$\boxed{1}$ INTEGRALS BY SUBSTITUTION:

(a) $\displaystyle \int x\sqrt{x^2+3}\,dx = \int \frac{f'(x)}{2}\sqrt{f(x)}\,dx = \int \frac{\sqrt{t}}{2}\,dt = \left.\frac{t^{3/2}}{2\cdot 3/2}\right|_{t=x^2+3} + c =$

$f(x) = x^2+3$
$f'(x) = 2x$

SUBSTITUTION:
$t = f(x)$
$dt = f'(x)\,dx$

$\displaystyle = \frac{(x^2+3)^{3/2}}{3} + c$

(b) $\displaystyle \int \frac{\cos(x)}{\sin^4(x)}\,dx = \int f'(x)(f(x))^{-4}\,dx = \int t^{-4}\,dt = \left.\frac{t^{-3}}{-3}\right|_{t=\sin x} + c =$

$f(x) = \sin(x)$
$f'(x) = \cos(x)$

SUBSTITUTION
$t = f(x)$
$dt = f'(x)\,dx$

$\displaystyle = -\frac{(\sin x)^{-3}}{3} + c$

(c) $\displaystyle \int \frac{1+e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \left(1+e^{f(x)}\right)2f'(x)\,dx = 2\int\left(1+e^{t}\right)dt =$

$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2\sqrt{x}}$

SUBSTITUTION
$t = f(x)$
$dt = f'(x)\,dx$

$\displaystyle = \left.2t + 2e^{t}\right|_{t=\sqrt{x}} + c = 2\sqrt{x} + 2e^{\sqrt{x}} + c$

(d) $\displaystyle \int 5x^2\cos(x^3+5)\,dx = \int \frac{5}{3}f'(x)\cdot\cos(f(x))\,dx = \frac{5}{3}\int \cos t\,dt$

$f(x) = x^3+5$
$f'(x) = 3x^2$

SUBSTITUTION
$t = f(x)$
$dt = f'(x)\,dx$

$\displaystyle = \left.\frac{5}{3}\sin t\right|_{t=x^3+5} = \frac{5}{3}\sin(x^3+5)$

(e) $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx = \int f(x)\cdot f'(x)\,dx = \int t\,dt =$

$f(x) = \arctan(x)$

$f'(x) = \dfrac{1}{1+x^2}$

SUBSTITUTION

$t = f(x)$

$dt = f'(x)\,dx$

$= \left.\dfrac{t^2}{2} + c\right|_{t=\arctan x} =$

$= \dfrac{(\arctan x)^2}{2} + c$

(f) $\displaystyle\int \frac{1}{(1+x^2)\arctan x}\,dx = \int (f(x))^{-1} f'(x)\,dx = \int t^{-1}\,dt =$

$f(x) = \arctan x$

$f'(x) = \dfrac{1}{1+x^2}$

$t = f(x)$

$dt = f'(x)\,dx$

$= \ln|t| + c\Big|_{t=\arctan x} =$

$= \ln|\arctan x| + c$

$\boxed{2}$ INTEGRALS BY PARTS

FORMULA: $\displaystyle\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$

MAIN APPLICATIONS:

- $\displaystyle\int e^{\alpha x}\cdot P(x)$ $\quad P(x)$ a POLYNOMIAL WITH DEGREE $= n$

THE CHOICE: $f(x) = P(x)$, $g'(x) = e^{\alpha x}$

WE HAVE TO DO INTEGRATION BY PARTS n-TIMES

- $\int \sin(\alpha x)\, P(x)$ OR $\int \cos(\alpha x)\, P(x)$

THE CHOICE: $f(x) = P(x)$, $g'(x) = \sin(\alpha x)$

WE HAVE TO DO INTEGRATION BY PARTS n-TIMES

- $\int \log(\alpha x)\, P(x)$

THE CHOICE: $f(x) = \log(\alpha x)$, $g'(x) = P(x)$

INTEGRATION BY PARTS ONLY ONCE

- $\int e^x \sin(x)$ OR $\int e^x \cos(x)$ OR SIMILAR

THE CHOICE $f(x) = e^x$, $g'(x) = \sin x$ $\rightsquigarrow$ $f'(x) = e^x$, $g(x) = -\cos x$

$$\Rightarrow \int e^x \sin x = -e^x \cos x + \int e^x \cos x$$

DON'T GIVE UP AND REAPPLY THE INTEGRATION BY PARTS

$f(x) = e^x$, $g'(x) = \cos(x)$ $\qquad$ $f'(x) = e^x$, $g(x) = \sin x$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x$$

SO WE HAVE:

$$\int e^x \sin x = -e^x \cos x + e^x \sin x - \int e^x \sin x$$

$$\Rightarrow \quad 2\int e^x \sin x = e^x(\sin x - \cos x)$$

$$\Rightarrow \quad \int e^x \sin x = \frac{e^x(\sin x - \cos x)}{2} + c$$

(a) $\int x \sin(x)\, dx = -x\cos(x) + \int \cos(x) = -x\cos(x) + \sin(x) + c$

BY PARTS

$f(x) = x \quad g'(x) = \sin(x)$

$f'(x) = 1 \quad g(x) = -\cos(x)$

(b) $\int (x^2+1)\cos(x)\, dx = (x^2+1)\sin x - 2\int x \sin x =$

BY PARTS

$f(x) = x^2+1 \quad g'(x) = \cos(x)$

$f'(x) = 2x \quad g(x) = \sin(x)$

$= (x^2+1)\sin x - 2\left[-x\cos x - \int \cos x\right] = (x^2+1)\sin x + 2x\cos x$

$+ 2\sin x + C$

BY PARTS

$f(x) = x \quad g'(x) = \sin x$

$f'(x) = 1 \quad g(x) = -\cos x$

(c) $\int (3x+2)e^x\, dx = (3x+2)e^x - \int 3e^x dx = (3x+2)e^x - 3e^x + c$

BY PARTS $\quad f(x) = 3x+2 \quad g'(x) = e^x$

(d) $\int \frac{2x+1}{e^x}\,dx = \int (2x+1)e^{-x}\,dx \underset{\text{BY PARTS}}{=} -e^{-x}(2x+1) + 2\int e^{-x} =$

BY PARTS

$f(x) = 2x+1 \quad g'(x) = e^{-x}$

$f'(x) = 2 \quad g(x) = -e^{-x}$

REMEMBER

$\int f(2x) = \frac{F(2x)}{2}$ $\longrightarrow$

WHERE $F = \int f$

$= -e^{-x}(2x+1) - 2e^{-x} + c$

(e) $\int x^8 \log x\,dx \underset{\text{BY PARTS}}{=} x^9/9 \cdot \log x - \int \frac{x^8}{9} = x^9/9 \log x - \frac{x^9}{81} + c$

BY PARTS

$f(x) = \log x \quad g'(x) = x^8$

$f'(x) = \frac{1}{x} \quad g(x) = \frac{x^9}{9}$

(F) $\int \log(x)\,dx \underset{\text{BY PARTS}}{=} x\log x - \int 1\,dx = x\log x - x + c$

BY PARTS

$f(x) = \log x \quad g'(x) = 1$

$f'(x) = \frac{1}{x} \quad g(x) = x$

$\boxed{3}$ (a) $\int \frac{x}{1+x}\,dx = \int \frac{x+1-1}{1+x}\,dx = \int \frac{x+1}{x+1}dx - \int \frac{1}{x+1}dx =$

$= \int 1\,dx - \int \frac{1}{x+1}\,dx = x - \ln|x+1| + c$

(b) $\int \frac{x^3+x}{\sqrt{2x^2+x^4}}\,dx = \int (x^3+x)\,(x^4+2x^2)^{-\frac{1}{2}}\,dx =$

$= \int f(x)^{-\frac{1}{2}}\,\frac{f'(x)}{4}\,dx = \int \frac{t^{-\frac{1}{2}}}{4}\,dt =$

$f(x) = x^4+2x^2$
$f'(x) = 4x^3+4x$

BY SUB:
$t = f(x)$
$dt = f'(x)\,dx$

$= \left.\frac{t^{\frac{1}{2}}}{4\cdot\frac{1}{2}}\right|_{t=x^4+2x^2} + c = \frac{(x^4+2x^2)^{\frac{1}{2}}}{2} + c$

(c) $\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+t}\,2t\,dt = 2\int \frac{t}{1+t}\,dt = 2\left[\int 1\,dt - \int \frac{dt}{1+t}\right] =$

BY SUB
$t = \sqrt{x}$
$\leadsto x = t^2 \leadsto dx = 2t\,dt$

$= \left. 2t - 2\ln|1+t| + c \right|_{t=\sqrt{x}} = 2\sqrt{x} - 2\ln|1+\sqrt{x}| + c$

(d) $\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx =$

BY PARTS
$f(x) = x^3 \quad g'(x) = e^x$
$f'(x) = 3x^2 \quad g(x) = e^x$

BY PARTS
$f(x) = x^2 \quad g'(x) = e^x$
$f'(x) = 2x \quad g(x) = e^x$

$= x^3 e^x - 3\left[x^2 e^x - 2\int x e^x\,dx\right] = x^3 e^x - 3x^2 e^x + 6\int x e^x\,dx =$

BY PARTS

$f(x) = x \quad g'(x) = e^x$

$f'(x) = 1 \quad g(x) = e^x$

$= x^3 e^x - 3x^2 e^x + 6\left[x e^x - \int e^x \, dx \right] =$

$= x^3 e^x - 3x^2 e^x + 6x e^x - e^x + C.$

(e) $\int x^3 e^{x^2} \, dx = \int x^2 \cdot x e^{x^2} \, dx =$

BY PARTS

$f(x) = x^2 \quad f'(x) = 2x$

$g'(x) = x e^{x^2} \xrightarrow[\text{SUB}]{\text{BY}} g(x) = \frac{e^{x^2}}{2}$

$= x^2 \, e^{x^2}/2 - \int x e^{x^2} \, dx =$

$= x^2 \, e^{x^2}/2 - e^{x^2}/2 + C$

(f) $\int \frac{x \, e^{\sqrt{1+x^2}}}{\sqrt{1+x^2}} \, dx = \int f(x) \cdot f'(x) \, dx =$

$f(x) = e^{\sqrt{1+x^2}}$

$f'(x) = e^{\sqrt{1+x^2}} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}$

$$= \int t\, dt = \frac{t^2}{2} + c \Big|_{t = e^{\sqrt{1+x^2}}} = \frac{e^{2\sqrt{1+x^2}}}{2} + c$$

BY SUBSTITUTION

$t = f(x) \quad dt = f'(x)\,dx$

(g) $$\int \cos^3 dx = \int \cos x \cdot \cos^2 x \, dx =$$

$$= \int \cos x \,(1 - \sin^2 x)\, dx = \int f'(x)\,(1 - f^2(x))\,dx$$

$f(x) = \sin x$

$f'(x) = \cos x$

$$= \int (1 - t^2)\, dt =$$

BY SUB

$t = f(x)$

$dt = f'(x)\,dx$

$$= t - \frac{t^3}{3} + c \Big|_{t = \sin x} =$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

(h) $$\int \cos^2(x)\, dx$$

FIRST METHOD :

BY PARTS

$$\int \cos x \cdot \cos \, dx = \cos x \cdot \sin x + \int \sin^2 x =$$

$f(x) = \cos x \qquad g'(x) = \cos x$

$f'(x) = -\sin x \qquad g(x) = \sin x$

$$= \cos x \cdot \sin x + \int (1 - \cos^2 x)\, dx =$$

$$= \cos x \cdot \sin x + x - \int \cos^2 x \, dx$$

SO WE HAVE:

$$\int \cos^2 x \, dx = \cos x \cdot \sin x + x - \int \cos^2 x \, dx$$

$$\Rightarrow 2 \int \cos^2 x \, dx = \cos x \cdot \sin x + x$$

$$\Rightarrow \int \cos^2 x \, dx = \frac{\cos x \cdot \sin x}{2} + x/2 + c$$

SECOND METHOD:

BY TRIGONOMETRIC SUBSTITUTION

$$\cos^2(x) = \frac{1 + \cos 2x}{2}$$ FROM BISECTION'S FORMULA

$$\sin^2(x) = \frac{1 - \cos 2x}{2}$$

$$\int \cos^2(x)\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx + \int \frac{\cos 2x}{2}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \cos(2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + c$$

(i) $\int e^x \cos x = \sin x \, e^x - \int e^x \sin x =$

BY PARTS

$f(x) = e^x \quad g'(x) = \cos x$

$f'(x) = e^x \quad g(x) = \sin x$

$$= \sin x \cdot e^x - \left[-e^x \cdot \cos x + \int e^x \cos x \right] =$$

BY PARTS

DON'T GIVE UP!

$$= \sin x \, e^x + e^x \cos x - \int e^x \cos x \, dx$$

$f(x) = e^x \quad g'(x) = \sin x$

$f'(x) = e^x \quad g(x) = -\cos x$

$$\Rightarrow \int e^x \cos x = \sin x \, e^x + e^x \cos x - \int e^x \cos x \, dx$$

$$\Rightarrow 2\int e^x \cos x = e^x(\sin x + \cos x)$$

$$\Rightarrow \int e^x \cos x = \frac{e^x(\sin x + \cos x)}{2} + c$$

$\boxed{4}$

$$\int f(x) = F(x)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(a) $$\int_0^1 \frac{4x}{\sqrt{(x^2+8)^3}}\,dx = \int_0^1 4x\cdot(x^2+8)^{-3/2}\,dx =$$

$$= \int_0^1 2f'(x)\,f(x)^{-3/2}\,dx = \int_8^9 2t^{-3/2}\,dt = \left[\frac{2t^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_8^9 =$$

BY SUB

$t = f(x)$

$dt = f'(x)\,dx$

$$= \left[-4t^{-\frac{1}{2}}\right]_8^9 =$$

PAY ATTENTION :

$0 \xrightarrow[t=f(0)]{} 8$

$1 \xrightarrow[t=f(1)]{} 9$

$$= -4(9)^{-\frac{1}{2}} + 4(8)^{-\frac{1}{2}} =$$

$$= \frac{4}{\sqrt{8}} - \frac{4}{\sqrt{9}}$$

(b) $$\int_6^7 x(x-6)^9\,dx = \left[\frac{x(x-6)^{10}}{10}\right]_6^7 - \int_6^7 \frac{(x-6)^{10}}{10}\,dx =$$

BY PARTS

$f(x) = x \quad g'(x) = (x-6)^9$

$f'(x) = 1 \quad g(x) = \frac{(x-6)^{10}}{10}$

$$= 7/10 - 0 - \left[\frac{(x-6)^{11}}{110}\right]_6^7 = 7/10 - \left(\frac{1}{110} - 0\right) =$$

$$= 7/10 - \frac{1}{110} = \frac{77-1}{110} = \frac{76}{110} = \frac{38}{55}$$

(c) $$\int_0^{\pi/2} \frac{\cos^3(x)}{1+\sin^2(x)}\, dx = \int_0^{\pi/2} \frac{\cos^2(x)\cdot\cos(x)}{1+\sin^2(x)}\, dx =$$

BY SUB

$t = \sin x \qquad 0 \longmapsto 0$

$dt = \cos x\, dx \qquad \pi/2 \longmapsto 1$

$$= \int_0^1 \frac{(1-t^2)}{1+t^2}\, dt =$$

$$= \int_0^1 \frac{1}{1+t^2}\, dt - \int_0^1 \frac{t^2}{1+t^2}\, dt = \Big[\operatorname{ARCTAN} t\Big]_0^1 - \int_0^1 \frac{t^2+1-1}{t^2+1}$$

$$= \pi/4 - 0 - \left[\int_0^1 1\, dt - \int_0^1 \frac{dt}{t^2+1}\right] =$$

$$= \pi/4 - 1 + \Big[\operatorname{ARCTAN} t\Big]_0^1 = \pi/4 - 1 + \pi/4 = \pi/2 - 1$$

$\boxed{5}$ $$F(x) = \int_0^{x^2-x} \frac{e^{t^2+2t}}{\sqrt{t^2+3}}\, dt$$

CAN BE SEEN LIKE THE COMPOSITION BETWEEN:

$$G(x) = \int_0^{x} \frac{e^{t^2+2t}}{\sqrt{t^2+3}}\, dt \quad \text{AND} \quad g(x) = x^2 - x$$

$\rightarrow \dfrac{e^{t^2+2t}}{\sqrt{t^2+3}}$ IS CONTINUOS

SO $G(x)$ IS C^1

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

$$\Rightarrow G(g(x))$$

HOW IS THE DERIVATIVE OF THE COMPOSITION?

$$\frac{d}{dx} G(g(x)) = G'(g(x)) \cdot g'(x)$$

$$g'(x) = 2x - 1$$

$$G'(x) = \frac{e^{x^2+2x}}{\sqrt{x^2+3}} \longrightarrow G'(g(x)) = \frac{e^{(x^2-x)^2+2(x^2-x)}}{\sqrt{(x^2-x)^2+3}}$$

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

$$\Rightarrow \frac{d}{dx} F(x) = \frac{e^{(x^2-x)^2+2(x^2-x)}}{\sqrt{(x^2-x)^2+3}} (2x-1)$$

$F'(x) = 0$ ← IN ORDER TO FIND CRITICAL POINTS WE HAVE TO RESOLVE THIS EQUATION

$$\frac{e^{(x^2-x)^2+2(x^2-x)}}{\sqrt{(x^2-x)^2+3}}(2x-1) = 0$$

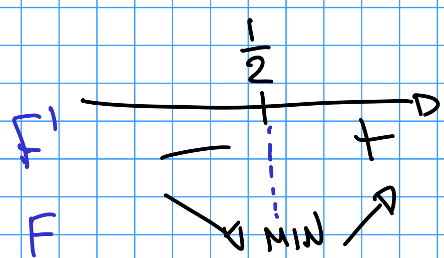
$$\Leftrightarrow \quad 2x-1=0 \quad \Leftrightarrow \quad \boxed{x=\frac{1}{2}}$$

CRITICAL POINT

BECAUSE $\frac{e^{(x^2-x)^2+2(x^2-x)}}{\sqrt{(x^2-x)^2+3}}$ IS EVER DIFFERENT FROM 0 (>0)

IN ORDER TO STUDY THE INTERVALS ON WICH F IS INCREASING AND DECREASING, WE HAVE TO STUDY

$$F'(x) \geq 0 \quad \Leftrightarrow \quad 2x-1 \geq 0 \quad \Leftrightarrow \quad x \geq \frac{1}{2}$$



F IS INCREASING IN $\left(\frac{1}{2}, +\infty\right)$

F IS DECREASING IN $\left(-\infty, \frac{1}{2}\right)$

$\boxed{6}$ (a) $\lim\limits_{x\to 0} \dfrac{\int_0^x e^{t^2}\,dt}{\sin(x)}$ (b) $\lim\limits_{x\to 0} \dfrac{\int_0^x \sqrt{t+1}\,\cos(t)\,dt}{x^2+x}$

BOTH THE LIMITS ARE INDETERMINATE FORMS $\frac{0}{0}$, SO WE CAN TRY TO APPLY DE L'HOSPITAL'S THEOREM. INFACT:

- NUMERATOR AND DENOMINATOR ARE DERIVABLE
- MOREOVER THE DENOMINATOR'S DERIVATIVE IS NOT NULL NEAR 0

THEN

(a) $$\lim_{x \to 0} \frac{\left(\int_0^x e^{t^2}\,dt\right)'}{(\sin x)'} = \lim_{x \to 0} \frac{e^{x^2}}{\cos(x)} = \frac{1}{1} = \boxed{1}$$

ON THE NUMERATOR WE HAVE APPLIED THE FUNDAMENTAL THEOREM

(b) $$\lim_{x \to 0} \frac{\left(\int_0^x \sqrt{t+1}\,\cos(t)\,dt\right)'}{(x^2+x)'} = \lim_{x \to 0} \frac{\sqrt{x+1}\,\cos x}{2x+1} = \frac{1}{1} = \boxed{1}$$

SO, SINCE THE LIMITS OF THE RATIO BETWEEN THE DERIVATIVES EXIST WE HAVE FOUND THE INITIAL LIMITS.