Week 3

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Mathematics I

University of Rome Tor Vergata

28 September - 4 October, 2025

Quadratic functions

$$f : \mathbb{R} \to \mathbb{R} \;\text{ such that }\; f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R},\; a \neq 0$$

- Graphically, this is the equation of a parabola.
- The parabola is convex if $a > 0$
- The parabola is concave if $a < 0$
- the vertex of the parabola is the point with coordinates $V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$, with $\Delta = b^2 - 4ac$

Position of a parabola with respect to the x-axis

Let $\Delta = b^2 - 4ac$. Suppose that $a > 0$

1. If $\Delta > 0$ The parabola intercepts the x-axis at two points, which are the solutions of
$$ax^2 + bx + c = 0$$

2. If $\Delta = 0$ The parabola intercepts the x-axis at one point, which is the unique solution of
$$ax^2 + bx + c = 0$$

3. If $\Delta < 0$ The parabola stays always above the x-axis: the equation $ax^2 + bx + c = 0$ does not have any solution

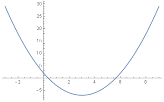

Figure: $a > 0, \quad \Delta > 0$

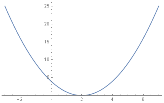

Figure: $a > 0, \quad \Delta = 0$

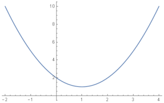

Figure: $a > 0, \quad \Delta < 0$

Position of a parabola with respect to the x-axis

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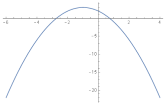

Figure: $a < 0, \quad \Delta > 0$

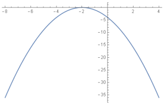

Figure: $a < 0, \quad \Delta = 0$

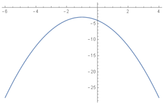

Figure: $a < 0, \quad \Delta < 0$

The function “Absolute Value”

Definition

The function “absolute value” of x, is given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ & \end{cases}$$

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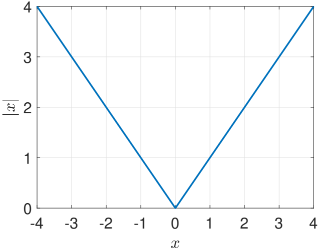


- $D = \mathbb{R}$, $R_f = [0, +\infty)$

Power functions

For all $n \in \mathbb{N}$ we define the function:

$$f(x) = x^n$$

which is nothing but the multiplication of x by itself n times

- This function is defined for all $x \in \mathbb{R}$, $D = \mathbb{R}$
- If n is even, the range is $R_f = [0, +\infty)$
- If n is odd, the range is $R_f = \mathbb{R}$

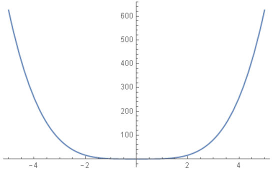


Figure: $f(x) = x^4$

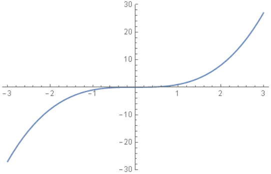


Figure: $f(x) = x^3$

A few characteristics of Power function with exponent $n \in \mathbb{N}$

If n is even, the function is not globally invertible. However if we consider only

$$f(x) : [0, +\infty) \to [0, +\infty)$$

the function is invertible and

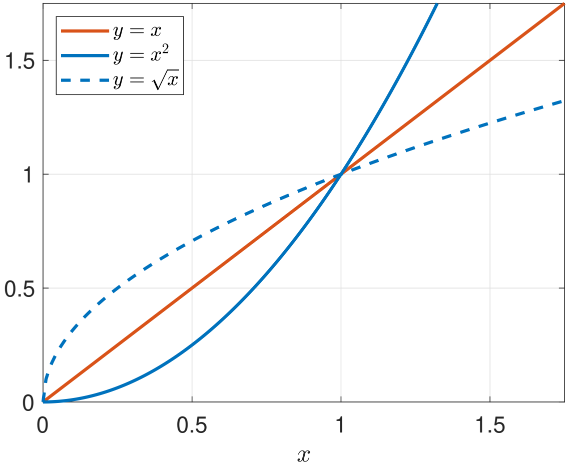
$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

If n is odd, the function is globally invertible and

$$f^{-1}(y) = y^{\frac{1}{n}} = \sqrt[n]{y}$$

The inverse function: graphical representation

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ over the line $y = x$.



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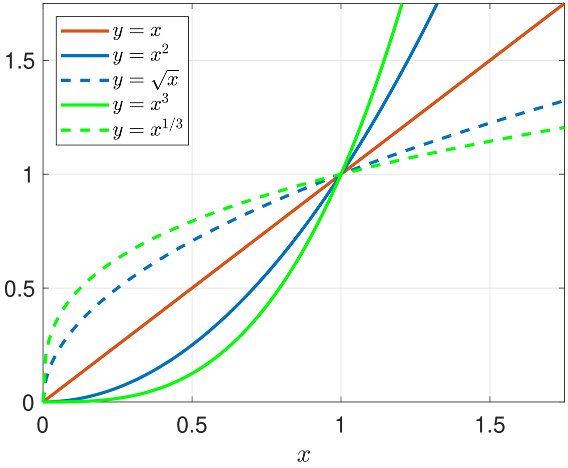

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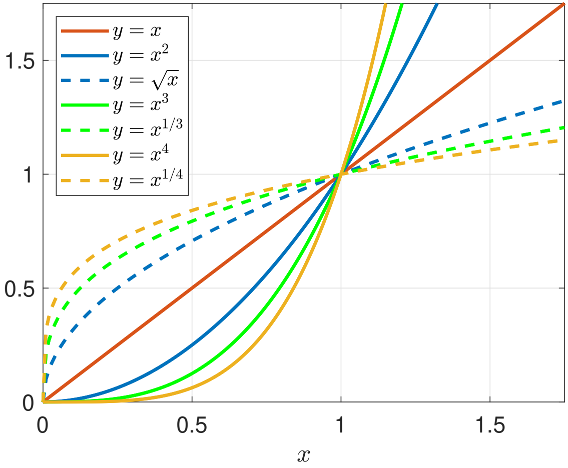

Power functions

Consider the function:

$$f(x) = x^r, \quad r \in \mathbb{R}$$

This is a power function with real exponent (which generalizes the case of a power function with natural exponent)
A few examples

1. $f(x) = x^{-1} = \dfrac{1}{x}$
2. $f(x) = x^{\frac{1}{2}} = \sqrt{x}$
3. $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$
4. $f(x) = x^{1.3}$

Notice that an extra care must be applied in computing the domain power functions with real exponent. In particular they are well defined when $x > 0$, but they may be undefined for $x = 0$ or $x < 0$. For instance function 1 is not defined when $x = 0$, functions 3 and 4 are not defined when $x < 0$.

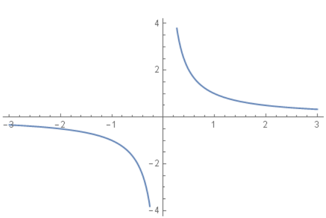

Figure: $f(x) = \frac{1}{x}$

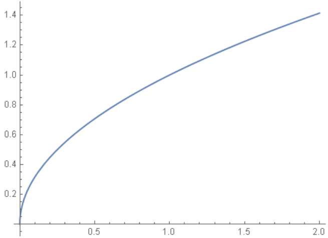

Figure: $f(x) = \sqrt{x}$

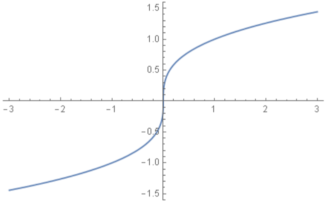

Figure: $f(x) = \sqrt[3]{x}$

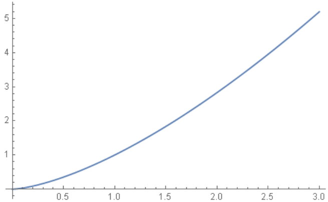

Figure: $f(x) = x^{1.3}$

The exponential function

$$f(x) = a^x, \qquad a > 0$$

Main characteristics:

- $D = \mathbb{R}$
- $R_f = (0, +\infty)$ meaning that $a^x > 0$ for all $x \in \mathbb{R}$
- $f(0) = a^0 = 1$
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing
- if $a = 1$ we get the flat line

The exponential function

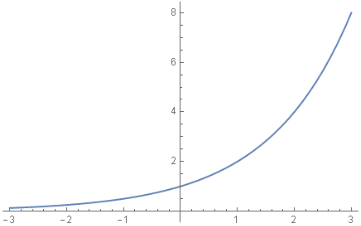

Figure: $f(x) = a^x$, $a > 1$

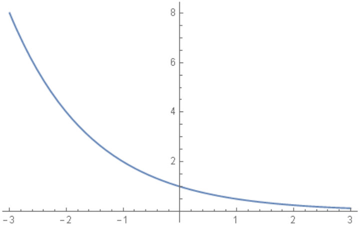

Figure: $f(x) = a^x$, $0 < a < 1$

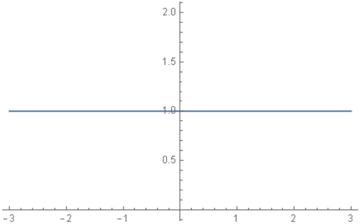

Figure: $f(x) = 1^x$

The logarithmic function

$$f(x) = \log_a(x), \qquad a > 0,\ a \neq 1$$

This is the inverse of the exponential function.

- $D = (0, +\infty)$,
- $R_f = \mathbb{R}$
- $f(1) = \log_a(1) = 0$ (this is a consequence of the fact that $a^0 = 1$)
- if $a > 0$ the function is monotonic strictly increasing
- if $0 < a < 1$ the function is monotonic strictly decreasing

The logarithmic function

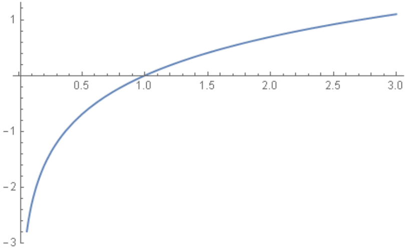

Figure: $f(x) = \log_a(x),\ a > 1$

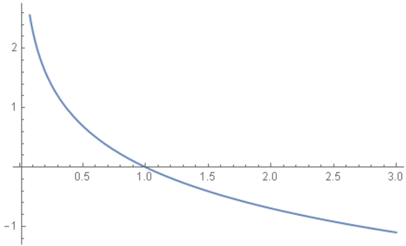

Figure: $f(x) = \log_a(x),\ 0 < a < 1$

The composite function: the intuition

Intuitively, the composition of two functions, $f : X \to Y$ and $g : Y \to Z$, is a function $h : X \to Z$ such that applying h to $x \in X$ produces the same results as applying first f to $x \in X$ and then applying g to $f(x) \in Y$.

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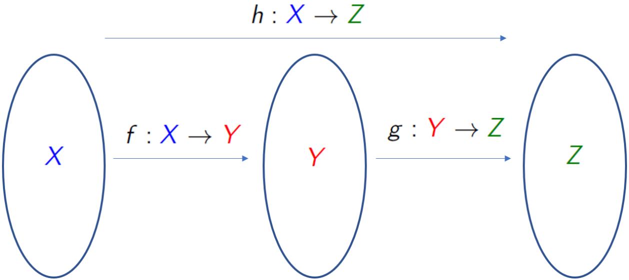

The composite function: the definition

Definition

Consider a function $f : X \to Y$ and another function $g : Y \to Z$. The composite function, denoted by $g \circ f$, is defined as:

$$g \circ f : X \to Z$$

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in X$$

Important: The order of composition matters. That is, in general,

$$g(f(x)) \neq f(g(x))$$

The composite function: examples

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, $\quad g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$

  We set $y = f(x) = x + 1$

$$(g \circ f)(x) = g(f(x)) = g(y) = y^2 - 1 =$$

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- $f : \mathbb{R} \to (0, +\infty)$, $f(x) = x^2 + 1$, $\quad g : (0, +\infty) \to (0, +\infty)$, $g(y) = 1/y$

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  We set $y = f(x) = x^2 + 1$. Thus:

  $$(g \circ f)(x) = g(f(x)) = g(y) =$$

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Intuitive plots of functions

Let $g : D \to \mathbb{R}$ be a function. For instance $g(x) = x^3 + 6x^2 - 15$.
How can we plot few composite function starting form the plot of $g(x)$?

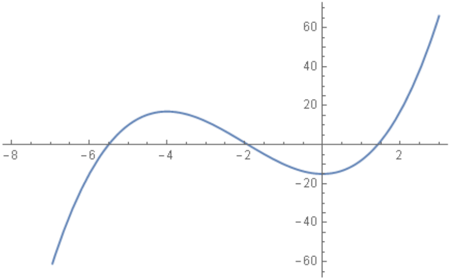

Figure: $g(x) = x^3 + 6x^2 - 15$

We want to derive the plot of a few composite functions, from the plot of $g(x)$.

Intuitive plots of functions

To plot $-g(x)$ we invert the graph of $g(x)$ along the x-axis: that is the negative part becomes positive and the positive part becomes negative.

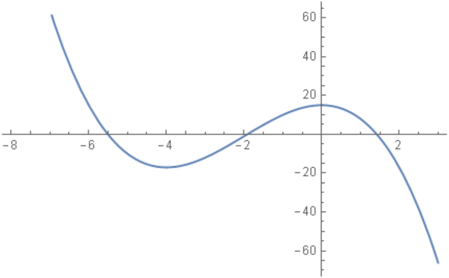

Figure: $-g(x) = -(x^3 + 6x^2 - 15)$

Intuitive plots of functions

To plot $|g(x)|$ we recall the definition of the absolute value:

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) < 0 \end{cases}$$

Then to plot $|g(x)|$ it is enough to overturn the negative part of the function above the x-axis

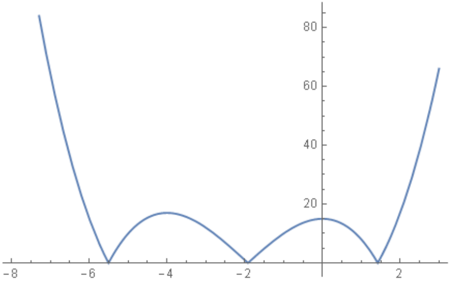

Figure: $|g(x)| = |x^3 + 6x^2 - 15|$

Intuitive plots of functions

To plot $g(x) + c$ we shift the plot of $g(x)$ up by the quantity c, if c is positive and down if c is negative.

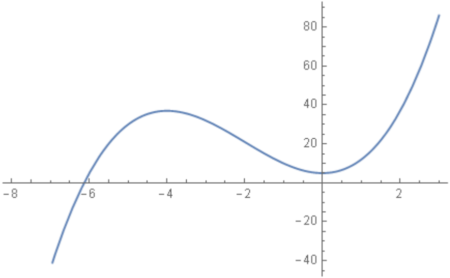

Figure: $g(x) + 20 = x^3 + 6x^2 - 15 + 20$

Intuitive plots of functions

To plot $g(x+a)$ we shift the plot of $g(x)$ on the left by the quantity c, if c is positive and on the right if c is negative.

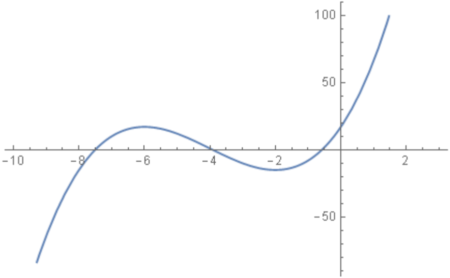

Figure: $g(x+2) = (x+2)^3 + 6(x+2)^2 - 15$

Sequences: the intuition

Intuitively, a sequence is a function which associates to each **natural** number $n \in \mathbb{N}$ a **real** number $s_n \in \mathbb{R}$.

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- $s_n = \sqrt{n}$

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Examples

- $s_n = \frac{1}{n} \quad \Rightarrow \quad s_1 = 1, \quad s_2 = \frac{1}{2}, \quad s_3 = \frac{1}{3}, \quad s_4 = \frac{1}{4}, \quad \ldots$
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Sequences: the definition

Definition

A sequence is any function $s : \mathbb{N} \to \mathbb{R}$. A sequence is denoted by $(s_n)_{n \in \mathbb{N}}$, whereas we denote by s_n the n-th element of the sequence.

Limit of sequences

What happens when n is "very large"?

Limit of sequences

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Examples

$$s_n = \frac{1}{n}$$

n	1	2	3	4	5	...
$\frac{1}{n}$	1	$\frac{1}{2} = 0.5$	$\frac{1}{3} = 0.3333$	$\frac{1}{4} = 0.25$	$\frac{1}{5} = 0.2$	...

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$$s_n = \frac{n}{n+1}$$

n	1	2	3	4	5	...
$\frac{n}{n+1}$	$\frac{1}{2} = 0.5$	$\frac{2}{3} = 0.6667$	$\frac{3}{4} = 0.75$	$\frac{4}{5} = 0.8$	$\frac{5}{6} = 0.8333$	...

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$$s_n = (-1)^n$$

n	1	2	3	4	5	...
$(-1)^n$	-1	1	-1	1	-1	...

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- The elements of $s_n = \frac{1}{n}$ approach 0 as n grows

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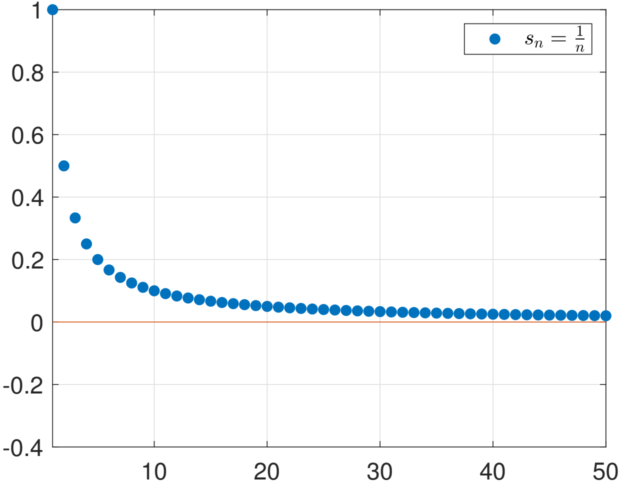
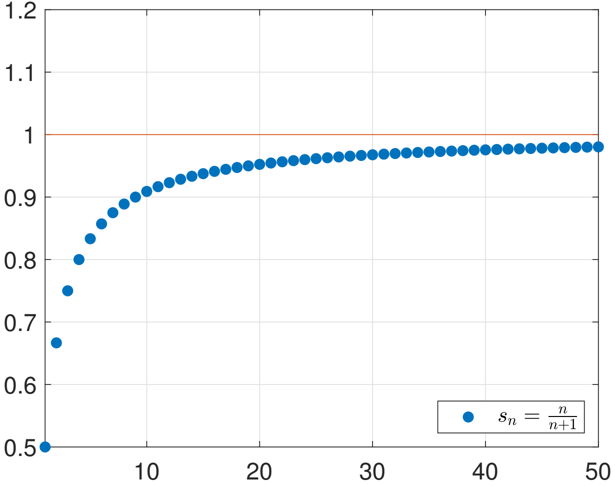
- The elements of $s_n = \frac{n}{n+1}$ approach 1 as n grows
- The elements of $s_n = (-1)^n$ swing between 1 and -1

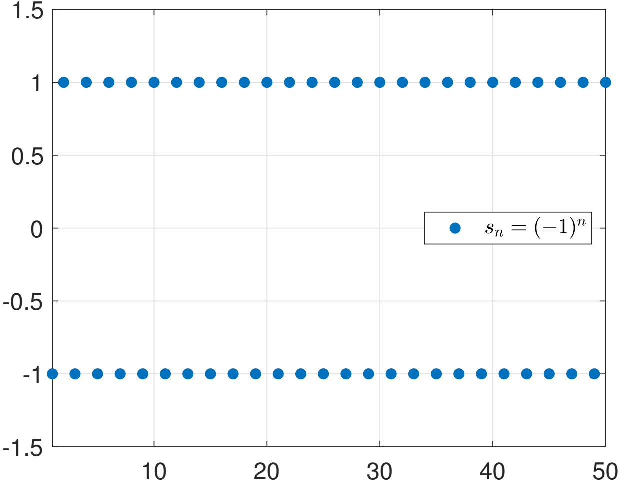
Limit of sequences, cont'd



Limit of sequences, cont'd



Limit of sequences, cont'd



Limit of sequences: the definition

Definition (Convergent sequence)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $(s_n)_{n\in\mathbb{N}}$ converges and we write

$$\lim_{n\to\infty} s_n = \ell$$

where $\ell \in \mathbb{R}$ is a finite real number, if:

$$\forall \epsilon > 0 \quad \exists\ n^* \in \mathbb{N}\ :\ \forall n > n^* \ \Rightarrow\ |s_n - \ell| < \epsilon$$

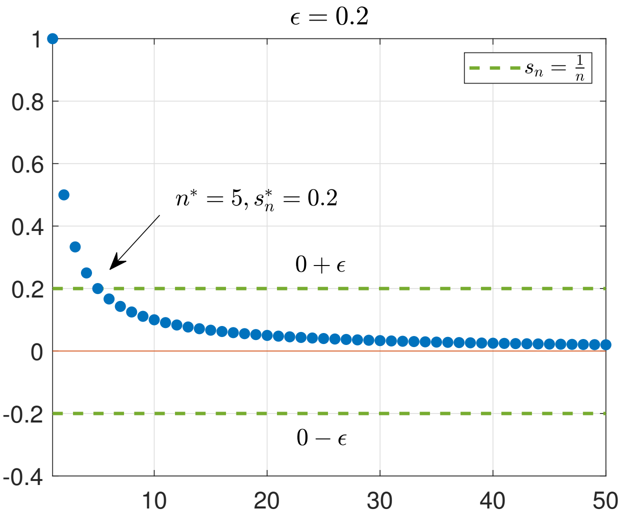
Sometimes we also use the notation $s_n \to \ell$ to indicate $\lim_{n\to\infty} s_n = \ell$.

Remark The above definition says that,

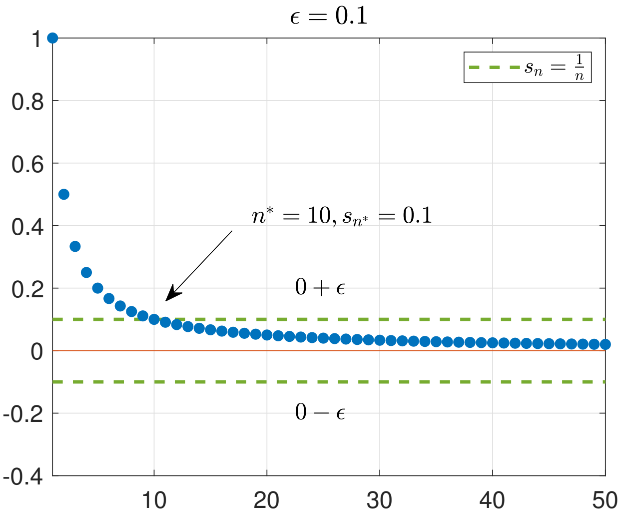
- for all length ϵ (as small as we like)
- we can find a natural number n^*
- such that for all indices n that are larger than n^*
- the distance between s_n and the limit ℓ is smaller than ϵ

That means, s_n approaches ℓ, when n is very large.

Limit of sequences, cont'd



Limit of sequences, cont'd



If we choose a smaller ϵ, the number n^* for which the definition is true gets larger!

Limit of sequences: exercises

Prove that:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

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Solution

Limit of sequences: exercises

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$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Solution

We have to find an n^* such that, for $n > n^*$, we have

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

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Solution

We have to find an n^* such that, for $n > n^*$, we have

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Let's solve the above inequality:

$$\left| \frac{1}{n} - 0 \right| < \epsilon \;\; \Leftrightarrow$$

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We fix $n^* = \left[\frac{1}{\epsilon}\right]$. Then for all $n > n^*$ we get that $\left|\frac{1}{n} - 0\right| < \epsilon$, and hence the definition is verified.

For example, if $\epsilon = 0.2$, then $n^* = 5$;

Limit of sequences: exercises

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For example, if $\epsilon = 0.2$, then $n^* = 5$; if $\epsilon = 0.1$, then $n^* = 10$; if $\epsilon = 0.014$, then $n^* = 72$, etc.

Limit of sequences: exercises, cont'd

Prove that:

$$\lim_{n \to \infty} \frac{n}{n+1} = 1$$

Limit of sequences: exercises, cont'd

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Let's solve the above inequality:

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - n - 1}{n+1} \right| =$$

Limit of sequences: exercises, cont'd

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which is solved for $n > \frac{1-\epsilon}{\epsilon}$.

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which is solved for $n > \frac{1-\epsilon}{\epsilon}$. We can set $n^* = \left[\frac{1-\epsilon}{\epsilon}\right]$. Then, for all $n > n^*$ it holds that $\left| \frac{n}{n+1} - 1 \right| < \epsilon$, and hence the definition is verified.

Limit of sequences: cont'd

Consider the following sequence:

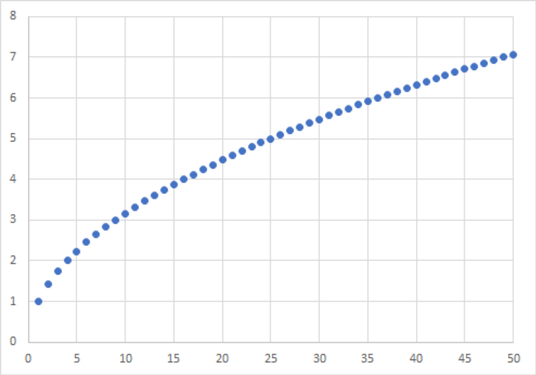
$$s_n = \sqrt{n}$$

Limit of sequences: cont'd

Consider the following sequence:

$$s_n = \sqrt{n}$$

Does it have a limit?



Limit of sequences, cont'd

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $(s_n)_{n\in\mathbb{N}}$ diverges, and we write

$$\lim_{n\to+\infty} s_n = \infty \quad (\text{ it could be } +\infty \text{ or } -\infty)$$

if:

$$\forall M > 0, \quad \exists\ n^* \ :\ \forall n > n^* \ \Rightarrow\ |s_n| > M$$

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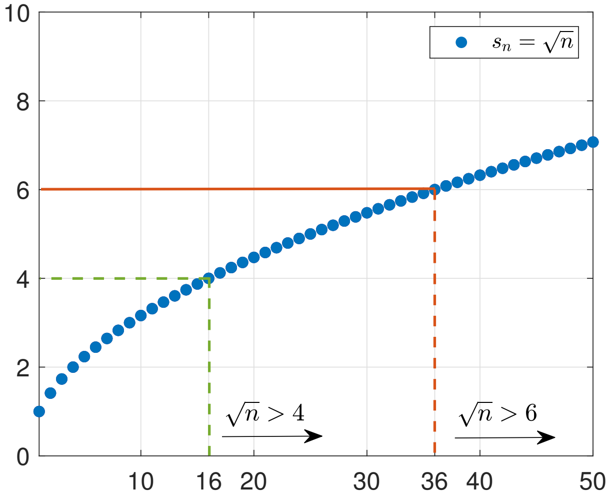
Notice that if $\lim_{n\to+\infty} s_n = +\infty$ then the absolute value is unnecessary and the inequality becomes

$$s_n > M.$$

If $\lim_{n\to+\infty} s_n = -\infty$ we need to use the absolute value or the equivalent exprssion

$$s_n < -M$$

Limit of sequences, cont'd



Sequences: the limit may not exist

The limit of a sequence may not exist.

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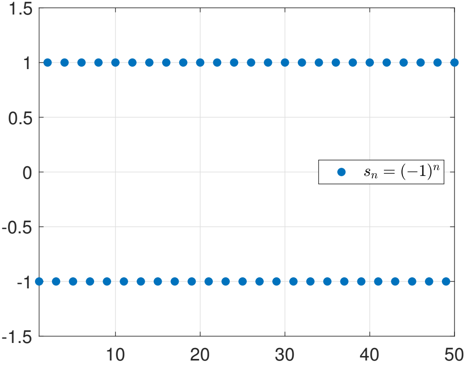
The limit of a sequence may not exist.
For instance consider the sequence $(s_n)_{n\in\mathbb{N}}$, with $s_n = (-1)^n$



When n is large this sequence does not approach any specific value.

Sequences: the limit may not exist

To understand when a sequence does not admit a limit we introduce the definition of subsequence.

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence, and let $(k_n)_{n\in\mathbb{N}}$ be a collection of sorted indices. The sequence $(s_{k_n})_{n\in\mathbb{N}}$ is called a subsequence of $(s_n)_{n\in\mathbb{N}}$.

Example Let $s_n = \frac{1}{n}$, and consider the collection of all even indices, that is $k_n = \{2, 4, 6, 8, 10, 12, \dots\}$.
The even subsequence is the subsequence where we only take the elements with even index:

$$s_2 = \frac{1}{2}, \quad s_4 = \frac{1}{4}, \quad s_6 = \frac{1}{6}, \quad s_8 = \frac{1}{8}, \dots$$

and it is denotes as $(s_{2n})_{n\in\mathbb{N}}$.
The odd subsequence is the subsequence where we only take the elements with odd index:

$$s_3 = \frac{1}{3}, \quad s_5 = \frac{1}{5}, \quad s_7 = \frac{1}{7}, \quad s_9 = \frac{1}{9}, \dots$$

Sequences: the limit may not exist

The following theorem allows us to understand whether the limit exists or not.

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If $\exists \ell \in \mathbb{R}$ such that:

$$\lim_{n\to\infty} s_n = \ell$$

then **for all subsequences** $(s_{k_n})_{n\in\mathbb{N}}$ *it holds that*

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then **for all subsequences** $(s_{k_n})_{n\in\mathbb{N}}$ *it holds that*

$$\lim_{n\to\infty} s_{k_n} = \ell$$

Simplified formulation which is used in exercises:

Corollary

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If

$$\lim_{n\to\infty} s_{2n} = \ell_1 \quad \textit{and} \quad \lim_{n\to\infty} s_{2n+1} = \ell_2$$

Sequences: the limit may not exist, cont'd

This means that, if the limit exists, it does **not** change when considering "sub-sequences".

We can also say that if we find two subsequences (for instance the even subsequence and the odd subsequence) that converge to different limits, then the sequence does not converge.

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Consider the following sequence:

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Sequences: some useful theorems

Theorem (Uniqueness of the limit)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If the sequence converges then the limit is unique.

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Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If the sequence converges then the limit is unique.

This theorem says that it is impossible that a sequence converges to two different limits.

Sequences: some useful theorems, cont'd

Theorem (Absolute value theorem for sequences)

Let $(s_n)_{n\in\mathbb{N}}$ *be a sequence. If* $|s_n| \to 0$*, then* $s_n \to 0$*.*

Sequences: some useful theorems, cont'd

Theorem (Absolute value theorem for sequences)

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence. If $|s_n| \to 0$, then $s_n \to 0$.

Important: This theorem holds only if the limit is zero!

Exercise

Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

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But we know that $\frac{1}{n} \to 0$.

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Exercise

Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

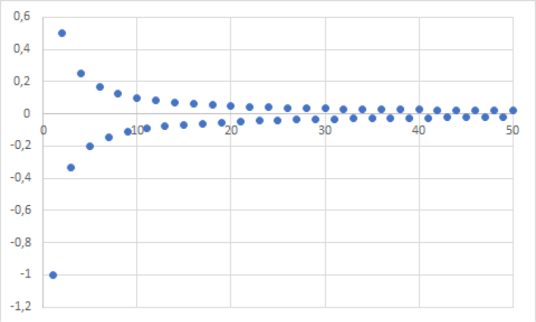
Let's compute first $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right|$. Observe that:

$$\left|\frac{(-1)^n}{n}\right| = \frac{|(-1)^n|}{|n|} = \frac{1}{n}$$

But we know that $\frac{1}{n} \to 0$. Thus, by the absolute value theorem, we conclude that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Sequences: some useful theorems, cont'd

$$s_n = \frac{(-1)^n}{n}$$



We easily see from the plot that the sequence converges to zero.

Sequences: some useful theorems, cont'd

Theorem (The comparison theorem)

Let $(a_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *be three sequences such that*

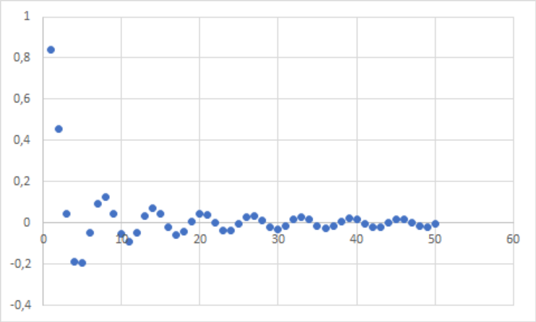
- $a_n \leq s_n \leq b_n$ *for every* n
- $\lim_{n\to+\infty} a_n = \ell$ *and* $\lim_{n\to+\infty} b_n = \ell$

Then

$$\lim_{n\to+\infty} s_n = \ell$$

Example

Consider the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \frac{\sin(n)}{n}$



Sequences: some useful theorems, cont'd

Example

Consider the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \frac{\sin(n)}{n}$.
Since

$$-1 \leq \sin(n) \leq 1,$$

then

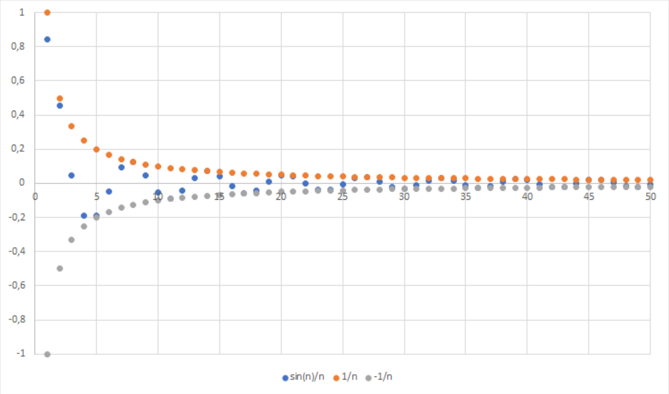
$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}, \qquad \text{for every } n.$$

We call $a_n = \frac{-1}{n}$ and $b_n = \frac{1}{n}$ and observe that $\lim_{n\to+\infty} \frac{-1}{n} = 0$ and $\lim_{n\to+\infty} \frac{1}{n} = 0$. By the comparison theorem also

$$\lim_{n\to+\infty} \frac{\sin(n)}{n} = 0$$

Sequences: some useful theorems, cont'd

Example, cont'd



Sequences: some useful theorems, cont'd

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ *and* $(q_n)_{n\in\mathbb{N}}$ *be two sequences with* $s_n \to \ell$ *and* $q_n \to \ell'$, $\ell, \ell' \in \mathbb{R}$ *(finite numbers). Then:*

- $s_n + q_n \to \ell + \ell'$

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Moreover,

- *If* $s_n \to +\infty$ *and* $q_n \to +\infty$, *then* $s_n + q_n \to +\infty$ *and* $s_n \cdot q_n \to +\infty$.

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- *If* $s_n \to -\infty$ *and* $q_n \to -\infty$, *then* $s_n + q_n \to -\infty$ *and* $s_n \cdot q_n \to +\infty$.
- *If* $s_n \to +\infty$ *and* $q_n \to -\infty$, *then* $s_n + q_n$ *is* **undetermined** *and* $s_n \cdot q_n \to -\infty$.

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Can we always use these rules?

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- ∞^0 and 0^0 and 1^∞ are INDETERMINATE