0
Δt
0
5
10
15
20
time (years)

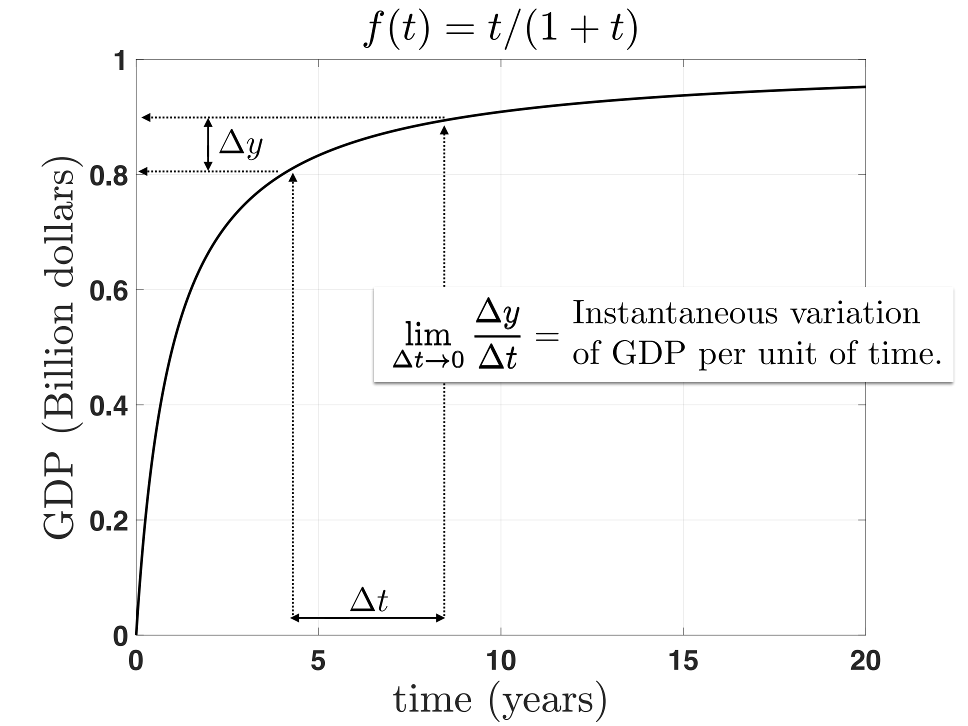


$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δt
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

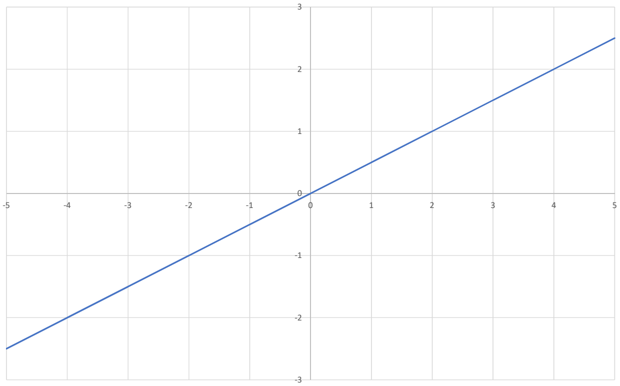


$f(t) = t/(1+t)$
Δy
Δt
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
GDP (Billion dollars)
time (years)

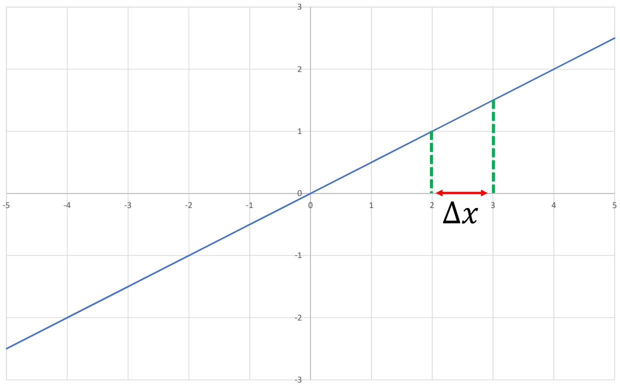

$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δy
Δt
$\frac{\Delta y}{\Delta t} =$ Variation of GDP per unit of time.
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20



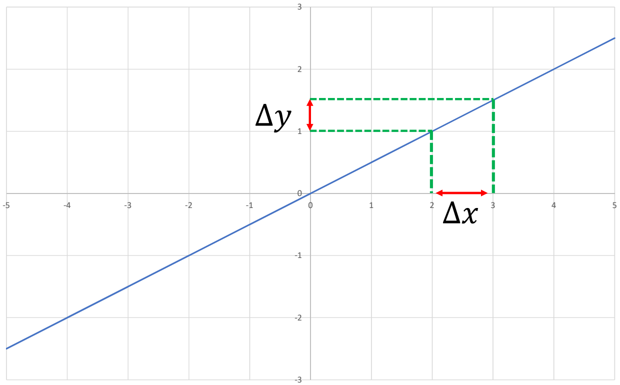
$f(t) = t/(1+t)$
GDP (Billion dollars)
time (years)
Δy
Δt
$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} =$ Instantaneous variation of GDP per unit of time.
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20



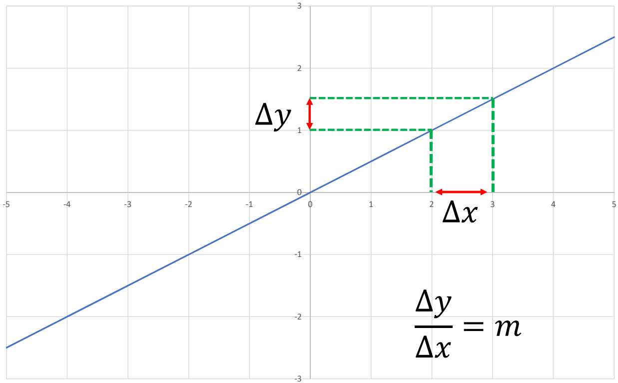
3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5



3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Δx



3
2
1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Δy
Δx



$$\frac{\Delta y}{\Delta x} = m$$

Slope of the line indicates the rate of change

Incremental Ratio

Definition

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0, x_1 \in (a, b)$. We call incremental ratio the ratio

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_1; f(x_1))$

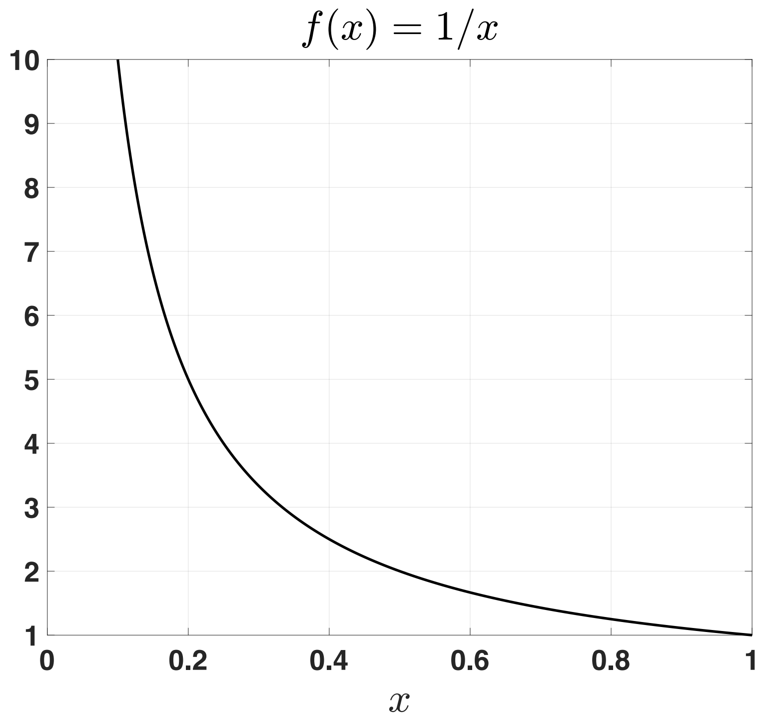
Incremental Ratio

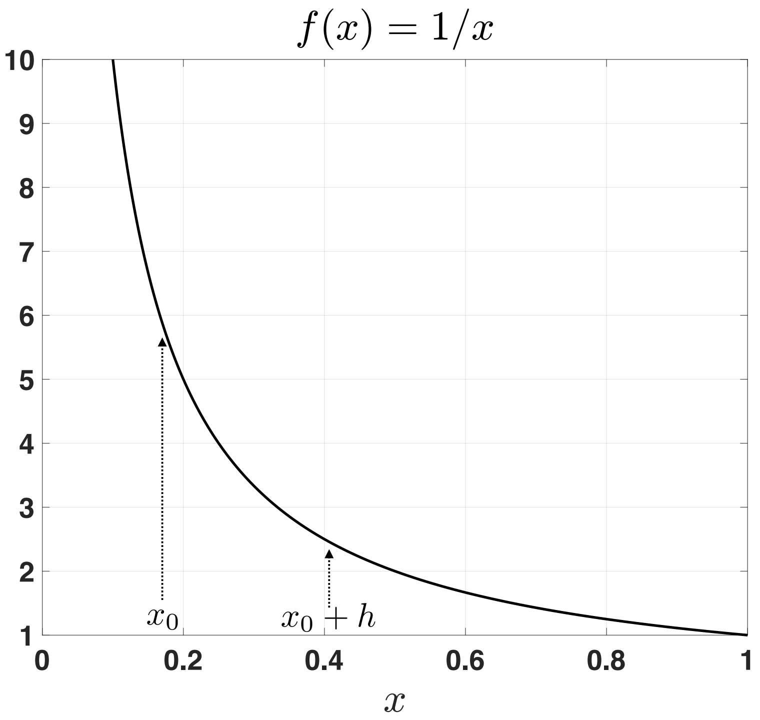
Definition

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let $x_0 \in (a, b)$ and let $h \in \mathbb{R}$ such that $x_0 + h \in (a, b)$. We call incremental ratio the ratio

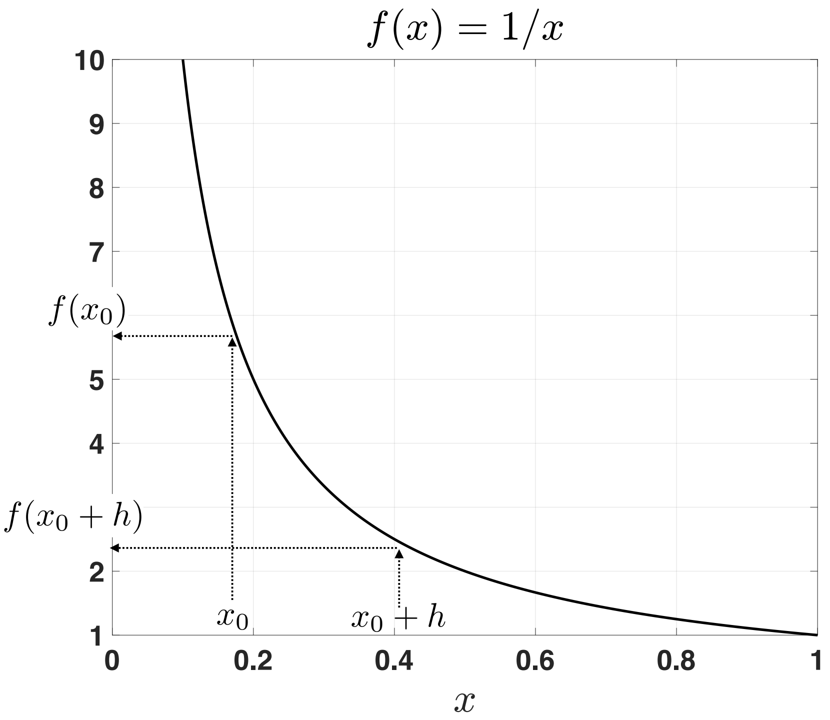
$$\frac{f(x_0 + h) - f(x_0)}{h}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_0 + h; f(x_0 + h))$

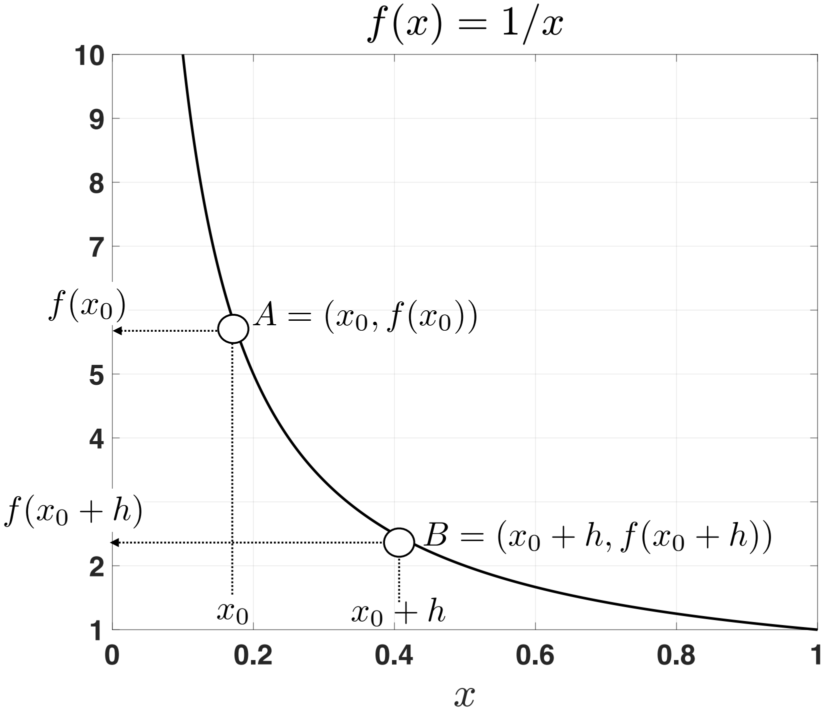

$f(x) = 1/x$
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x



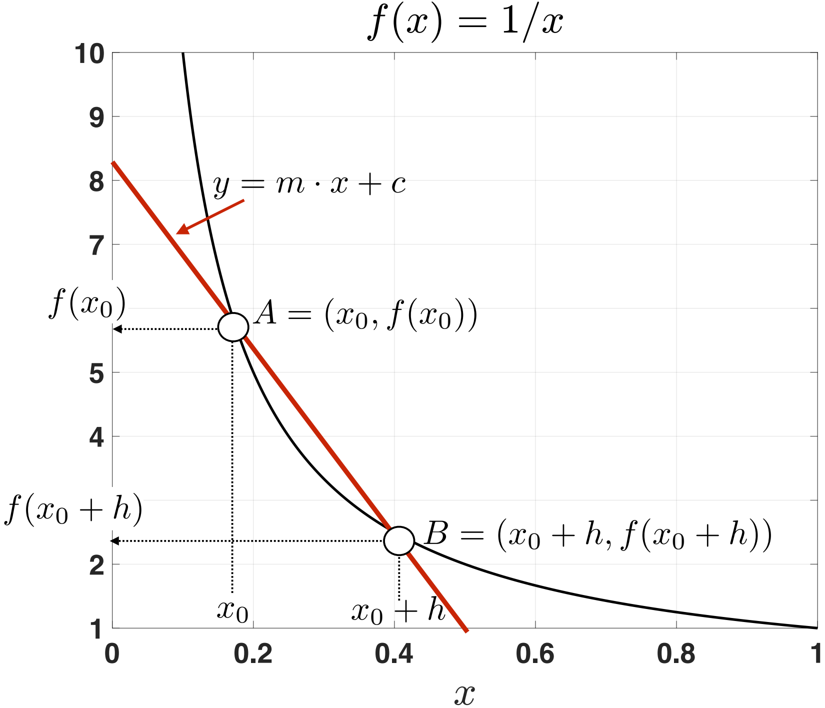
$f(x) = 1/x$
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x_0
$x_0 + h$
x



$f(x) = 1/x$
10
9
8
7
$f(x_0)$
5
4
$f(x_0 + h)$
2
1
x_0
$x_0 + h$
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
10
9
8
7
$f(x_0)$
$A = (x_0, f(x_0))$
5
4
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
2
1
x_0
$x_0 + h$
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
$y = m \cdot x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
x_0
$x_0 + h$
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1
x

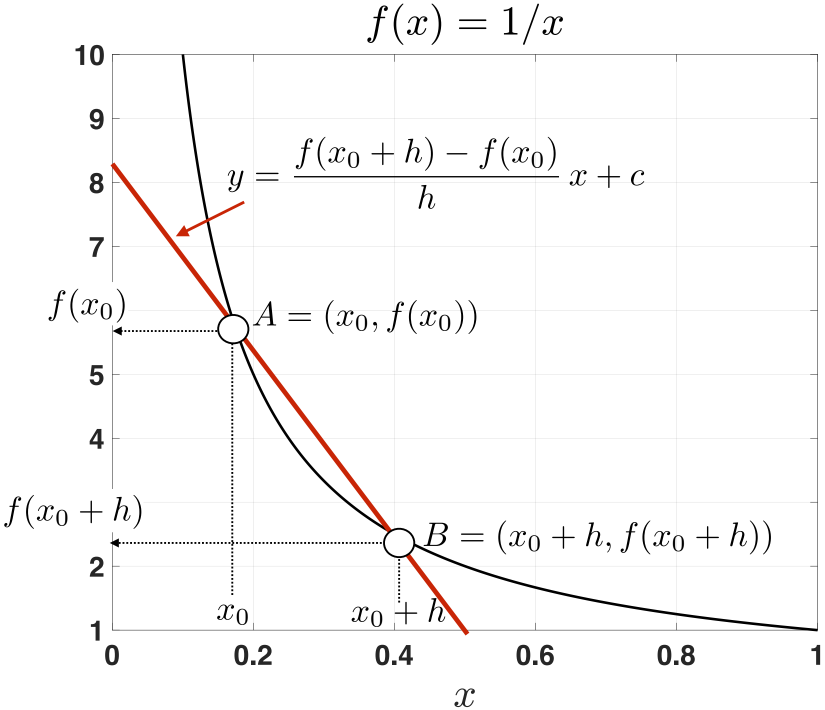
Derivatives

Definition

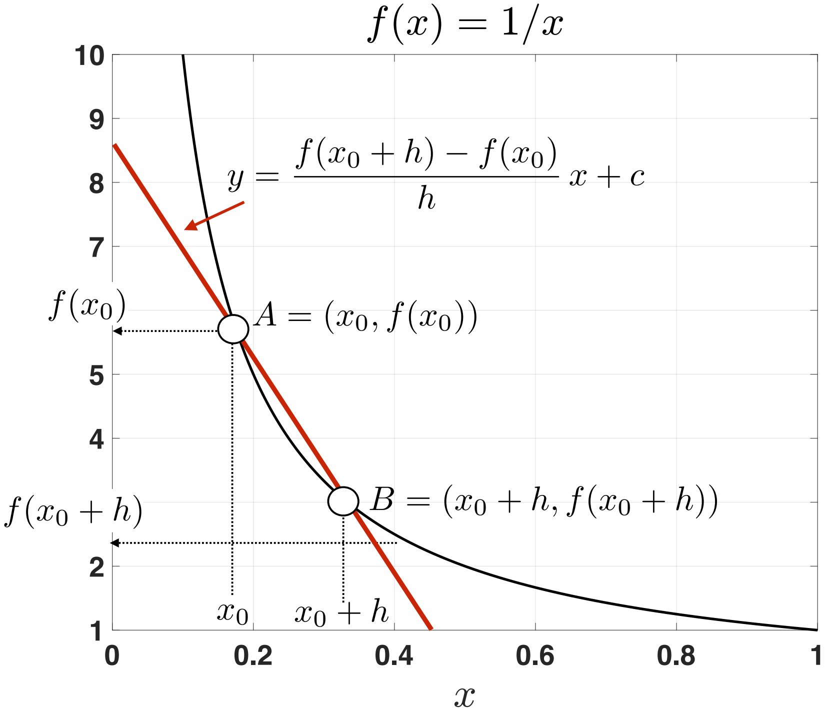
Let $f : D \to \mathbb{R}$ be a function and let $[a, b] \subset D$. We say that f is differentiable at x_0 if:

- $x_0 \in (a, b)$
- $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h} = d$ exists and it is finite.

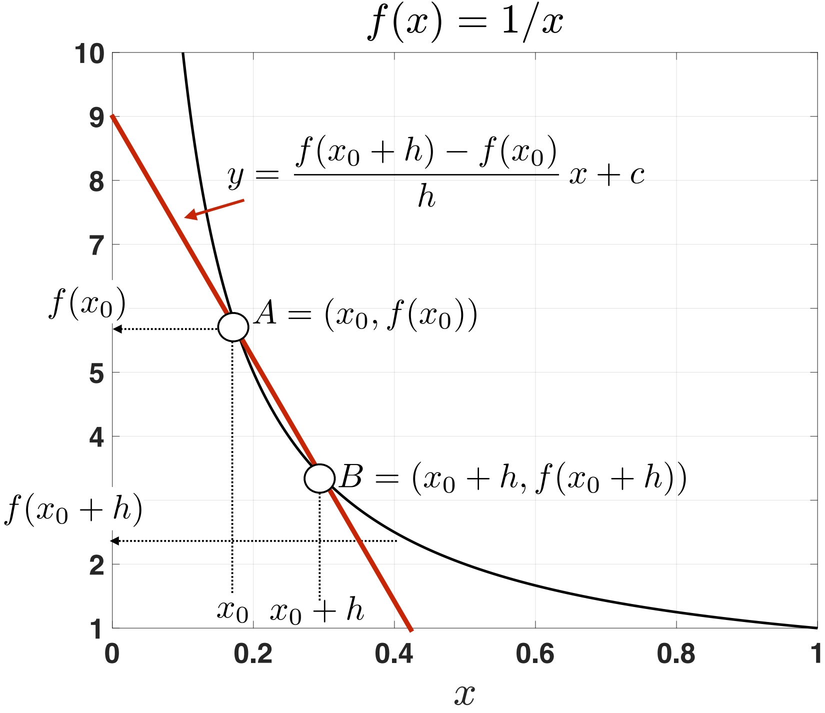
If this is the case we call $d = f'(x_0)$ and we say that d is the derivative of f at the point x_0.

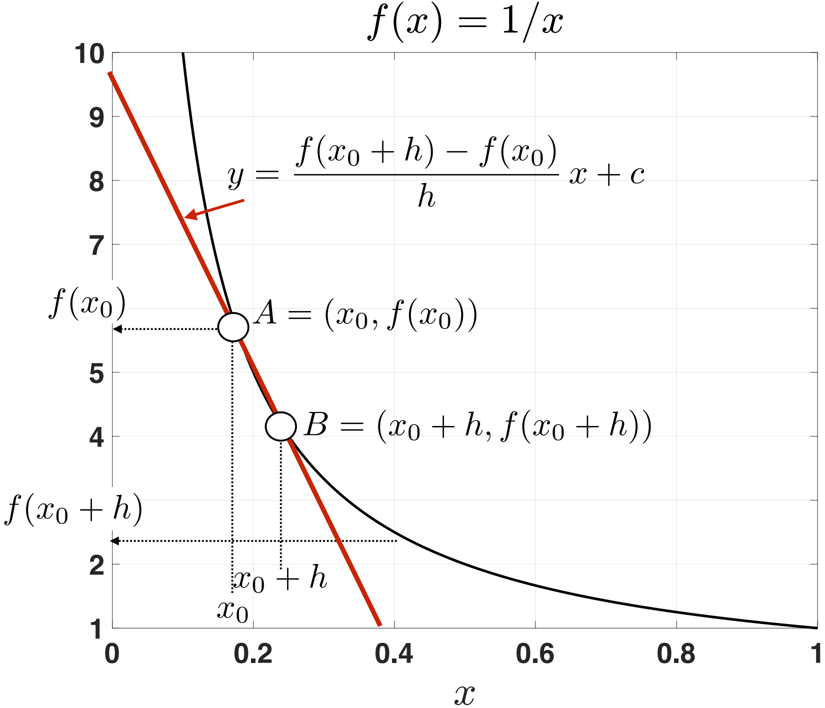


$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0+h)$
$B = (x_0+h, f(x_0+h))$
x_0
$x_0 + h$
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$B = (x_0+h, f(x_0+h))$
$f(x_0+h)$
x_0
$x_0 + h$
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1
x


$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$B = (x_0+h, f(x_0+h))$
$f(x_0+h)$
x_0
$x_0 + h$
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1
x



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$B = (x_0 + h, f(x_0 + h))$
$f(x_0 + h)$
$x_0 + h$
x_0
x
10
9
8
7
5
4
2
1
0
0.2
0.4
0.6
0.8
1

Derivatives of elementary functions

- Let $f(x) = k$, with $k \in \mathbb{R}$ (i.e. the constant function). $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

  $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = 0$$

  The last equality is not an undetermined form since the numerator is equal to 0 no matter the value of h.

- Let $f(x) = x$. $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

  $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Derivatives of elementary functions

- Let $f(x) = x^2$. $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = 2x$$

- Let $f(x) = e^x$. $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{hto0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \underbrace{\frac{e^h - 1}{h}}_{1} = e^x$$

Derivatives of elementary functions

- Let $f(x) = \log(x)$. $D = (0, +\infty)$.
  Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\log(x+h) - \log(x)}{h}$$
$$= \lim_{h\to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{h} = \lim_{h\to 0} \underbrace{\frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}}_{1} \frac{1}{x} = \frac{1}{x}$$

- Let $f(x) = \sin(x)$. $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{hto0} \frac{\sin(x+h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \underbrace{\frac{\sin(x)(\cos(h) - 1)}{h}}_{0} + \lim_{h\to 0} \cos(x) \underbrace{\frac{\sin(h)}{h}}_{1} = \cos(x)$$

Derivatives of elementary functions

- Let $f(x) = \cos(x)$. $D = \mathbb{R}$.
  Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{hto0} \frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(x) - \cos(x)}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\cos(x)(\cos(h)-1)}{h}}_{0} - \lim_{h\to 0} \sin(x) \underbrace{\frac{\sin(h)}{h}}_{1} = -\sin(x)$$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$
exponential	e^x	e^x

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	x^α, $\alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$
exponential	e^x	e^x
logarithm	$\log x$	$\frac{1}{x}$

Operations with derivatives

Theorem

Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ be two functions such that $f'(x)$ and $g'(x)$ exists. Then

$$\begin{aligned}
D(f(x)+g(x)) &= f'(x)+g'(x) \\
D(f(x)-g(x)) &= f'(x)-g'(x) \\
D(kf(x)) &= kf'(x) \\
D(f(x)\cdot g(x)) &= f'(x)\cdot g(x)+f(x)\cdot g'(x) \\
D\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)} \\
D((f\circ g)(x)) &= D(f(g(x))) = f'(g(x))\cdot g'(x)
\end{aligned}$$

Exercises: Compute the derivatives

1. $f(x) = 4x^3 - 3x^2 + x + 3$
2. $f(x) = \sin x + \cos x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \tan x$
5. $f(x) = x^{\sqrt{2}}$
6. $f(x) = \log x + \log_5 x$
7. $f(x) = 2x + 3e^x$
8. $f(x) = (2x+3)^4 + e^{2x}$
9. $f(x) = \cos(x^2) + \cos^2(x)$
10. $f(x) = x^2 \sin(3x)$
11. $f(x) = \frac{x \log(x)}{\sin(x)}$
12. $f(x) = \frac{x^2+\sqrt{x}}{(2x-1)^3}$

Checklist

Let $f : D \to \mathbb{F}$ and let $x_0 \in D$. Then f is differentiable at x_0 if the following three conditions hold:

1. f is continuous in x_0
2. left and right limits of the incremental ratio must coincide
3. the limit must be finite

Non-differentiability points

- if f is not defined at x_0, that is $x_0 \notin D$, then clearly it cannot be differentiable (because in this case the function does not exists at the point x_0)
- if

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = d_1, \qquad d_1 \neq \pm\infty$$
$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = d_2, \qquad d_2 \neq \pm\infty$$

but $d_1 \neq d_2$, then the function is NOT differentiable at x_0, and $(x_0, f(x_0))$ is called an **angle point**

Non-differentiability points

- if

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty, \qquad (-\infty)$$
$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = +\infty, \qquad (-\infty)$$

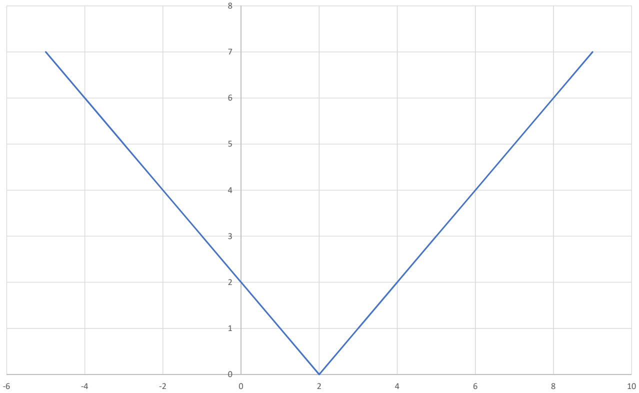
(i.e. left and right limits are infinite of the same sign!) then the function is NOT differentiable at x_0, and $(x_0, f(x_0))$ is called an **inflection point with vertical tangent**

- if

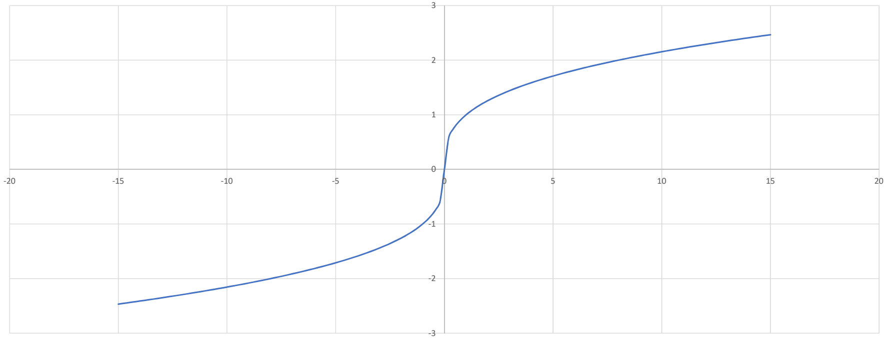
$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty \qquad (-\infty)$$
$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = -\infty, \qquad (+\infty)$$

(i.e. left and right limits are infinite of the different sign!) then the function is NOT differentiable at x_0, and $(x_0, f(x_0))$ is called a **cusp point**

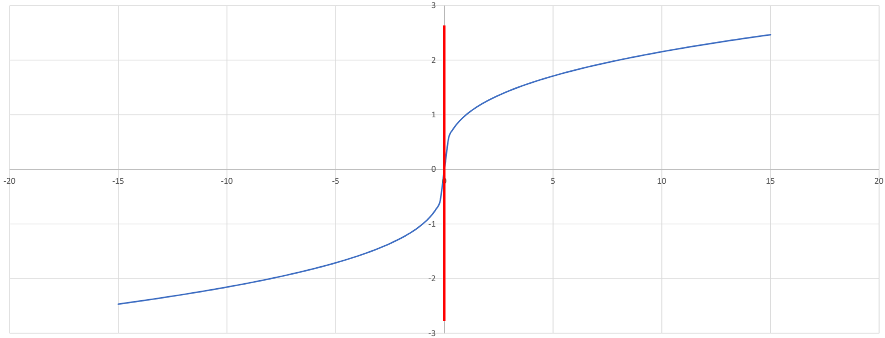
$$f(x) = |x - 2|$$



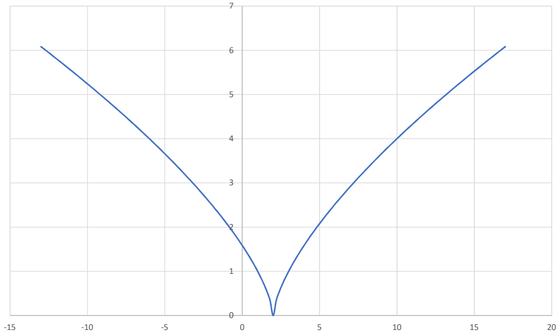
$$f(x) = \sqrt[3]{x}$$



$$f(x) = \sqrt[3]{x}$$



$$f(x) = \sqrt[3]{(x-2)^2}$$



First order Taylor approximation

Suppose that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} = f'(x_0) \tag{1}$$

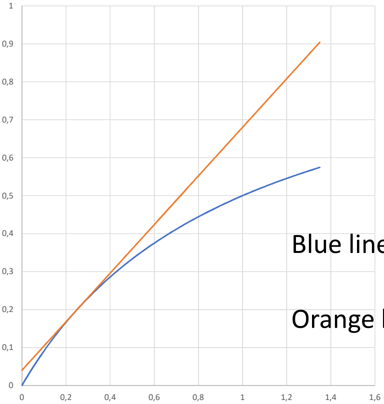
Then if x is close to x_0, we can say that

$$f'(x_0) \sim \frac{f(x) - f(x_0)}{(x - x_0)}$$

Equivalently

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0)$$



Blue line: $f(x)$

Orange line: tangent at $x_0 = \frac{1}{4}$

First order Taylor approximation

- The function

$$P(x) = f(x_0) + f'(x_0)(x - x_0)$$

  is called the first order Taylor approximation of f or the linearization of f or the first order Taylor polynomial of f.

- The quantity

$$R(x_1) = f(x_1) - P(x_1)$$

  is the reminder or the (absolute) error, for every $x_1 \neq x_0$.

- The relative error is $\epsilon(x_1) = \frac{R(x_1)}{f(x_1)}$

The approximation $P(x)$ is good if

- if x is close to x_0, i.e. $|x - x_0| < \delta$
- f is almost flat

Differentiability and continuity

Theorem

Let $f : D \to \mathbb{R}$ and let $x_0 \in D$. If f is differentiable at x_0 then it is continuous in x_0.

Proof: Notice that we already know that $x_0 \in D$. Therefore we need to show that $\lim_{x\to x_0} f(x)$ exists and it is equal to $f(x_0)$.
For all $x \neq x_0$ we can write

$$f(x) = f(x) - f(x_0) + f(x_0)$$
$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0)$$

Now, we take the limit as $x \to x_0$ on both sides:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} \underbrace{(x - x_0)}_{0} + f(x_0) = f(x_0)$$

This concludes the proof.

Differentiability and continuity

Problem If a function is continuous, is it also differentiable ? **No!**

Differentiability and continuity

Problem If a function is continuous, is it also differentiable ? **No!**
Example: The function absolute value

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases},$$

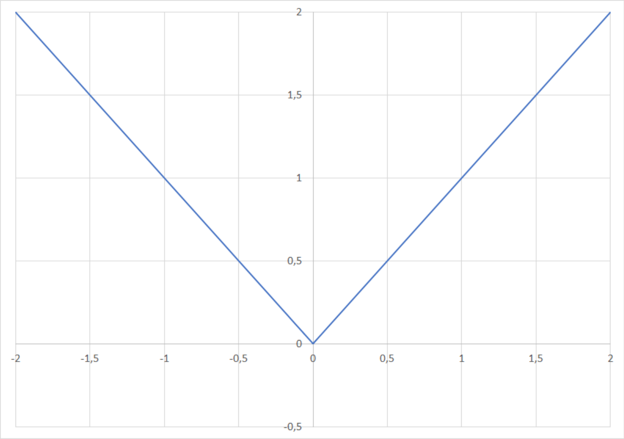
This function is continuous in for all $x \in \mathbb{R}$.
However, if we compute the left and right limit of incremental ratio at $x_0 = 0$ we get

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$
$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

Hence f is not differentiable at $x_0 = 0$ and $x_0 = 0$ is an angle point.

Example

$$f(x) = |x|$$



Derivatives: Rolle's Theorem

Theorem

If a function $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b)

Derivatives: Rolle's Theorem

Theorem

If a function $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b) *and* $f(a) = f(b)$ *then* $\exists x_0 \in (a, b)$ *such that* $f'(x_0) = 0$.

Derivatives: Rolle's Theorem

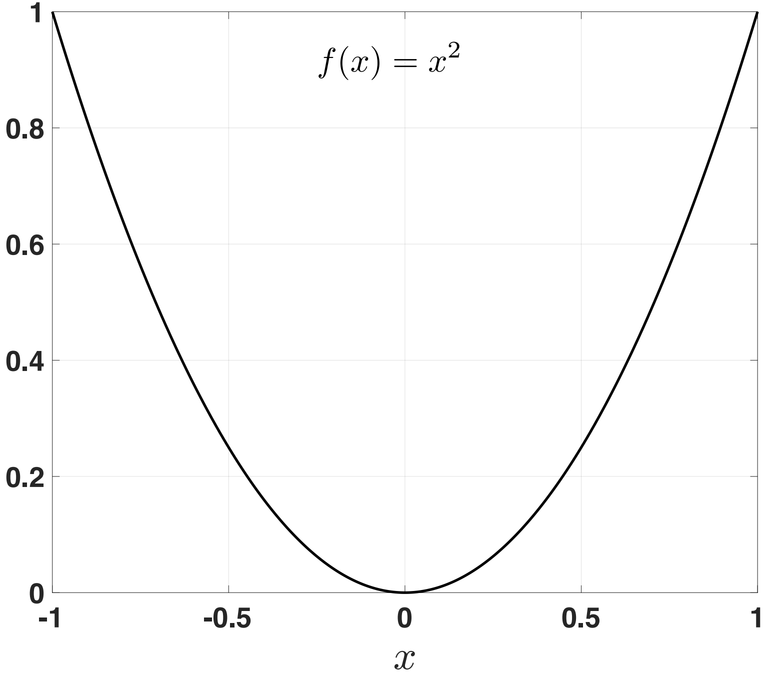
Theorem

If a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

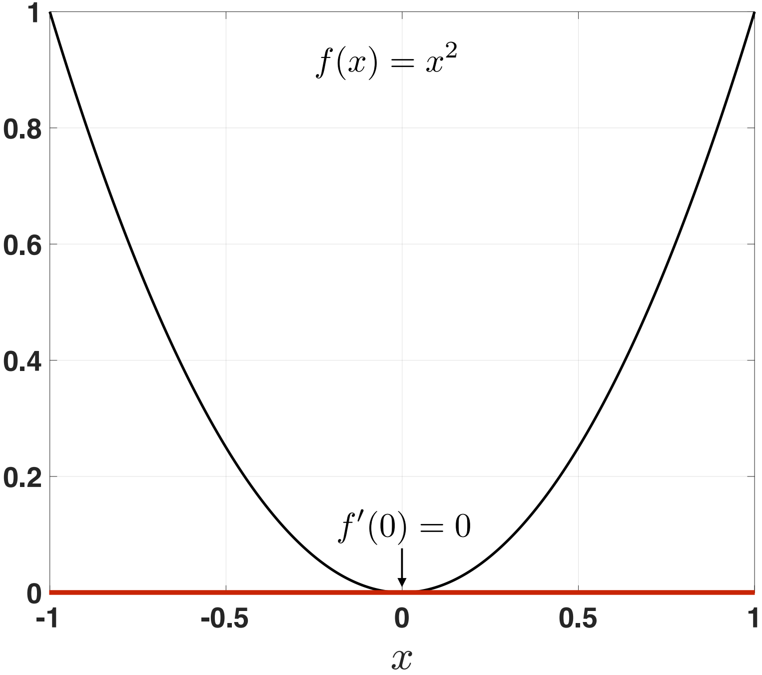
Rolle Theorem says that there exists at least ONE point in $[a, b]$ with horizontal tangent line!

Checklist:

- $[a, b]$ is closed and bounded
- The function is continuous in $[a, b]$
- The domain of the derivative MUST include the set (a, b)
- $f(a) = f(b)$



$f(x) = x^2$
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x



$f(x) = x^2$
$f'(0) = 0$
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x

Derivatives: Lagrange's Mean Value Theorem.

Theorem

Let $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable in* (a, b)*.*

Derivatives: Lagrange's Mean Value Theorem.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Then there exists at least one point $x_0 \in (a, b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

That means there is at least one point x_0 where the tangent to the graph of f is parallel to the line from $(a, f(a))$ to $(b, f(b))$

Derivatives: Lagrange's Mean Value Theorem.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Then there exists at least one point $x_0 \in (a, b)$ such that

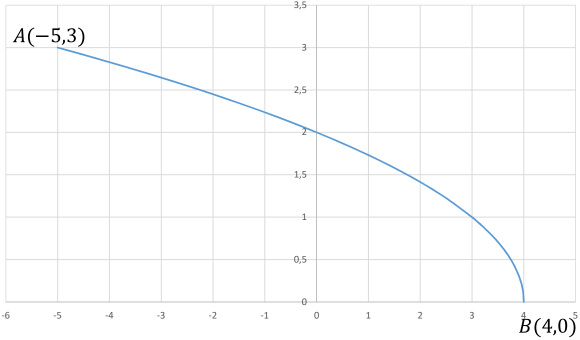
$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

That means there is at least one point x_0 where the tangent to the graph of f is parallel to the line from $(a, f(a))$ to $(b, f(b))$

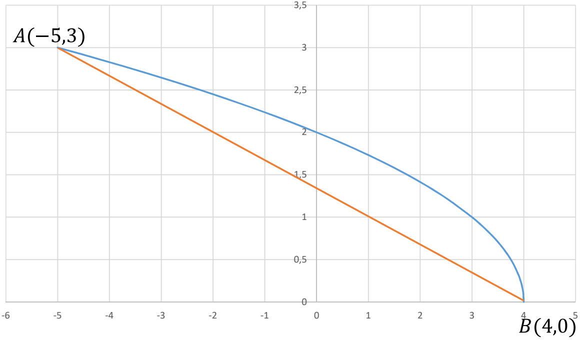
Checklist:

- $[a, b]$ is closed and bounded
- The function is continuous in $[a, b]$
- The domain of the derivative MUST include the set (a, b)

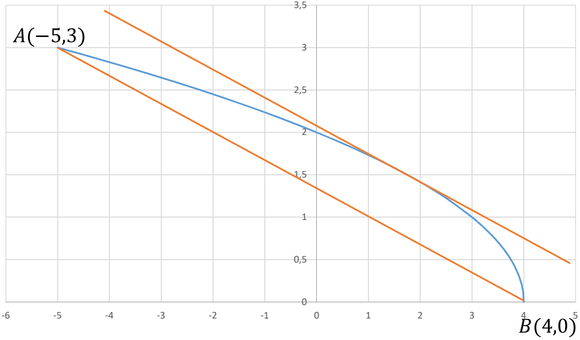
$$f(x) = \sqrt{4 - x} \text{ on } [-5,4]$$



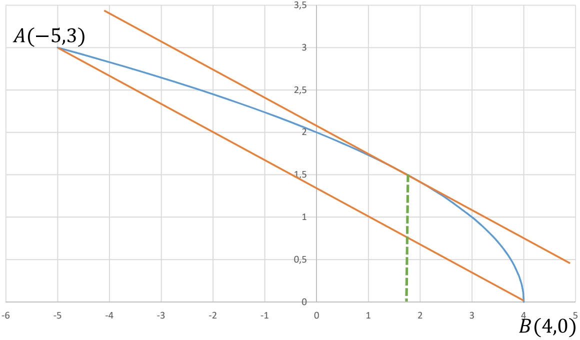
$$f(x) = \sqrt{4 - x} \text{ on } [-5,4]$$



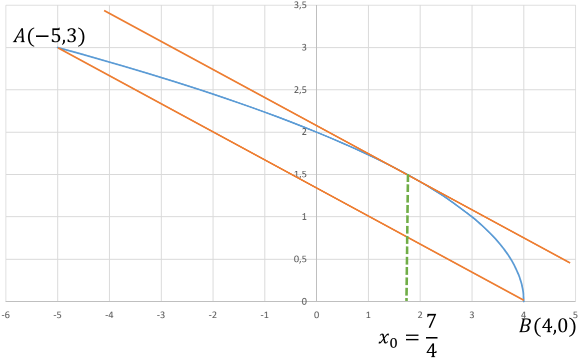
$$f(x) = \sqrt{4 - x} \text{ on } [-5,4]$$



$$f(x) = \sqrt{4 - x} \text{ on } [-5,4]$$



$$f(x) = \sqrt{4 - x} \text{ on } [-5,4]$$



Increasing and Decreasing functions

Theorem

Necessary and sufficient conditions for monotonicity of differentiable functions
Let $f : D \to \mathbb{R}$ be a function and assume that f is differentiable in any open interval $I \subseteq D$. Then:

- *f is increasing in I if and only if $f'(x) \geq 0$ for all $x \in I$*

Increasing and Decreasing functions

Theorem

Necessary and sufficient conditions for monotonicity of differentiable functions
Let $f : D \to \mathbb{R}$ be a function and assume that f is differentiable in any open interval $I \subseteq D$. Then:

- *f is increasing in I if and only if $f'(x) \geq 0$ for all $x \in I$*
- *f is strictly increasing in I if and only if $f'(x) > 0$ for all $x \in I$*

Increasing and Decreasing functions

Theorem

Necessary and sufficient conditions for monotonicity of differentiable functions
Let $f : D \to \mathbb{R}$ be a function and assume that f is differentiable in any open interval $I \subseteq D$. Then:

- *f is increasing in I if and only if $f'(x) \geq 0$ for all $x \in I$*
- *f is strictly increasing in I if and only if $f'(x) > 0$ for all $x \in I$*
- *f is decreasing in I if and only if $f'(x) \leq 0$ for all $x \in I$*

Increasing and Decreasing functions

Theorem

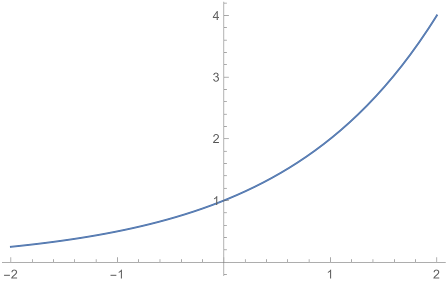
Necessary and sufficient conditions for monotonicity of differentiable functions
Let $f : D \to \mathbb{R}$ be a function and assume that f is differentiable in any open interval $I \subseteq D$. Then:

- *f is increasing in I if and only if $f'(x) \geq 0$ for all $x \in I$*
- *f is strictly increasing in I if and only if $f'(x) > 0$ for all $x \in I$*
- *f is decreasing in I if and only if $f'(x) \leq 0$ for all $x \in I$*
- *f is strictly decreasing in I if and only if $f'(x) < 0$ for all $x \in I$*

Pay attention: The interval I can be unbounded.

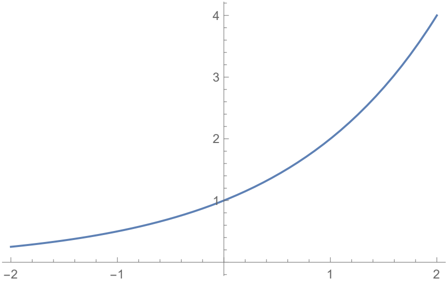
Increasing and decreasing functions: examples

$$f(x) = 2^x$$



Increasing and decreasing functions: examples

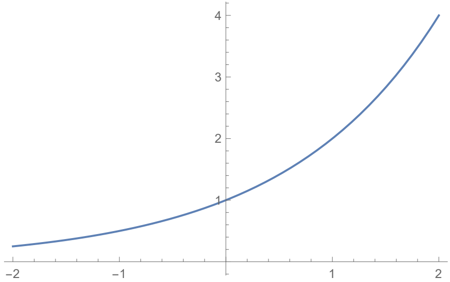
$$f(x) = 2^x$$



The function $f(x) = 2^x$ is differentiable in $\mathbb{R}$.

Increasing and decreasing functions: examples

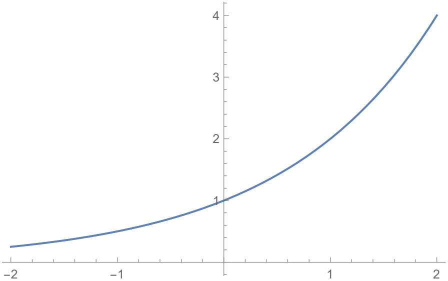
$$f(x) = 2^x$$



The function $f(x) = 2^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2^x \log 2$

Increasing and decreasing functions: examples

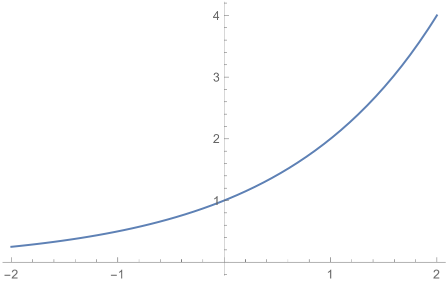
$$f(x) = 2^x$$



The function $f(x) = 2^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2^x \log 2 > 0$, $\forall x \in \mathbb{R}$.

Increasing and decreasing functions: examples

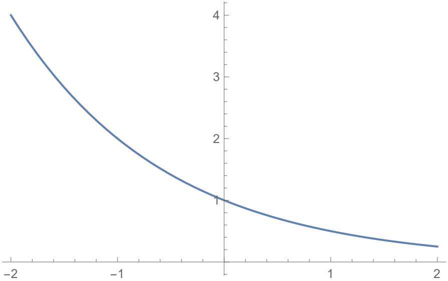
$$f(x) = 2^x$$



The function $f(x) = 2^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2^x \log 2 > 0$, $\forall x \in \mathbb{R}$. Thus, the function is strictly increasing in $\mathbb{R}$.

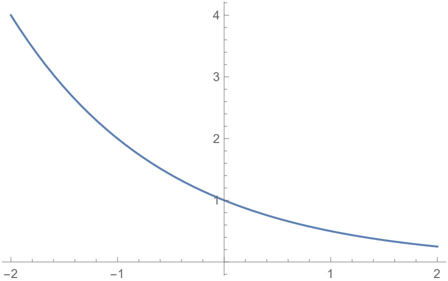
Increasing and decreasing functions: examples, cont'd

$$f(x) = \left(\frac{1}{2}\right)^x$$



Increasing and decreasing functions: examples, cont'd

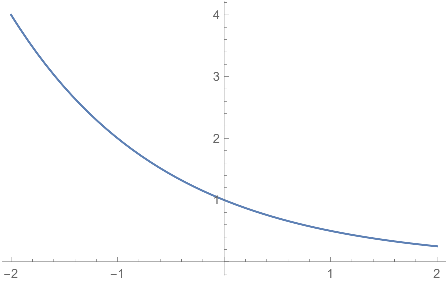
$$f(x) = \left(\frac{1}{2}\right)^x$$



The function $f(x) = \left(\frac{1}{2}\right)^x$ is differentiable in $\mathbb{R}$.

Increasing and decreasing functions: examples, cont'd

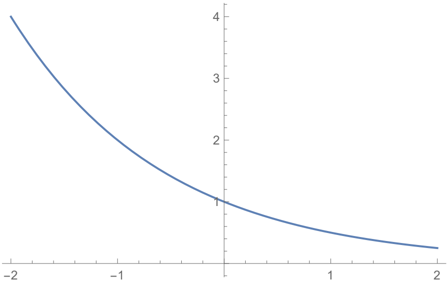
$$f(x) = \left(\frac{1}{2}\right)^x$$



The function $f(x) = \left(\frac{1}{2}\right)^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = \left(\frac{1}{2}\right)^x \log \frac{1}{2}$

Increasing and decreasing functions: examples, cont'd

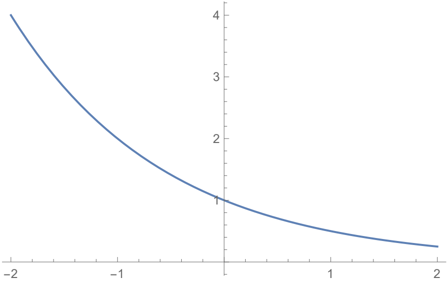
$$f(x) = \left(\frac{1}{2}\right)^x$$



The function $f(x) = \left(\frac{1}{2}\right)^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = \left(\frac{1}{2}\right)^x \log\frac{1}{2} < 0$, $\forall x \in \mathbb{R}$.

Increasing and decreasing functions: examples, cont'd

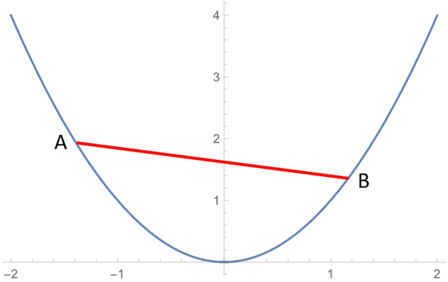
$$f(x) = \left(\frac{1}{2}\right)^x$$



The function $f(x) = \left(\frac{1}{2}\right)^x$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = \left(\frac{1}{2}\right)^x \log \frac{1}{2} < 0$, $\forall x \in \mathbb{R}$. Thus, the function is strictly decreasing in $\mathbb{R}$.

Increasing and decreasing functions: examples

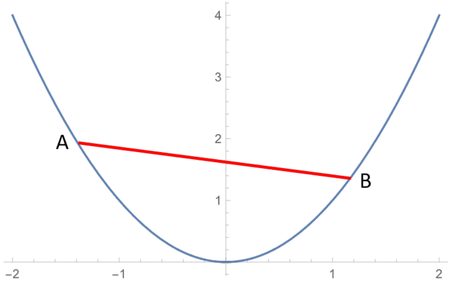
$$f(x) = x^2$$



The function $f(x) = x^2$ is differentiable in $\mathbb{R}$.

Increasing and decreasing functions: examples

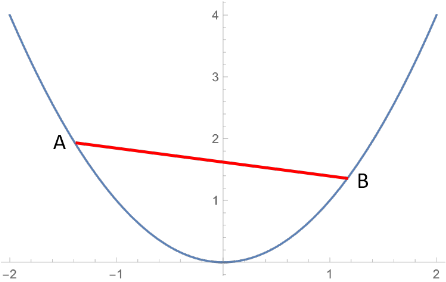
$$f(x) = x^2$$



The function $f(x) = x^2$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2x$ which is positive for $x > 0$ and negative for $x < 0$.

Increasing and decreasing functions: examples

$$f(x) = x^2$$



The function $f(x) = x^2$ is differentiable in $\mathbb{R}$. The derivative is $f'(x) = 2x$ which is positive for $x > 0$ and negative for $x < 0$. Thus, the function is strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$.

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$
2. $f(x) = \log x - x$

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$
2. $f(x) = \log x - x$
3. $f(x) = \sin(x)$

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$
2. $f(x) = \log x - x$
3. $f(x) = \sin(x)$
4. $f(x) = x^3 - 6x^2 + 4x + 12$

Increasing and decreasing functions: exercises

Determine in which subsets of their domain the following functions are increasing and decreasing:

1. $f(x) = \log x$
2. $f(x) = \log x - x$
3. $f(x) = \sin(x)$
4. $f(x) = x^3 - 6x^2 + 4x + 12$
5. $f(x) = \log x - x^2$

Local maxima and local minima

Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

Local maxima and local minima

Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

- A point $x_0 \in D$ is a local maximum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \leq f(x_0)$ $\forall x \in N_\epsilon(x_0)$

Local maxima and local minima

Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

- A point $x_0 \in D$ is a local maximum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \leq f(x_0)$ $\forall x \in N_\epsilon(x_0)$
- A point $x_0 \in D$ is a local minimum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \geq f(x_0)$ $\forall x \in N_\epsilon(x_0)$

Local maxima and local minima

Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

- A point $x_0 \in D$ is a local maximum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \leq f(x_0)$ $\forall x \in N_\epsilon(x_0)$
- A point $x_0 \in D$ is a local minimum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \geq f(x_0)$ $\forall x \in N_\epsilon(x_0)$

Pay attention: Note the difference between local maxima/minima and maxima/minima in an interval that we saw in the Weierstrass theorem (see Class 12).

Local maxima and local minima

Definition

Let $f : D \to \mathbb{R}$ be a function. Local maxima and minima are defined as follows:

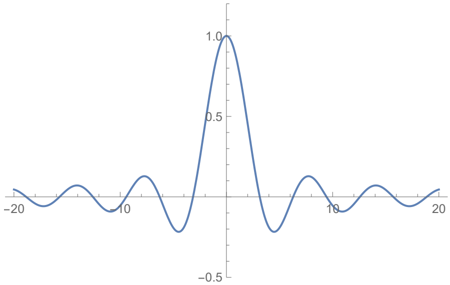
- A point $x_0 \in D$ is a local maximum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \leq f(x_0)$ $\forall x \in N_\epsilon(x_0)$
- A point $x_0 \in D$ is a local minimum if we can find at least one neighborhood $N_\epsilon(x_0)$ such that $f(x) \geq f(x_0)$ $\forall x \in N_\epsilon(x_0)$

Pay attention: Note the difference between local maxima/minima and maxima/minima in an interval that we saw in the Weierstrass theorem (see Class 12).

In an interval $[a, b]$ we can have multiple local maxima/minima but ONLY one maximum/minimum.

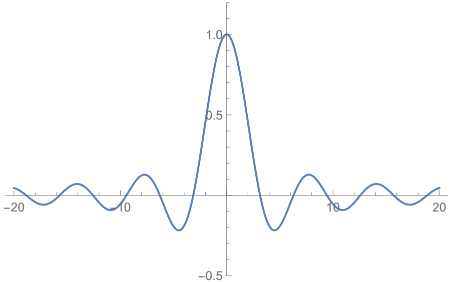
Local maxima and local minima: examples

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



Local maxima and local minima: examples

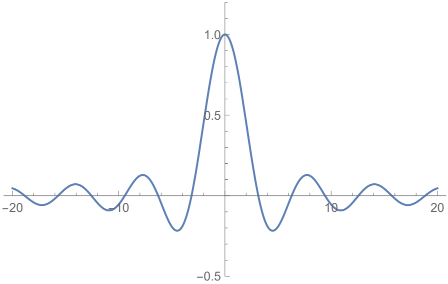
$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



This function is continuous in $\mathbb{R}$.

Local maxima and local minima: examples

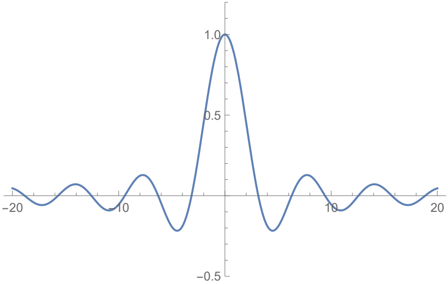
$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



This function is continuous in $\mathbb{R}$. By the Weierstrass theorem, it admits a maximum and a minimum in every closed bounded interval $[a, b]$.

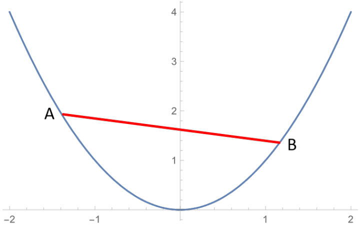
Local maxima and local minima: examples

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



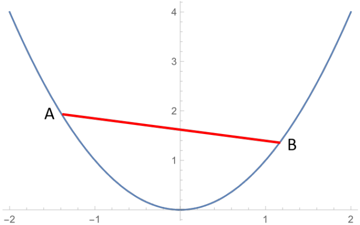
This function is continuous in $\mathbb{R}$. By the Weierstrass theorem, it admits a maximum and a minimum in every closed bounded interval $[a, b]$. However, the function can have multiple local maxima and minima in $[a, b]$.

Fermat's theorem: the intuition

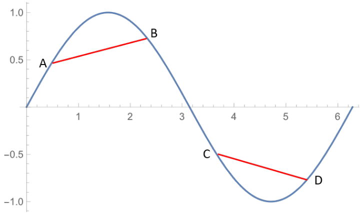


$$f(x) = x^2$$

Fermat's theorem: the intuition

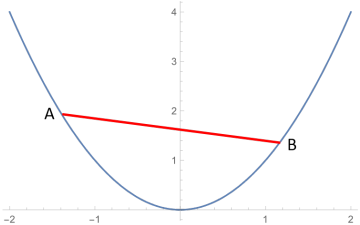


$f(x) = x^2$

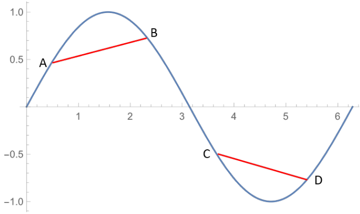


$g(x) = \sin x$

Fermat's theorem: the intuition



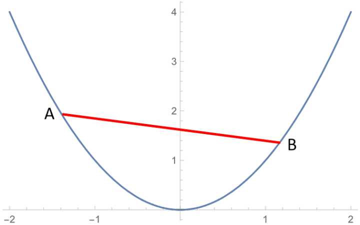
$f(x) = x^2$



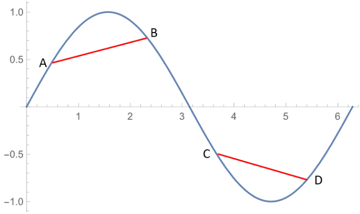
$g(x) = \sin x$

Both functions are differentiable in $\mathbb{R}$.

Fermat's theorem: the intuition



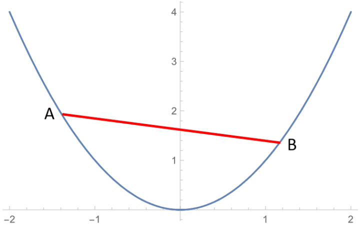
$f(x) = x^2$



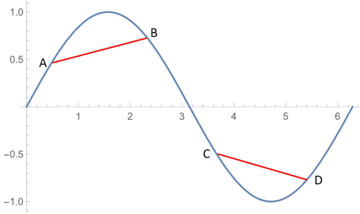
$g(x) = \sin x$

Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$,

Fermat's theorem: the intuition



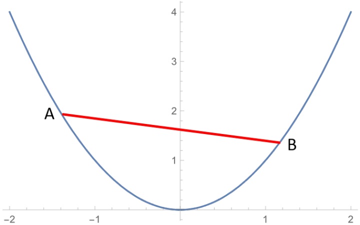
$f(x) = x^2$



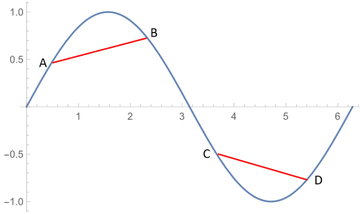
$g(x) = \sin x$

Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$, whereas $g(x)$ has a local maximum for $x = \frac{\pi}{2}$ and a local minimum for $x = \frac{3}{2}\pi$.

Fermat's theorem: the intuition



$f(x) = x^2$

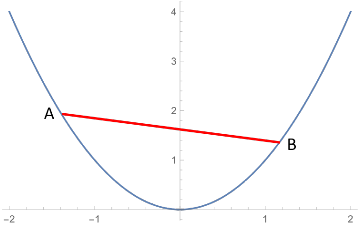


$g(x) = \sin x$

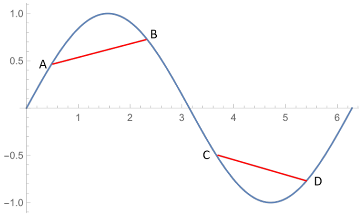
Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$, whereas $g(x)$ has a local maximum for $x = \frac{\pi}{2}$ and a local minimum for $x = \frac{3}{2}\pi$.

- $f'(x) = 2x$ then $f'(0) = 0$

Fermat's theorem: the intuition



$f(x) = x^2$

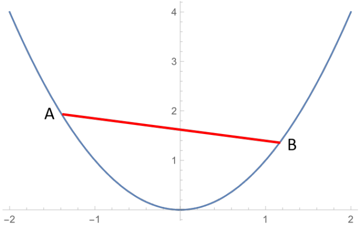


$g(x) = \sin x$

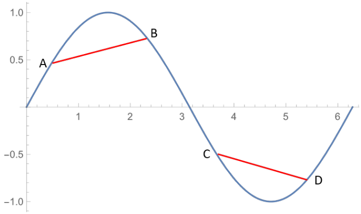
Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$, whereas $g(x)$ has a local maximum for $x = \frac{\pi}{2}$ and a local minimum for $x = \frac{3}{2}\pi$.

- $f'(x) = 2x$ then $f'(0) = 0$
- $g'(x) = \cos x$ then $f'(\frac{\pi}{2}) = 0$, $f'(\frac{3}{2}\pi) = 0$

Fermat's theorem: the intuition



$f(x) = x^2$



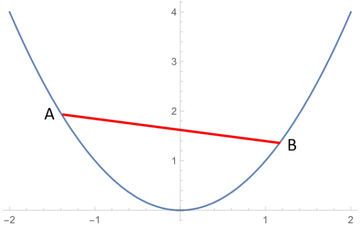
$g(x) = \sin x$

Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$, whereas $g(x)$ has a local maximum for $x = \frac{\pi}{2}$ and a local minimum for $x = \frac{3}{2}\pi$.

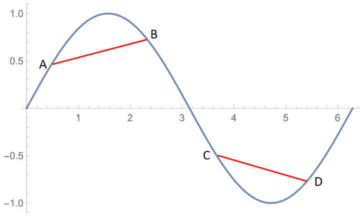
- $f'(x) = 2x$ then $f'(0) = 0$
- $g'(x) = \cos x$ then $f'(\frac{\pi}{2}) = 0$, $f'(\frac{3}{2}\pi) = 0$

Thus, if a function is differentiable, the derivative computed in its local maxima and minima is equal to zero

Fermat's theorem: the intuition



$f(x) = x^2$



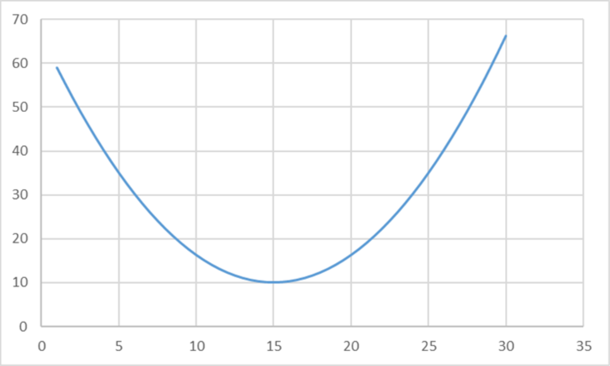
$g(x) = \sin x$

Both functions are differentiable in $\mathbb{R}$. $f(x)$ has a local minimum for $x = 0$, whereas $g(x)$ has a local maximum for $x = \frac{\pi}{2}$ and a local minimum for $x = \frac{3}{2}\pi$.

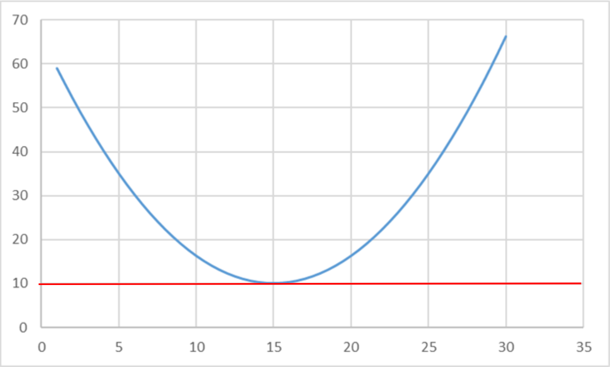
- $f'(x) = 2x$ then $f'(0) = 0$
- $g'(x) = \cos x$ then $f'(\frac{\pi}{2}) = 0,\ f'(\frac{3}{2}\pi) = 0$

Thus, if a function is differentiable, the derivative computed in its local maxima and minima is equal to zero then the tangent line is horizontal

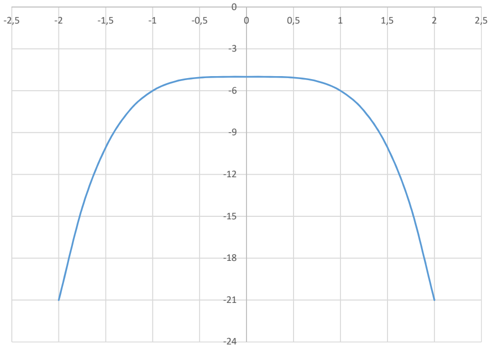
$$f(x) = x^2 - 2x + 35$$



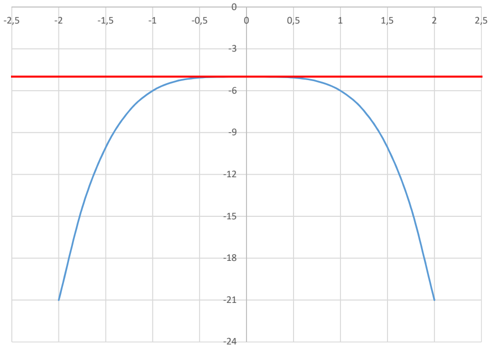
$$f(x) = x^2 - 2x + 35$$



$$f(x) = -5 - x^4$$



$$f(x) = -5 - x^4$$



Fermat's theorem

Theorem

Let $f : D \to \mathbb{R}$ *and let* f *be differentiable in* $x_0 \in D$.

Fermat's theorem

Theorem

Let $f : D \to \mathbb{R}$ *and let* f *be differentiable in* $x_0 \in D$. *If* x_0 *is a local maximum or a local minimum then* $f'(x_0) = 0$.

Fermat's theorem

Theorem

Let $f : D \to \mathbb{R}$ and let f be differentiable in $x_0 \in D$. If x_0 is a local maximum or a local minimum then $f'(x_0) = 0$.

Does the converse hold?

Fermat's theorem

Theorem

Let $f : D \to \mathbb{R}$ and let f be differentiable in $x_0 \in D$. If x_0 is a local maximum or a local minimum then $f'(x_0) = 0$.

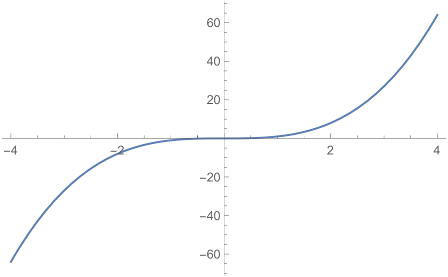
Does the converse hold? Namely, if a function $f : D \to \mathbb{R}$ is differentiable in $x_0 \in D$ and if $f'(x_0) = 0$, can we conclude that x_0 is a local maximum or a local minimum?

Fermat's theorem, cont'd

NO!

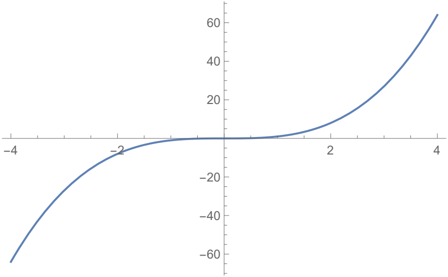
Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:



Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:

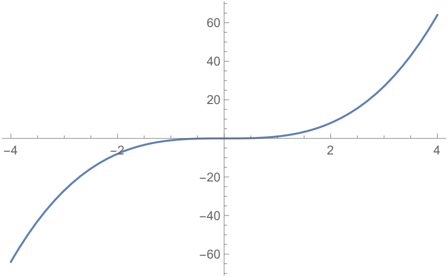


This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \to$$

Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:

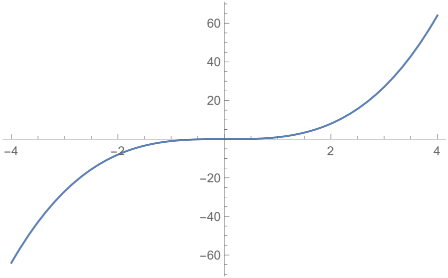


This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \rightarrow \quad f'(0) = 0$$

Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:



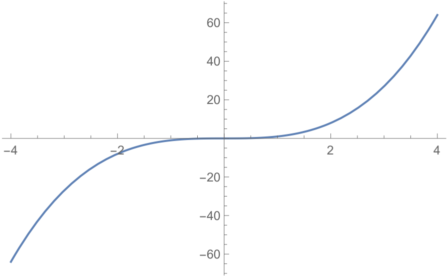
This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \rightarrow \quad f'(0) = 0$$

However, $x = 0$ is NOT a local maximum/minimum.

Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:



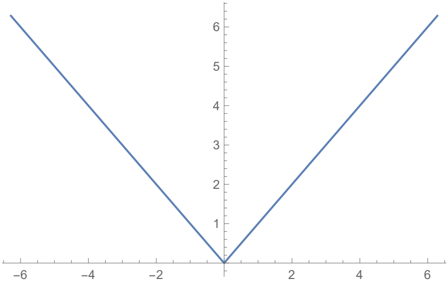
This function is differentiable in $\mathbb{R}$ and we have:

$$f'(x) = 3x^2 \quad \to \quad f'(0) = 0$$

However, $x = 0$ is NOT a local maximum/minimum. It is an **inflection point with an horizontal tangent**.

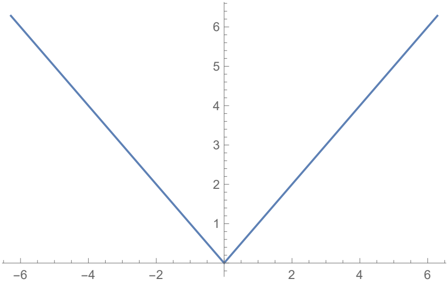
Fermat's theorem, cont'd

What if the function is not differentiable?



Fermat's theorem, cont'd

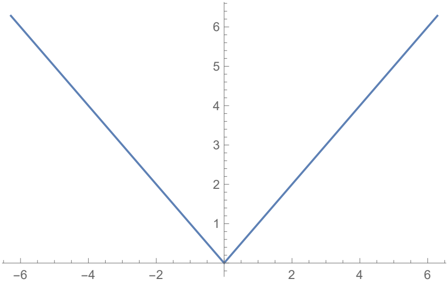
What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist.

Fermat's theorem, cont'd

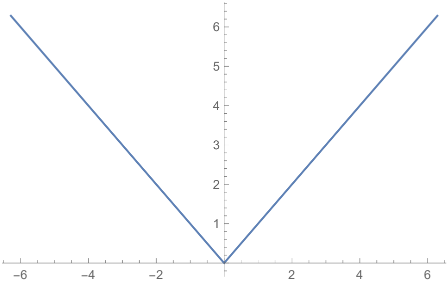
What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist. However note that $x = 0$ is a local minumum.

Fermat's theorem, cont'd

What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist. However note that $x = 0$ is a local minumum. **We CANNOT use the methods described in this course, which are based on derivatives, to find the local maxima/minima of a non-differentiable function.**

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$.

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Based on what we have seen so far, critical points can be:

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Based on what we have seen so far, critical points can be:

- Local maxima

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Based on what we have seen so far, critical points can be:

- Local maxima
- Local minima

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Based on what we have seen so far, critical points can be:

- Local maxima
- Local minima
- Inflection points with horizontal tangent

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

Based on what we have seen so far, critical points can be:

- Local maxima
- Local minima
- Inflection points with horizontal tangent

Thus, the condition $f'(x_0) = 0$ is only **necessary but not sufficient** for x_0 to be a local maximum or minimum for f.

Critical points

Definition

Let $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. If $f'(x_0) = 0$, the point x_0 is called a **critical point**.

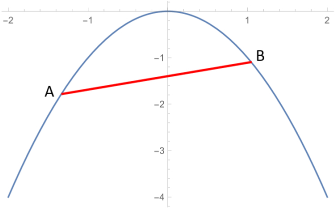
Based on what we have seen so far, critical points can be:

- Local maxima
- Local minima
- Inflection points with horizontal tangent

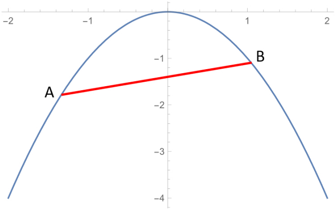
Thus, the condition $f'(x_0) = 0$ is only **necessary but not sufficient** for x_0 to be a local maximum or minimum for f.

In order to understand whether x_0 is a local minimum, a local maximum or an inflection point, we need additional conditions that involve the **second derivative** .

Concavity and Convexity: the intuition

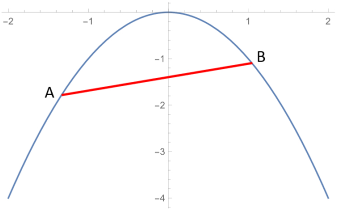


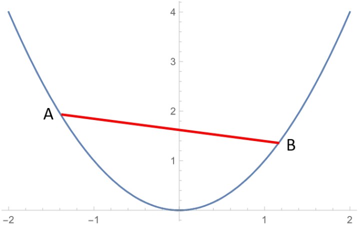
Concavity and Convexity: the intuition



The function $f(x) = -x^2$ is concave because any segment joining two points A and B of the graph is below the function

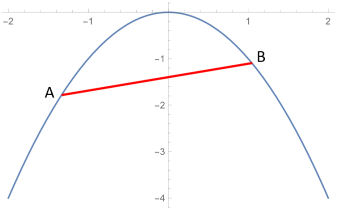
Concavity and Convexity: the intuition

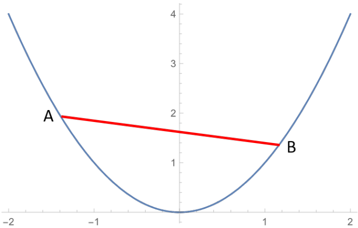




The function $f(x) = -x^2$ is concave because any segment joining two points A and B of the graph is below the function

Concavity and Convexity: the intuition

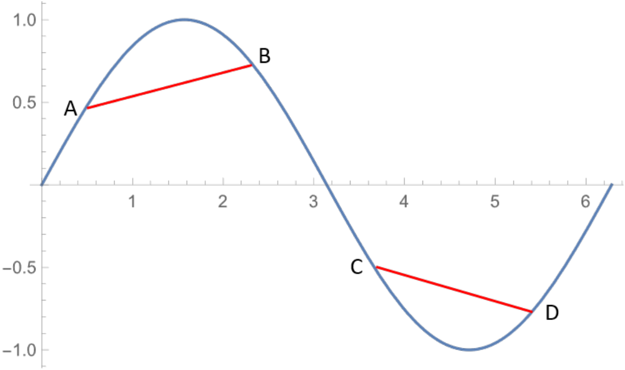




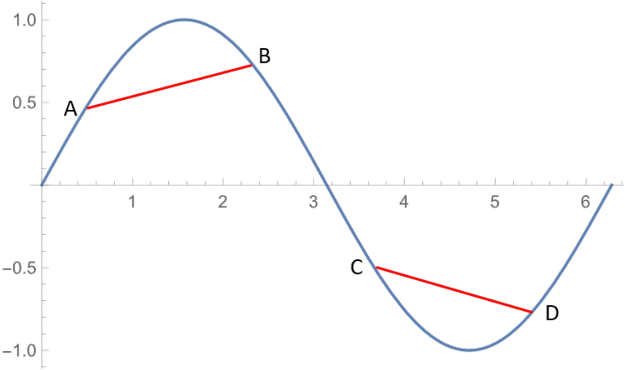
The function $f(x) = -x^2$ is concave because any segment joining two points A and B of the graph is below the function

The function $f(x) = x^2$ is convex because any segment joining two points A and B of the graph is above the function

Concavity and Convexity: the intuition, cont'd



Concavity and Convexity: the intuition, cont'd



The function $f(x) = \sin(x)$ is concave in $(0, \pi)$ and convex in $(\pi, 2\pi)$

Concavity and Convexity: the definition

Definition

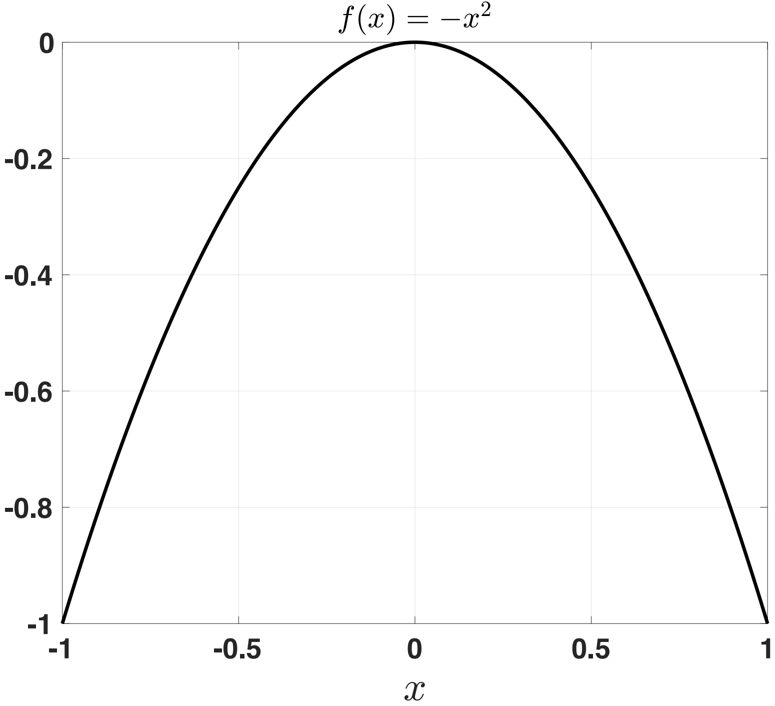
A function $f : D \to \mathbb{R}$ is said to be concave in $(a, b) \subseteq D$ if for all $x_1, x_2 \in (a, b)$ the segment from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$ lies below the graph of $f(x)$ in the interval (x_1, x_2).

Concavity and Convexity: the definition

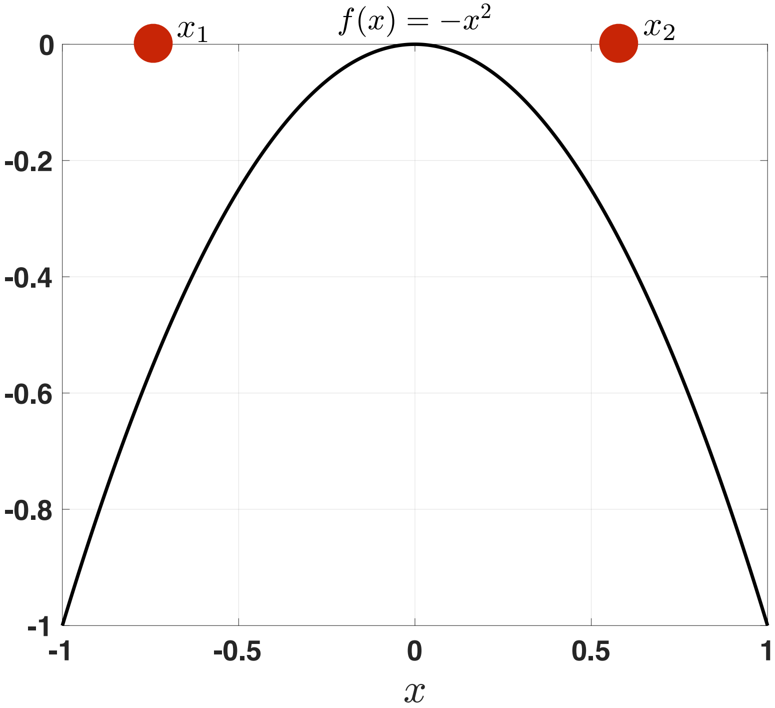
Definition

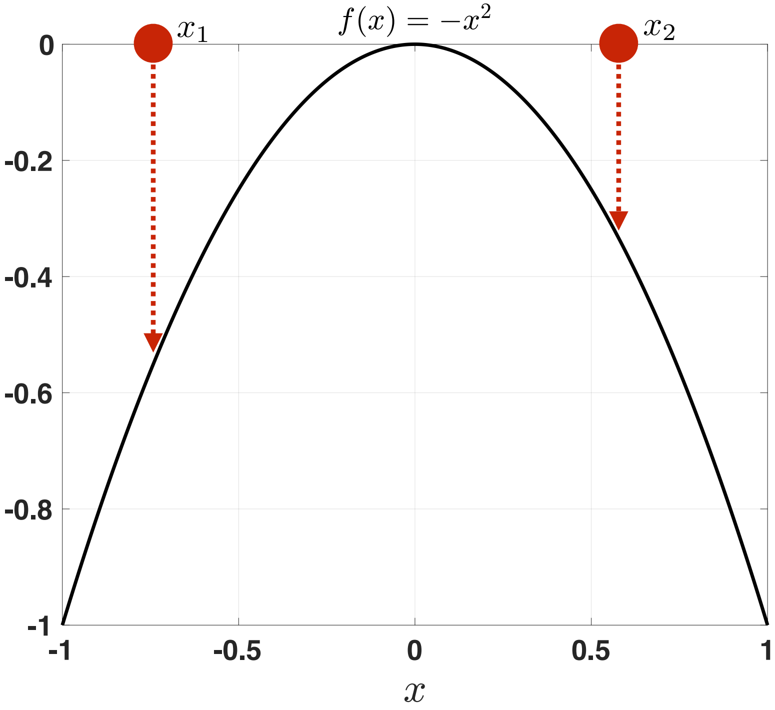
A function $f : D \to \mathbb{R}$ is said to be concave in $(a, b) \subseteq D$ if for all $x_1, x_2 \in (a, b)$ the segment from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$ lies below the graph of $f(x)$ in the interval (x_1, x_2).

A function $f : D \to \mathbb{R}$ is said to be convex in $(a, b) \subseteq D$ if for all $x_1, x_2 \in (a, b)$ the segment from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$ lies above the graph of $f(x)$ in the interval (x_1, x_2).

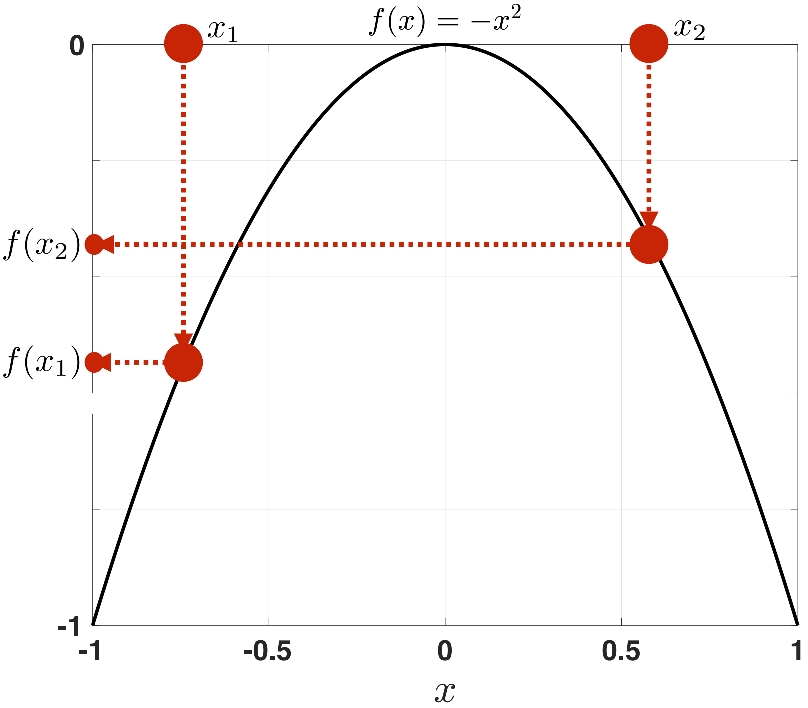


$f(x) = -x^2$
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x

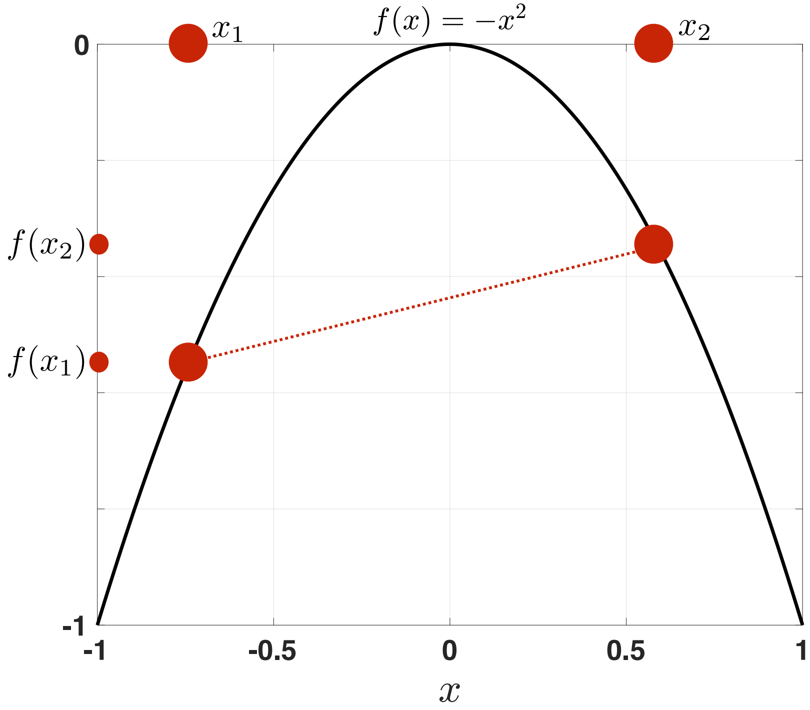

$f(x) = -x^2$
x_1
x_2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x



$f(x) = -x^2$
x_1
x_2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x



$f(x) = -x^2$
x_1
x_2
0
-1
$f(x_2)$
$f(x_1)$
-1
-0.5
0
0.5
1
x



$f(x) = -x^2$
x_1
x_2
0
-1
$f(x_2)$
$f(x_1)$
-1
-0.5
0
0.5
1
x

The derivative of concave and convex functions

Theorem

Let $f : D \to \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- *f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*

The derivative of concave and convex functions

Theorem

Let $f : D \to \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- *f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*
- *f is convex on (a, b) if and only if f' is strictly increasing on (a, b)*

The derivative of concave and convex functions

Theorem

Let $f : D \to \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- *f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*
- *f is convex on (a, b) if and only if f' is strictly increasing on (a, b)*

The have an intuition on this theorem, observe that, if a function is concave, the slope of the line tangent to a point decreases.

The derivative of concave and convex functions

Theorem

Let $f : D \to \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- *f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*
- *f is convex on (a, b) if and only if f' is strictly increasing on (a, b)*

The have an intuition on this theorem, observe that, if a function is concave, the slope of the line tangent to a point decreases. Instead, if a function is convex, the slope of the line tangent to a point increases.

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$.

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$
- $f(x) = \sin x$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$
- $f(x) = \sin x$, $f'(x) = \cos x$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$
- $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$,

Higher order derivatives

Definition

Let $f : D \to \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$. The second derivative of $f(x)$, denoted by $f''(x)$, is defined as the derivative of $f'(x)$, that is:

$$f''(x) = [f'(x)]'$$

In a similar manner, we can define third derivatives, forth derivatives, etc.

Examples

- $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) = 0$
- $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well.

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

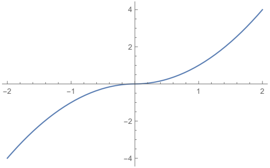
.

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

.

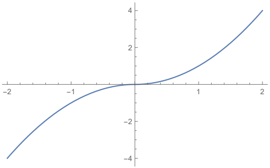


Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

.



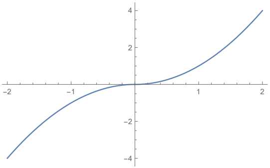
This function is differentiable in $\mathbb{R}$,

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

.



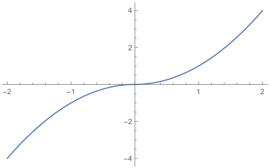
This function is differentiable in $\mathbb{R}$, but in $x = 0$ $f''(x)$ does not exist because the left and right limits of $f''(x)$ are -2 and 2, respectively.

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

.



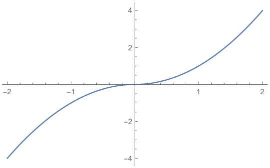
This function is differentiable in $\mathbb{R}$, but in $x = 0$ $f''(x)$ does not exist because the left and right limits of $f''(x)$ are -2 and 2, respectively. Thus, $f'(x)$ has an angle point in $x = 0$ and is not differentiable.

Higher order derivatives, cont'd

If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

.



This function is differentiable in $\mathbb{R}$, but in $x = 0$ $f''(x)$ does not exist because the left and right limits of $f''(x)$ are -2 and 2, respectively. Thus, $f'(x)$ has an angle point in $x = 0$ and is not differentiable.

If $f''(x)$ exists everywhere in the domain of the function, we say that the function is **twice** differentiable.

The derivative of concave and convex functions, cont'd

We have seen that:

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

But this implies that:

- A **twice** differentiable function is concave if and only if

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

But this implies that:

- A **twice** differentiable function is concave if and only if $f''(x) < 0$.

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

But this implies that:

- A **twice** differentiable function is concave if and only if $f''(x) < 0$.
- A **twice** differentiable function is convex if and only if

The derivative of concave and convex functions, cont'd

We have see that:

- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
- A differentiable function is convex if and only if $f'(x)$ is strictly increasing.

But this implies that:

- A **twice** differentiable function is concave if and only if $f''(x) < 0$.
- A **twice** differentiable function is convex if and only if $f''(x) > 0$.

The second derivative test for local maxima and minima

Theorem

Let $f : D \to \mathbb{R}$ *be twice differentiable on* $(a, b) \subseteq D$ *and let* $x_0 \in (a, b)$ *be such that* $f'(x_0) = 0$*, that is,* x_0 *is a critical point. Then:*

The second derivative test for local maxima and minima

Theorem

Let $f : D \to \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

- *If $f''(x_0) < 0$ then x_0 is a local maximum.*

The second derivative test for local maxima and minima

Theorem

Let $f : D \to \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

- *If $f''(x_0) < 0$ then x_0 is a local maximum.*
- *If $f''(x_0) > 0$ then x_0 is a local minimum.*

The second derivative test for local maxima and minima

Theorem

Let $f : D \to \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

- *If $f''(x_0) < 0$ then x_0 is a local maximum.*
- *If $f''(x_0) > 0$ then x_0 is a local minimum.*

To have an intuition on this theorem, observe that, if x_0 is a local maximum, the function is concave in a neighborhood of x_0 and therefore $f''(x) < 0$.

The second derivative test for local maxima and minima

Theorem

Let $f : D \to \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

- *If $f''(x_0) < 0$ then x_0 is a local maximum.*
- *If $f''(x_0) > 0$ then x_0 is a local minimum.*

To have an intuition on this theorem, observe that, if x_0 is a local maximum, the function is concave in a neighborhood of x_0 and therefore $f''(x) < 0$.

Similarly, if x_0 is a local minimum, the function is convex in a neighborhood of x_0 and therefore $f''(x_0) > 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$. Moreover, $f''(0) = 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \to \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$. Moreover, $f''(0) = 0$. Since $f'''(0) = 6 \neq 0$, $x = 0$ is an inflection point.

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$
7. $f(x) = \frac{\log x}{x}$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$
7. $f(x) = \frac{\log x}{x}$
8. $f(x) = \frac{\log^2 x}{x}$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$
7. $f(x) = \frac{\log x}{x}$
8. $f(x) = \frac{\log^2 x}{x}$
9. $f(x) = x + e^{-3x}$

The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

1. $f(x) = \log x - x$
2. $f(x) = \sin(x)$
3. $f(x) = x^3 - 6x^2 + 4x + 12$
4. $f(x) = \log x - x^2$
5. $f(x) = x - 9x^3$
6. $f(x) = \log(4x - x^2)$
7. $f(x) = \frac{\log x}{x}$
8. $f(x) = \frac{\log^2 x}{x}$
9. $f(x) = x + e^{-3x}$
10. $f(x) = \log(1 + \log(x)) - \log(x)$ (difficult)

Derivative of the inverse function

Theorem

Let $f : X \to Y$ be differentiable in X.

Derivative of the inverse function

Theorem

Let $f : X \to Y$ be differentiable in X. Assume f is invertible and call $f^{(-1)} : Y \to X$ the inverse function.

Derivative of the inverse function

Theorem

Let $f : X \to Y$ be differentiable in X. Assume f is invertible and call $f^{(-1)} : Y \to X$ the inverse function. Then $f^{(-1)}$ is differentiable in Y and

Derivative of the inverse function

Theorem

Let $f : X \to Y$ be differentiable in X. Assume f is invertible and call $f^{(-1)} : Y \to X$ the inverse function. Then $f^{(-1)}$ is differentiable in Y and

$$[f^{(-1)}]'(y) = \frac{1}{f'\left(f^{(-1)}(y)\right)}$$

for all $y \in Y$ such that $f'\left(f^{(-1)}(y)\right) \neq 0$.

Derivative of the inverse function: an example

The function $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is injective and surjective and therefore it can be inverted. The inverse is called "arcsin":

Derivative of the inverse function: an example

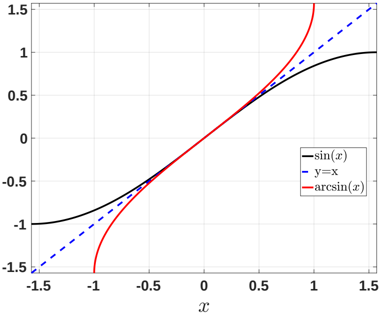
The function $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is injective and surjective and therefore it can be inverted. The inverse is called "arcsin":

$$\arcsin(x) : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Derivative of the inverse function: an example

The function $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is injective and surjective and therefore it can be inverted. The inverse is called "arcsin":

$$\arcsin(x) : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



Derivative of the inverse function: an example

Remind that, by definition of inverse function:

$$x_0 \xrightarrow{f} f(x_0) = y_0 \;\Leftrightarrow$$

Derivative of the inverse function: an example

Remind that, by definition of inverse function:

$$x_0 \xrightarrow{f} f\left(x_0\right) = y_0 \;\Leftrightarrow\; y_0 \xrightarrow{f^{(-1)}} f^{(-1)}\left(y_0\right) =$$

Derivative of the inverse function: an example

Remind that, by definition of inverse function:

$$x_0 \xrightarrow{f} f\left(x_0\right) = y_0 \;\Leftrightarrow\; y_0 \xrightarrow{f^{(-1)}} f^{(-1)}\left(y_0\right) = x_0$$

Derivative of the inverse function: an example

Remind that, by definition of inverse function:

$$x_0 \xrightarrow{f} f(x_0) = y_0 \;\Leftrightarrow\; y_0 \xrightarrow{f^{(-1)}} f^{(-1)}(y_0) = x_0$$

Then we have:

$$\sin(0) = 0 \Rightarrow \arcsin(0) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow \arcsin(1) = \frac{\pi}{2}$$

$$\sin\left(-\frac{\pi}{2}\right) = -1 \Rightarrow \arcsin(-1) = -\frac{\pi}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \Rightarrow \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \Rightarrow \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \Rightarrow \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Rightarrow \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \Rightarrow \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem.

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem. Recall the formula for the derivative of the inverse:

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{-1}(y))}$$

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem. Recall the formula for the derivative of the inverse:

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{-1}(y))}$$

In our case, we have:

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$$

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem. Recall the formula for the derivative of the inverse:

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{-1}(y))}$$

In our case, we have:

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$$

Using the fact that, in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos x = \sqrt{1-\sin^2 x}$, we have:

$$\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} =$$

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem. Recall the formula for the derivative of the inverse:

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{-1}(y))}$$

In our case, we have:

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$$

Using the fact that, in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos x = \sqrt{1 - \sin^2 x}$, we have:

$$\cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2}$$

Derivative of the inverse function: an example

Let's compute the derivative of $\arcsin(x)$ using the previous theorem. Recall the formula for the derivative of the inverse:

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{-1}(y))}$$

In our case, we have:

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}$$

Using the fact that, in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos x = \sqrt{1 - \sin^2 x}$, we have:

$$\cos(\arcsin(x)) = \sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2}$$

and thus:

$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}$$

Derivative of the inverse function: other examples

Using the same method it is possible to compute the derivative of the inverse of the cosine and tangent.

Derivative of the inverse function: other examples

Using the same method it is possible to compute the derivative of the inverse of the cosine and tangent.

The inverse of the function $\cos x : [0, \pi] \to [-1, 1]$ is the function

$$\arccos x : [-1, 1] \to [0, \pi]$$

Derivative of the inverse function: other examples

Using the same method it is possible to compute the derivative of the inverse of the cosine and tangent.

The inverse of the function $\cos x : [0, \pi] \to [-1, 1]$ is the function

$$\arccos x : [-1, 1] \to [0, \pi]$$

which is differentiable in $[-1, 1]$ and its derivative is given by:

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

Derivative of the inverse function: other examples

Using the same method it is possible to compute the derivative of the inverse of the cosine and tangent.

The inverse of the function $\cos x : [0, \pi] \to [-1, 1]$ is the function

$$\arccos x : [-1, 1] \to [0, \pi]$$

which is differentiable in $[-1, 1]$ and its derivative is given by:

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

The inverse of the function $\tan x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ is the function

$$\arctan x : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Derivative of the inverse function: other examples

Using the same method it is possible to compute the derivative of the inverse of the cosine and tangent.

The inverse of the function $\cos x : [0, \pi] \to [-1, 1]$ is the function

$$\arccos x : [-1, 1] \to [0, \pi]$$

which is differentiable in $[-1, 1]$ and its derivative is given by:

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

The inverse of the function $\tan x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ is the function

$$\arctan x : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

which is differentiable in $\mathbb{R}$ and its derivative is given by:

$$(\arctan x)' = \frac{1}{1 + x^2}$$

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ *and* $g : D \to \mathbb{R}$ *be continuous on* D *and let* x_0 *be a limit point of* D.

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D. Assume that f and g are both differentiable in $D/\{x_0\}$ and $g(x_0) \neq 0$.

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ *and* $g : D \to \mathbb{R}$ *be continuous on* D *and let* x_0 *be a limit point of* D. *Assume that* f *and* g *are both differentiable in* $D/\{x_0\}$ *and* $g(x_0) \neq 0$. *If:*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D. Assume that f and g are both differentiable in $D/\{x_0\}$ and $g(x_0) \neq 0$. If:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \to x_0} f(x) = \pm\infty, \qquad \lim_{x \to x_0} g(x) = \pm\infty \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D. Assume that f and g are both differentiable in $D/\{x_0\}$ and $g(x_0) \neq 0$. If:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \to x_0} f(x) = \pm\infty, \qquad \lim_{x \to x_0} g(x) = \pm\infty \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

De L'Hôpital rule

Theorem

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous on D and let x_0 be a limit point of D. Assume that f and g are both differentiable in $D/\{x_0\}$ and $g(x_0) \neq 0$. If:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \to x_0} f(x) = \pm\infty, \qquad \lim_{x \to x_0} g(x) = \pm\infty \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

The point x_0 can be either finite or $\pm\infty$.

De L'Hôpital rule: examples

Compute the following limit:

$$\lim_{x \to 0^+} x \log x$$

De L'Hôpital rule: examples

Compute the following limit:

$$\lim_{x \to 0^+} x \log x$$

We have an indeterminate form $0 \times (-\infty)$.

De L'Hôpital rule: examples

Compute the following limit:

$$\lim_{x \to 0^+} x \log x$$

We have an indeterminate form $0 \times (-\infty)$. However, applying the De L'Hôpital rule we have:

$$\begin{aligned}
\lim_{x \to 0^+} x \log(x) &= \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \\
&= \lim_{x \to 0^+} \frac{[\ln(x)]'}{\left[\frac{1}{x}\right]'} \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x \to 0^+} -\frac{x^2}{x} = \lim_{x \to 0^+} -x = 0
\end{aligned}$$

De L'Hôpital rule

Compute the following limit:

$$\lim_{x \to +\infty} \frac{\ln(x)}{\sqrt{x}}$$

$$\begin{aligned}
\lim_{x \to +\infty} \frac{\ln(x)}{\sqrt{x}} &= \lim_{x \to +\infty} \frac{[\ln(x)]'}{\left[\sqrt{x}\right]'} \\
&= \lim_{x \to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\
&= \lim_{x \to +\infty} \frac{2\sqrt{x}}{x} = \lim_{x \to +\infty} \frac{2}{\sqrt{x}} = 0
\end{aligned}$$

De L'Hôpital rule

Compute the following limit:

$$\lim_{x \to \infty} \left(\arctan x - \frac{\pi}{2} \right) \, e^x$$

De L'Hôpital rule

Compute the following limit:

$$\lim_{x\to\infty}\left(\arctan x - \frac{\pi}{2}\right)\ e^x$$

Since the arc whose tangent is $+\infty$ is $\frac{\pi}{2}$ we have:

$$\lim_{x\to+\infty}\arctan(x) = +\frac{\pi}{2}$$

De L'Hôpital rule

Compute the following limit:

$$\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)\,e^x$$

Since the arc whose tangent is $+\infty$ is $\frac{\pi}{2}$ we have:

$$\lim_{x\to+\infty}\arctan(x)=+\frac{\pi}{2}$$

and thus:

$$\begin{aligned}\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)\,e^x &= 0\times(+\infty)=\lim_{x\to\infty}\left(\frac{\arctan x-\dfrac{\pi}{2}}{e^{-x}}\right).\\ &= \frac{0}{0}\overset{H}{=}\lim_{x\to\infty}\frac{\dfrac{1}{1+x^2}}{-e^{-x}}=-\lim_{x\to\infty}\frac{e^x}{1+x^2}\\ &= \frac{\infty}{\infty}\overset{H}{=}-\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty}\overset{H}{=}-\lim_{x\to\infty}\frac{e^x}{2}=-\infty\end{aligned}$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental.

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well.

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} =$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \stackrel{H}{=}$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} =$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists$$

Remind that the limit for $x \to +\infty$ of $\sin x$ and $\cos x$ do not exist because these functions oscillate between -1 and 1.

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \stackrel{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists$$

Remind that the limit for $x \to +\infty$ of $\sin x$ and $\cos x$ do not exist because these functions oscillate between -1 and 1. However

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} =$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists$$

Remind that the limit for $x \to +\infty$ of $\sin x$ and $\cos x$ do not exist because these functions oscillate between -1 and 1. However

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \lim_{x \to +\infty} \left(1 + \underbrace{\frac{\sin(x)}{x}}_{\to 0}\right) =$$

De L'Hôpital rule

Pay attention: The hypothesis:

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental. If this limit does not exist, we CANNOT say that the original limit does not exist as well. Let's consider this limit:

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists$$

Remind that the limit for $x \to +\infty$ of $\sin x$ and $\cos x$ do not exist because these functions oscillate between -1 and 1. However

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \lim_{x \to +\infty} \left(1 + \underbrace{\frac{\sin(x)}{x}}_{\to 0} \right) = 1$$