

Exercises, session 1 — Solutions

1. Determine the domain of the following functions

10–10 min

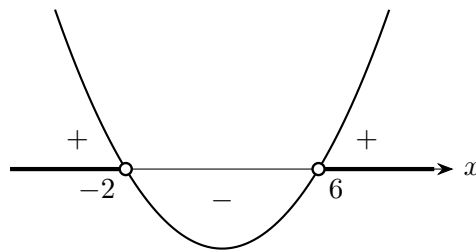
(a) $f(x) = 3x^2 - 4x + 7$

Polynomial function $\Rightarrow \boxed{D_f = \mathbb{R}}$

(b) $f(x) = \ln(x^2 - 4x - 12)$

$$x^2 - 4x - 12 = (x - 6)(x + 2) > 0 \Rightarrow x < -2 \text{ or } x > 6$$

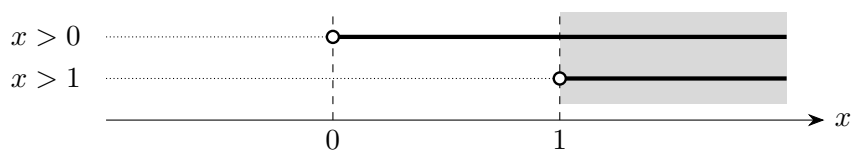
$$\boxed{D_f = (-\infty, -2) \cup (6, \infty)}$$



(c) $f(x) = \ln(\ln x)$

$$\begin{cases} x > 0 \\ \ln x > 0 \end{cases} \Rightarrow \begin{cases} x > 0 \\ x > 1 \end{cases} \Rightarrow x > 1$$

$$\boxed{D_f = (1, \infty)}$$



(d) $f(x) = \log(1 - 3^x)$

$$1 - 3^x > 0 \Rightarrow 3^x < 1 = 3^0 \Rightarrow x < 0$$

$$\boxed{D_f = (-\infty, 0)}$$

2. Solve the following exponential equations and inequalities

10–10 min

(a) $5^x \cdot 125 = 25$

$$5^x \cdot 5^3 = 5^2 \Rightarrow 5^{x+3} = 5^2 \Rightarrow \log_5(5^{x+3}) = \log_5(5^2) \Rightarrow x + 3 = 2 \Rightarrow \boxed{x = -1}$$

(b) $\exp(-2x) = e^5$

$$e^{-2x} = e^5 \Rightarrow \ln(e^{-2x}) = \ln(e^5) \Rightarrow -2x = 5 \Rightarrow \boxed{x = -\frac{5}{2}}$$

(c) $8 \cdot 16^x = 4^x \cdot 2$

$$\begin{aligned} 2^3 \cdot (2^4)^x &= (2^2)^x \cdot 2^1 \Rightarrow 2^3 \cdot 2^{4x} = 2^{2x} \cdot 2^1 \Rightarrow 2^{4x+3} = 2^{2x+1} \\ &\Rightarrow \log_2(2^{4x+3}) = \log_2(2^{2x+1}) \Rightarrow 4x + 3 = 2x + 1 \Rightarrow \boxed{x = -1} \end{aligned}$$

(d) $5^x + 3 \cdot 5^x > 2 \cdot 5^x + 10$

$$2 \cdot 5^x > 10 \Rightarrow 5^x > 5 \Rightarrow \boxed{x > 1}$$

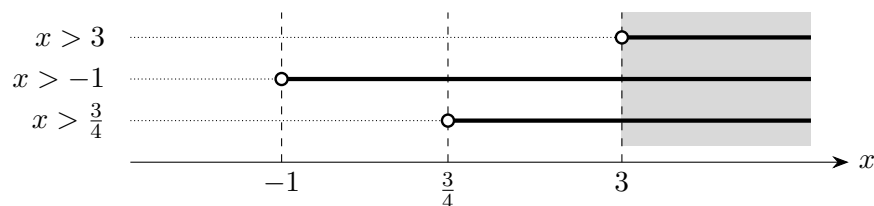
3. Solve the following logarithmic equations

20-20 min

(a) $\ln(x-3) + \ln(x+1) = \ln(4x-3)$

Domain:

$$\begin{cases} x-3 > 0 \\ x+1 > 0 \\ 4x-3 > 0 \end{cases} \Rightarrow \begin{cases} x > 3 \\ x > -1 \\ x > \frac{3}{4} \end{cases} \Rightarrow x > 3$$

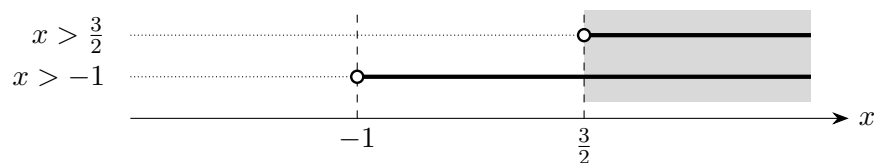


$$\begin{aligned} \ln((x-3)(x+1)) &= \ln(4x-3) \Rightarrow \exp(\ln((x-3)(x+1))) = \exp(\ln(4x-3)) \Rightarrow \\ (x-3)(x+1) &= 4x-3 \Rightarrow x^2 - 6x = 0 \Rightarrow x = 0 \text{ (not acceptable)} \text{ or } x = 6 \text{ (acceptable)} \\ &\Rightarrow \boxed{x = 6} \end{aligned}$$

(b) $\log_3(2x-3) = 2 + \log_3(x+1)$

Domain:

$$\begin{cases} 2x-3 > 0 \\ x+1 > 0 \end{cases} \Rightarrow \begin{cases} x > \frac{3}{2} \\ x > -1 \end{cases} \Rightarrow x > \frac{3}{2}$$



$$\begin{aligned} 3^{\log_3(2x-3)} &= 3^{2+\log_3(x+1)} (= 9 \cdot 3^{\log_3(x+1)}) \\ 2x-3 &= 9(x+1) \Rightarrow 7x = -12 \Rightarrow x = -\frac{12}{7} \text{ (not acceptable)} \\ &\Rightarrow \boxed{\text{No solutions}} \end{aligned}$$

(c) $2(\log_3 x)^2 + 3\log_3 x - 2 = 0$

Domain of $f(x) = \log_3(x)$: $x > 0 \Rightarrow D_f = (0, \infty)$.

Set $\log_3 x = t$:

$$2t^2 + 3t - 2 = 0 \Rightarrow (2t - 1)(t + 2) = 0 \Rightarrow t = \frac{1}{2} \quad \text{or} \quad t = -2$$

$$\log_3 x = \frac{1}{2} \Rightarrow x = \sqrt{3} \text{ (acceptable)} \quad \log_3 x = -2 \Rightarrow x = \frac{1}{9} \text{ (acceptable)}$$

$$\boxed{x = \sqrt{3} \quad \text{or} \quad x = \frac{1}{9}}$$

4. Solve the following equations with absolute values

20-20 min

(a) $|x^2 + 3x| = -2$

$$|x^2 + 3x| \geq 0, \text{ while } -2 < 0 \Rightarrow \boxed{\text{No solutions}}$$

(b) $|7x + 3| = 2 - x$

Since $|7x + 3| \geq 0$, we have to impose $2 - x \geq 0$, which means we have to look for solution in the region $x \leq 2$. Then, distinguish on the possible sign of $7x + 3$, we have

$$\begin{cases} 7x + 3 \geq 0 \\ 7x + 3 = 2 - x \end{cases} \Rightarrow \begin{cases} x \geq -\frac{3}{7} \\ 8x = -1 \end{cases} \Rightarrow \begin{cases} x \geq -\frac{3}{7} \\ x = -\frac{1}{8} \end{cases} \quad (\text{acceptable})$$

$$\begin{cases} 7x + 3 < 0 \\ -7x - 3 = 2 - x \end{cases} \Rightarrow \begin{cases} x < -\frac{3}{7} \\ -6x = 5 \end{cases} \Rightarrow \begin{cases} x < -\frac{3}{7} \\ x = -\frac{5}{6} \end{cases} \quad (\text{acceptable})$$

Since both the solutions are smaller than or equal to 2, we can accept both.

$$\boxed{x = -\frac{1}{8} \quad \text{or} \quad x = -\frac{5}{6}}$$

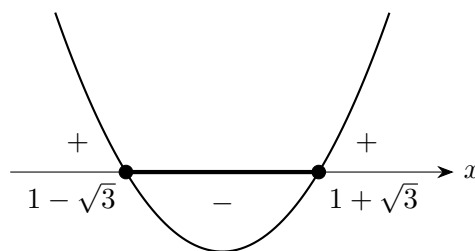
(c) $\left| \frac{x^2 + 2}{x + 2} \right| \leq 2$

Domain of $f(x) = \frac{x^2+2}{x+2}$: $x \neq -2 \Rightarrow D_f = (-\infty, -2) \cup (-2, \infty)$.

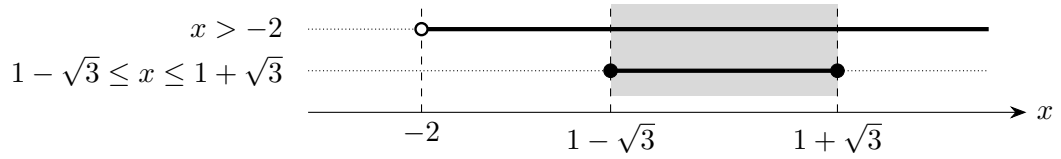
$$\left| \frac{x^2 + 2}{x + 2} \right| \leq 2 \iff \frac{x^2 + 2}{|x + 2|} \leq 2 \iff x^2 + 2 \leq 2|x + 2|$$

Case $x + 2 > 0$:

$$\begin{cases} x + 2 > 0 \\ x^2 + 2 \leq 2x + 4 \end{cases} \Rightarrow \begin{cases} x > -2 \\ x^2 - 2x - 2 \leq 0 \end{cases} \quad x^2 - 2x - 2 = 0 \Rightarrow x_{1,2} = 1 \pm \sqrt{3}$$

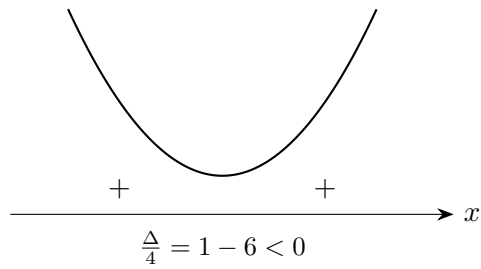


$$\begin{cases} x > -2 \\ 1 - \sqrt{3} \leq x \leq 1 + \sqrt{3} \end{cases} \Rightarrow 1 - \sqrt{3} \leq x \leq 1 + \sqrt{3}$$

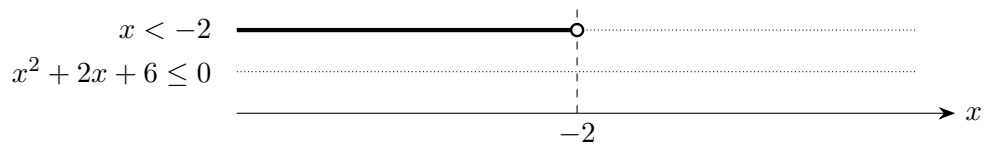


Case $x + 2 < 0$:

$$\begin{cases} x < -2 \\ x^2 + 2 \leq -2x - 4 \end{cases} \Rightarrow \begin{cases} x < -2 \\ x^2 + 2x + 6 \leq 0 \end{cases} \quad x_{1,2} = -1 \pm \sqrt{1-6} \notin \mathbb{R}$$



$$x^2 + 2x + 6 > 0 \quad \forall x \in \mathbb{R} \Rightarrow \text{No solutions}$$



Final answer:

$$\boxed{1 - \sqrt{3} \leq x \leq 1 + \sqrt{3}}$$