MATRICES

A MATRIX is a rectangular array of real numbers displayed in rows and columns

EXAMPLES:

i) $\begin{pmatrix} 1 & 0 & 2 & -3 \\ 8 & 4 & 5 & -2 \end{pmatrix}$ ii) $\begin{pmatrix} 0 & 2 & -3 \\ 1 & 1 & 1 \\ 4 & -1 & -2 \end{pmatrix}$

A matrix with m rows and n columns is called a "$m \times n$" (m by n) matrix.

The entry in the intersection of the i-th row ($i = 1, \dots, m$) and the j-th column ($j = 1, \dots, n$) is denoted with the symbol a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$

For example in the previous matrix i) the element $a_{23} = 5$.

So a general $m \times n$ matrix is an array of the following form

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{k1} & a_{k2} \cdots & a_{kn} \end{pmatrix}$$

MATRICES IN GENERAL ARE INDICATED BY CAPITAL LETTERS

$M_{m \times n}$ denotes the family of all $m \times n$ matrices.

PARTICULAR MATRICES

- A matrix made by one single row is called a ROW MATRIX, and it is a $1\times n$ matrix:

  Example: $\begin{pmatrix} -1 & 2 & 0 & 3 \end{pmatrix}$ is a 1×4 row matrix

- A matrix made by one single column is called a COLUMN MATRIX, and it is an $m\times1$ matrix

  Example: $\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$ is a 3×1 column matrix

- A matrix with same number of rows and columns, $m=n$, is called a SQUARE matrix of ORDER n

  Example: $A=\begin{pmatrix} 3 & 0 & 6 \\ 1 & 2 & 1 \\ 0 & 5 & 7 \end{pmatrix}$ is a square matrix of order 3

  In a square matrix the elements a_{ij} such that $i=j$ are said to belong to the PRINCIPAL DIAGONAL.

- A square matrix in which all the elements that don't belong to the principal diagonal are null, is called a DIAGONAL matrix:

if $A \in M_{n\times n}$ and $a_{ij}=0$ for $i\neq j$ then A is called DIAGONAL

EXAMPLE:

$$A=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in M_{4\times 4}$$

⊙ A diagonal matrix in which all $a_{ii}=1$ $\forall i=1,\dots,n$ is called IDENTITY or UNIT matrix, and it will be denoted by I or by I_n

EXAMPLES: $I_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

⊙ A square matrix where all the elements below the diagonal are null is called UPPER TRIANGULAR MATRIX

$U \in M_{n\times n}$ such that $u_{ij}=0$ for $i>j$ is called UPPER TRIANGULAR

EXAMPLE: $U=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 3 \end{pmatrix}$

whereas if $L \in M_{n \times n}$ such that $l_{ij} = 0$ for $i < j$ is called LOWER TRIANGULAR MATRIX (all elements above the diagonal are null)

EXAMPLE:

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 7 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 3 & 10 & 4 & 9 \end{pmatrix}$$

- A square matrix $A \in M_{n \times n}$ such that $\underbrace{a_{ij} = a_{ji}}_{}$ is called SYMMETRIC

ALL ELEMENTS SYMMETRIC WITH RESPECT TO THE DIAGONAL ARE EQUAL

EXAMPLE:

$$A = \begin{pmatrix} 1 & 2 & 6 & 5 \\ 2 & 5 & 7 & -3 \\ 6 & 7 & 4 & 0 \\ 5 & -3 & 0 & 3 \end{pmatrix} \in M_{4 \times 4}$$

OPERATIONS BETWEEN MATRICES

Matrices can be added, multiplied by a number, multiplied between them

REMARK: Two matrices are confrontable if and only if they have the same dimension. And they are equal if and only if every element of the first matrix is equal to the correspondent element of the other one.

EXAMPLE: the equality

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

implies that $x=2$, $y=0$, $z=3$, $t=1$.

ADDITION Two matrices can be added only if they have the same dimension $m \times n$. The result is still a matrix of the same dimension $m \times n$ obtained by adding the correspondent elements

$$A, B \in M_{m \times n} \Rightarrow C = A + B \in M_{m \times n} \text{ and}$$

$$c_{ij} = a_{ij} + b_{ij}.$$

THE NULL MATRIX (any matrix with all null entries) is the ADDITIVE UNITY

EXAMPLE:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & -2 & 0 & 3 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

$$\Rightarrow A + B = \begin{pmatrix} -1 & -1 & 2 & 6 \\ 6 & 2 & 2 & 4 \end{pmatrix}$$

MULTIPLICATION OF A MATRIX BY A NUMBER

Given a number $\lambda \in \mathbb{R}$ and $A \in M_{m \times n}$

$\Longrightarrow$ $P = \lambda \cdot A \in M_{m \times n}$ and is such that

$(p_{ij}) = (\lambda a_{ij})$

EXAMPLE: Given the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \\ 1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$$

Evaluate:

i) $2B$

$$2B = 2 \cdot \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 2 & 4 \\ 4 & 6 \end{pmatrix}$$

ii) $A - 3B = \begin{pmatrix} 1 & 0 \\ 3 & 0 \\ 1 & 4 \end{pmatrix} - 3 \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1-3\cdot 0 & 0-3\cdot 2 \\ 3-3\cdot 1 & 0-3\cdot 2 \\ 1-3\cdot 2 & 4-3\cdot 3 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & -6 \\ -5 & -5 \end{pmatrix}$$

PRODUCT "ROW BY COLUMN" BETWEEN MATRICES

The product between two matrices A and B (in this order) is possible if the number of columns of A is equal to the number of rows of B:

If $A \in M_{m \times n}$ and $B \in M_{n \times k}$

can be multiplied through the ROW BY COLUMN PRODUCT and the result is an $m \times k$ matrix

$$\Rightarrow C = A \cdot B \in M_{m \times k}$$

and the generic element c_{ij} of C is given by the sum of the products of the elements of the i-th row of A by the correspondent elements of the j-th column of B (the first by the first, the second by the second ...)

$$c_{ij} = (A \cdot B)_{ij} = (a_{i1} \; a_{i2} \ldots a_{im}) \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{im} b_{mj} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

REMARK: The product between matrices is not commutative, moreover if $A \cdot B$ is executable, not necessarily $B \cdot A$ is. Indeed if $k \neq m$, $B \cdot A$ cannot be calculated

EXERCISE: Calculate $A \cdot B$ and verify if $A \cdot B$ exists

$$A=\begin{pmatrix}2 & -1 & 0\\ 3 & 1 & -3\\ 1 & 0 & -1\end{pmatrix} \text{ and } B=\begin{pmatrix}-2 & 0\\ 2 & -3\\ 0 & 1\end{pmatrix}$$

Solution:

$$\underbrace{A\cdot B}_{A_{3\times 3}B_{3\times 2}=C_{3\times 2}}=\begin{pmatrix}2 & -1 & 0\\ 3 & 1 & -3\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}-2 & 0\\ 2 & -3\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2\cdot(-2)+(-1)\cdot 2+0\cdot 0 & 2\cdot 0+(-1)\cdot(-3)+0\cdot 1\\ 3\cdot(-2)+1\cdot 2+(-3)\cdot 0 & 3\cdot 0+1\cdot(-3)+(-3)\cdot 1\\ 1\cdot(-2)+0\cdot 2+(-1)\cdot 0 & 1\cdot 0+0\cdot(-3)+(-1)\cdot 1\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 3\\ -4 & -6\\ -2 & -1\end{pmatrix}$$

$B\cdot A$ is not executable as $B_{3\times 2}$ and $A_{3\times 3}$ (so number of columns of B is different from numbers of rows of A)

EXERCISE: Calculate CD and DC and verify that $CD \neq DC$

$$C=\begin{pmatrix}-1 & -1\\ 2 & 1\\ 1 & 0\end{pmatrix} \text{ and } D=\begin{pmatrix}-2 & 0 & 5\\ 1 & 4 & 1\end{pmatrix}$$

Solution:

$$\underbrace{\underset{3\times 2}{C}\;\underset{2\times 3}{D}}_{3\times 3} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 0 & 5 \\ 1 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1\cdot(-2)+(-1)\cdot 1 & -1\cdot 0+(-1)\cdot 4 & -1\cdot 5+(-1)\cdot 1 \\ 2\cdot(-2)+1\cdot 1 & 2\cdot 0+1\cdot 4 & 2\cdot 5+1\cdot 1 \\ 1\cdot(-2)+0\cdot 1 & 1\cdot 0+0\cdot 4 & 1\cdot 5+0\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -4 & -6 \\ -3 & 4 & 11 \\ -2 & 0 & 5 \end{pmatrix}$$

$$\underbrace{\underset{2\times 3}{D}\cdot\underset{3\times 2}{C}}_{2\times 2} = \begin{pmatrix} -2 & 0 & 5 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2\cdot(-1)+0\cdot 2+5\cdot 1 & -2\cdot(-1)+0\cdot 1+5\cdot 0 \\ 1\cdot(-1)+4\cdot 2+1\cdot 1 & 1\cdot(-1)+4\cdot 1+1\cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 2 \\ 8 & 3 \end{pmatrix} \Longrightarrow \quad C\cdot D \neq D\cdot C$$

PROPERTY OF THE IDENTITY MATRIX

Let I_n be the n-dimensional matrix, hence $\forall\, A \in M_{m\times n}$:

$$AI_n = A$$

and $\forall B \in M_{n \times k}$: $I_n B = B$

DEFINITION: A square matrix B is called IDEMPOTENT if $BB = B$

PROPERTIES OF MATRIX OPERATIONS

- ASSOCIATIVE PROPERTY

$$(A+B)+C = A+(B+C)$$

the sum or product of 3 or more matrices does not change if they are grouped in a different way

$$(AB)C = A(BC)$$

- COMMUTATIVE PROPERTY FOR ADDITION

$$A+B = B+A$$

the change in the order doesn't change the result

- DISTRIBUTIVE PROPERTY

$$A(B+C) = AB+AC$$

$$(A+B)\cdot C = AC+BC$$

REMARK: THE ZERO PRODUCT RULE DOESN'T HOLD FOR MATRIX MULTIPLICATION INDEED LET US SEE IT WITH AN EXAMPLE

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix}$$

$$\Rightarrow C = A \cdot B = \begin{pmatrix} 3\cdot 3 + 1\cdot(-9) & 3\cdot(-1)+1\cdot 3 \\ 6\cdot 3 + 2\cdot(-9) & 6\cdot(-1)+2\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ but nor A and B are null matrices

Hence in general, if $AX = AB$ this doesn't imply that $X = B$. Indeed

$$AX = AB \longrightarrow AX - AB = \underset{\text{null matrix}}{O} \longrightarrow A(X-B) = O$$

but if $A = O$ (← null matrix) this doesn't imply (as seen with the previous example) that $X = B$.

TRANSPOSITION OF MATRICES

DEFINITION: The TRANSPOSE of an $m \times n$ matrix A is an $n \times m$ matrix obtained by interchanging the rows and the columns of the matrix. This matrix is denoted by A^T.

This means that the (i,j)-th entry of A becomes the (j,i)-th entry of A^T

EXAMPLE:

i) $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \longrightarrow B^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

ii) $A = \begin{pmatrix} 1 & 3 & 5 \\ 4 & 6 & 1 \end{pmatrix} \longrightarrow A^T = \begin{pmatrix} 1 & 4 \\ 3 & 6 \\ 5 & 1 \end{pmatrix}$

PROPERTIES OF TRASPOSITION

i) $(A^T)^T = A$

ii) $(A+B)^T = A^T + B^T$

iii) $(A-B)^T = A^T - B^T$

iv) $(k \cdot A)^T = k \cdot A^T$

v) $(A \cdot B)^T = B^T \cdot A^T$

Let us verify the properties ii) and v) through an example

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \Longrightarrow$$

ii)

$$(A+B)^T = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$$

$$\|$$

$$A^T + B^T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$$

v)

$$(A \cdot B)^T = \begin{pmatrix} 1 \cdot 0 + 0 \cdot 2 & 1 \cdot 1 + 0 \cdot 2 \\ 2 \cdot 0 + 3 \cdot 2 & 2 \cdot 1 + 3 \cdot 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & 1 \\ 6 & 8 \end{pmatrix}^T = \begin{pmatrix} 0 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\|$$

$$B^T \cdot A^T = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 1 & 8 \end{pmatrix}$$

EXERCISE: Using the matrices

$$A=\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}, \quad D=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Calculate:

1. $A+B-2D$
2. $(2A-3B)^T D$
3. $(AD)^T C+B$

Solution:

1. $$A+B-2D=\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}+\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}-2\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 2-1-4 & 0+2-6 \\ -1+1-2 & 1-1-8 \end{pmatrix}=\begin{pmatrix} -3 & -4 \\ -2 & -8 \end{pmatrix}$$

2. $$(2A-3B)^T D=\left[2\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}-3\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\right]^T\cdot\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 4+3 & 0-6 \\ -2-3 & 2+3 \end{pmatrix}^T\cdot\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}=\begin{pmatrix} 7 & -6 \\ -5 & 5 \end{pmatrix}^T\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 7 & -5 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}=\begin{pmatrix} 9 & 1 \\ 7 & 2 \end{pmatrix}$$

EXERCISES:

1. Find real numbers a, b and x such that

$$\begin{pmatrix} a & b \\ x & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ x & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2a+b & a+b \\ 2x & x \end{pmatrix} - \begin{pmatrix} a & b \\ 2a+x & 2b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2a+b-a & a+b-b \\ 2x-2a-x & x-2b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} a+b & a \\ x-2a & x-2b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix} \Rightarrow$$

$$\begin{cases} a+b=2 & \to b=2-a=2-1=1 \\ a \quad = 1 & \to a=1 \\ x-2a=4 & \to x=6 \\ x-2b=4 & \to 6-2=4 \end{cases}$$

2. A square matrix B is ANTISYMMETRIC (or SKEW SYMMETRIC) if $B=-B^T$. Show that if A is any square matrix, then $A_1=\frac{1}{2}(A+A^T)$ is symmetric and $A_2=\frac{1}{2}(A-A^T)$ is anti-symmetric.

Verify that

Solution: Let us verify that A_1 is symmetric:

$$A_1^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}(A^T + (A^T)^T)$$

$$= \frac{1}{2}(A^T + A) = A_1$$

Let us prove that A_2 is antisymmetric

$$A_2^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A^T - (A^T)^T)$$

$$= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -A_2$$

Finally

$$A_1 + A_2 = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}A + \cancel{\frac{1}{2}A^T} + \frac{1}{2}A - \cancel{\frac{1}{2}A^T} = A$$

Hence any square matrix can be written as the sum of a symmetric matrix and of an antisymmetric matrix

3. If P and Q are $n \times n$ matrices with $PQ - QP = P$, prove that

$$P^2Q - QP^2 = 2P^2 \text{ and } P^3Q - QP^3 = 3P^3$$

Then use induction to prove that

$$P^kQ - QP^k = kP^k \text{ for } k=1,2,\dots$$

Solution:

$$P^2Q - QP^2 = P^2Q - PQP + PQP - QP^2$$
$$= P(PQ - QP) + (PQ - QP)P$$
$$= P \cdot P + P \cdot P = 2P^2$$

$$P^3Q - QP^3 = P^3Q - PQP^2 + PQP^2 - QP^3$$
$$= P(P^2Q - QP^2) + (PQ - QP)P^2$$
$$= P \cdot 2P^2 + P \cdot P^2 = 2P^3 + P^3 = 3P^3$$

By induction let us assume that $P^kQ - QP^k = kP^k$ holds and let us prove it for $k+1$

$$P^{k+1}Q - QP^{k+1} =$$
$$= P^{k+1}Q - PQP^k + PQP^k - QP^{k+1}$$
$$= P(P^kQ - QP^k) + (PQ - QP)P^k$$
$$= P \cdot kP^k + P \cdot P^k = kP^{k+1} + P^{k+1}$$
$$= (k+1)P^{k+1}$$

DETERMINANTS

To any square matrix $A \in M_n$ is associated a number, called DETERMINANT, denoted with the symbols $\det A$ or $|A|$.

This number can be obtained in a

recursive way. Let us see cases $n=1,2,3$

- $n=1$: A is a 1×1 matrix tha consist in a single number $A=(a)$. Hence
$$\det A = a$$

- $n=2$: A is a 2×2 matrix
$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
and we define
$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

- $n=3$: A is a 3×3 matrix
$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
only if $n=3$ we can apply the SARRUS RULE that can be memorized by rewriting A adding the first two columns on its right hand side.
$$\begin{array}{ccccc} & & - & - & - \\ \left(\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \left.\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}\right) & \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} \\ & & + & + & + \end{array}$$

$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
$- (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$

We can although generalize by a recursive method called the LAPLACE METHOD and that comprises all the 3 cases above. Before let us give some definitions

DEFINITION: Let A be an $n \times n$ matrix. Let A_{ij} be the $(n-1)\times(n-1)$ submatrix obtained by deleting row i and column j from A. Then the scalar

$$M_{ij} = \det(A_{ij})$$

is called the (i,j)th MINOR of A and the scalar

$$C_{ij} = (-1)^{i+j} M_{ij}$$

is called the (i,j)th COFACTOR

LAPLACE METHOD: Choosing any i-th row of $A \in M_n$ or any k-th column of A, the $\det(A)$ is given by

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$ EXPANSION FOLLOWING THE i-th ROW

$$= \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$ EXPANSION FOLLOWING THE k-th COLUMN

REMARK: According to this rule if we want to compute the det A if n=3, for example by expanding through the 1^{st} row, we get

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

that gives you exactly SARRUS formula

EXERCISES: Compute the determinants of the following matrices

i) $D = \begin{pmatrix} 2 & -1 \\ 5 & 4 \end{pmatrix}$

$$\det D = 2 \cdot 4 - (-1) \cdot 5 = 8 + 5 = 13$$

ii) $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$

with SARRUS

$$\begin{array}{ccccc} & & - & - & - \\ \end{array}$$
$$\left(\begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{matrix}\right) \begin{matrix} 0 & 1 \\ 3 & 4 \\ 6 & 7 \end{matrix}$$
$$\begin{array}{ccccc} & & + & + & + \\ \end{array}$$

$\det A = 0 \cdot 4 \cdot 8 + 1 \cdot 5 \cdot 6 + 2 \cdot 3 \cdot 7 - (2 \cdot 4 \cdot 6 + 0 \cdot 5 \cdot 7$
$+ 1 \cdot 3 \cdot 8) = 0 + 30 + 42 - (48 + 0 + 24)$
$= 72 - 72 = 0$

with LAPLACE expanding through the first row

$$\det A = (-1)^{1+1} \cdot 0 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} + (-1)^{1+2} \cdot 1 \begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix}$$
$$+ (-1)^{1+3} \cdot 2 \cdot \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$

$$= 0 - (24 - 30) + 2(21 - 24) = 6 - 6 = 0$$

iii) $B = \begin{pmatrix} 5 & 0 & -1 & 2 \\ 2 & 3 & 0 & 0 \\ -2 & 3 & -4 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

REMARK: IN THE LAPLACE METHOD CHOOSE ROW OR COLUMN THAT HOLDS MORE ZEROES going back to the calculation of det B, I will expand the det with respect to the 4^{th} column as it has all null elements except for $b_{14}=2$.

$$\det B = (-1)^{1+4} \cdot \begin{vmatrix} 2 & 3 & 0 \\ -2 & 3 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -(6-12+8+6) = -8$$

iv) $C = \begin{pmatrix} 0 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & -4 \end{pmatrix}$.

I want to expand with respect to the second row

$$\det C = (-1)^{2+1} \cdot 3 \cdot \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} + (-1)^{2+3} \cdot 1 \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= -3 \cdot (4-6) - 2 = 6 - 2 = 4$$

HOMEWORK: TRY WITH ~~SAR~~RUS

REMARK: it should be quite ~~evident that the~~, ~~determinant of triangu~~ lar, or diagonal matrices is equal to the product of the elements of its principal diagonal

FOR EX:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33}$$

It is provable that:

THEOREM: if $A \in M_n$. Then

i) $\det(A^T) = \det A$

ii) $\det(A \cdot B) = \det A \cdot \det B$

iii) $\det(kA) = k^n \det A$, $k \in \mathbb{R}$

CAREFUL:

$$\det(A+B) \neq \det A + \det B$$

SOME OTHER PROVABLE PROPERTIES USEFUL FOR DETERMINANT CALCULATION

- If all of the elements of a row or of a column are null, then the det is equal to zero
- if in a matrix two rows or columns are equal or proportional by a factor $k \in \mathbb{R}$, then its det is null

- if in a matrix a row is a linear combination of two or more rows of the same matrix (same for columns) then the det is null

EXAMPLE:

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 4 & 1 \\ 5 & 6 & 0 & 3 \\ 4 & 1 & 6 & 7 \end{pmatrix}$$

notice that the 4^{th} row is given by 2 times the 1^{st} row plus the second row

$$(4 \quad 1 \quad 6 \quad 7) = 2 \cdot (2 \quad 0 \quad 1 \quad 3) + (0 \quad 1 \quad 4 \quad 1)$$

hence $\det A = 0$ → HW: check by applying SARRUS method

DEFINITION: A square matrix $A \in M_n$, whose determinant is zero is called SINGULAR

INVERSE MATRIX

DEFINITION: Given a square matrix $A \in M_n$, a matrix $B \in M_n$ is an INVERSE for A if $A \cdot B = BA = I_n$

If the matrix B exists, we say that A is INVERTIBLE

THEOREM: A square matrix can have at the most one inverse

<u>proof</u>: Suppose that B and C are both inverse of A. Then we have

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

<u>NOTATION</u> if a matrix $A \in M_n$ is invertible, then its unique inverse is denoted by the symbol A^{-1}.

EXERCISES

i) Verify that $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is invertible and

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

ii) Verify that

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

is invertible and

$$B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

EXERCISE: Proof that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is invertible if $\det A = ad - bc \neq 0$ and that

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Solution: by uniqueness of the inverse it is sufficient to prove that A^{-1} as written in the exercise is the inverse

$$A^{-1}A = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

EXERCISE: Prove that if $A \in M_n$ is invertible, $b \in M_{n\times 1}$ (column matrix), $x \in M_{n\times 1}$ (column matrix of unknowns), then the equation

$$Ax = b$$

has the unique solution $x = A^{-1}b$.

Solution: Assume A is any invertible matrix and we wish to solve $Ax = b$. Then

$$A^{-1}Ax = A^{-1}b \longrightarrow I_n x = A^{-1}b$$

$$\longrightarrow x = A^{-1}b$$

Suppose $w \in M_{n\times 1}$ is also a

solution to $Ax=b$. Then $Aw=b$ and

$$A^{-1}Aw = A^{-1}b$$ which means

$w = A^{-1}b$. So $w = x$.

EXERCISE: Use the inverse of

$$A = \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}$$ to solve

$$\begin{cases} -7x_1 + 3x_2 = 2 \\ 5x_1 - 2x_2 = 1 \end{cases}$$ (HINT: write the system as $Ax=b$ where

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$)

RECALL: if $A \in M_n$, the (i,j)th COFACTOR is:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where $M_{ij} = \det(A_{ij})$ and A_{ij} is the submatrix of A obtained by deleting the i-th row and the j-th column

DEFINITION: The transpose of the $n \times n$ matrix whose (i,j)th entry C_{ij}, is called the ADJOINT (or ADJUGATE) and is denoted by $\mathrm{adj}(A)$

THEOREM: Let $A \in M_n$ be an invertible matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

Notice that a matrix is invertible if and only if its determinant is different from zero

THEOREM: Given $A, B \in M_n$ invertible, then the following results hold

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ and $\det(A^{-1}) = \frac{1}{\det A}$

b. (AB) is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

proof: a) as $AA^{-1} = I_n$ thus
$$\det(AA^{-1}) = \det(I_n) = 1$$
Since $\det(AA^{-1}) = \det(A)\det(A^{-1})$
$$\Longrightarrow \det(A)\det(A^{-1}) = 1$$
$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

b) As $\det(AB) = \det(A)\det(B)$ $\neq 0 \Rightarrow AB$ is invertible

Moreover

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1}$$

$$= AA^{-1} = I_n$$

$$\rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

c) As $\det(A^T) = \det(A) \neq 0 \Rightarrow A^T$ is invertible. Moreover

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$$

Hence $(A^T)^{-1} = (A^{-1})^T$.

EXERCISE: Find the inverse of the following matrix

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

Solution: As

$$\det A = 9 + 12 + 12 - 9 - 16 - 9 = -1 \neq 0$$

$\rightarrow A$ is invertible

Let us evaluate the cofactors

$$c_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$$

$$c_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 3$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

Hence $\operatorname{adj}(A) = \begin{pmatrix} -7 & 1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -7 & 3 & 3 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\Rightarrow A^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ (HOMEWORK: check that $AA^{-1} = A^{-1}A = I_3$)

HOMEWORK: Find the inverse (if it exists) of the following matrices

i) $B=\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$

ii) $C=\begin{pmatrix} 1 & 0 & 0 \\ -3 & -2 & 1 \\ 4 & -16 & 8 \end{pmatrix}$

HOMEWORK: Let

$$A_t=\begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) for what values of t does A_t have an inverse?

b) find a matrix X such that

$$B+XA_1^{-1}=A_1^{-1}.$$

HOMEWORK: Solve the equation

$$\begin{vmatrix} 1-x & 2 & 2 \\ 2 & 1-x & 2 \\ 2 & 2 & 1-x \end{vmatrix}=0$$

RANK OF A MATRIX

DEFINITION: The RANK r of a matrix $A \in M_{m\times n}$ is the highest order of her non singular square submatrices (in other words, it is the highest order of its minors different from zero)
The rank of A will be denoted by the symbol $rk(A)$.

REMARKS:

1. A matrix $A \in M_{m\times n}$ has rank r if all of her square submatrices of order $(r+1)$ are singular, whereas instead there is at least one non singular square matrix of order r
2. it is evident that if $A \in M_{m\times n}$, then

$$0 \leq rk(A) \leq \min\{m, n\}$$

3. If all the elements of the matrix are null then $rk(A) = 0$.

To find the rank of a matrix we can apply two methods:

a) we analyze the minors applying KRONECKER'S THEOREM (bordered submatrices)

b) Gauss elimination method

Let us go through the first method.
Recall that:

<u>DEFINITION</u>: A KTH ORDER MINOR of $A \in M_{m\times n}$ is the determinant of a square submatrix obtained intersecting k rows and k columns of A.

For example

$$A = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 3 & -1 & 4 & 2 \\ 1 & 5 & 2 & 3 \end{pmatrix}$$

this is a 3×4 matrix, hence there are minors of order 1, 2 and 3

and $2\cdot(-1)-1\cdot3=-5$ and $-1\cdot(2)-4\cdot5=-22$ are two minors of order 2 as they are the determinants of the two sub-matrices of order two highlighted in the previous matrix

<u>CAREFUL</u>: Not necessarily rows and columns have to be near to form a submatrix; for example, from the previous matrix we obtain a matrix of order 2 by intersecting the first row, the third row, the second column and the fourth column:

$$\begin{pmatrix} 2 & 1 & 3 & 0 \\ 3 & -1 & 4 & 2 \\ 1 & 5 & 2 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}$$ whose minor

is $1 \cdot 3 - 0 \cdot 5 = 3$

<u>REMARK</u>: $A \in M_{m \times n}$ has $rk(A) = k$ if
i) there is a non null minor of order k
ii) there are no minors of order $k+1$, or if they do exist, they are all null

$\Longrightarrow$ hence if i find a minor of order p, then $rk(A) \geq p$, and if I want to check if the rank is p, I'll have to consider all the minors of order $p+1$, and if they are all null, I conclude that the rank is p, otherwise, if at least one $(p+1)$-minor is non null I will repeat the same reasoning with $(p+2)$-minors.

The problem is that this research could result cumbersome, so we need a shortcut that comes from the following theorem:

<u>KRONECKER THEOREM (or OF THE BORDERED MINORS</u>

A matrix $A \in M_{m \times n}$ has $rk(A) = k$

if and only if the following 2 properties hold

i) a non null minor of order k exists

ii) all the minors of order $(k+1)$ obtained from the minor in i, bordering the correspondent submatrix by adding whatever other row and column

The theorem states hence that I don't have to evaluate all the $(k+1)$ minors but it is sufficient to consider the bordered ones

<u>For example</u>:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix}$$

as this isn't a null matrix, hence

$$1 \leq rk(A) \leq 4$$

$\Rightarrow$ I consider the determinant of the submatrix of order 2 obtained by intersecting the 1^{st} and 2^{nd} rows with the 3^{rd} and 4^{th} column

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$$

and

$$\begin{vmatrix} 0 & 1 \\ 3 & 3 \end{vmatrix} = -3 \neq 0$$

hence the rank is at least 2. But if I consider the submatrices of order 3 obtained by bordering the previous submatrix of order 2, and their determinants are all null, then I can conclude that the rank is 2.

I will list all the bordered submatrices

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix} \Longrightarrow \begin{vmatrix} 2 & 0 & 1 \\ 4 & 3 & 3 \\ 0 & 6 & 2 \end{vmatrix} = $$

$$= 12 + 24 - 36 = 0$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix} \Longrightarrow \begin{vmatrix} 2 & 0 & 1 \\ 4 & 3 & 3 \\ 2 & -3 & 0 \end{vmatrix} = -12 - 6 + 18 = 0$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix} \Longrightarrow \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 3 \\ 0 & 6 & 2 \end{vmatrix} = 6 + 12 - 18 = 0$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 3 & 3 \\ 0 & 0 & 6 & 2 \\ 1 & 2 & -3 & 0 \end{pmatrix} \Longrightarrow \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 3 \\ 1 & -3 & 0 \end{vmatrix} = -6 - 3 + 9 = 0$$

hence all the bordered minors of order 3, obtained by adding to the order 2 submatrix $\begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$ a row and a column are null, and as a consequence of Kronecker's theorem $\rightarrow rk(A) = 2$

The theorem simplifies the calculus of the rank: without it I would have to calculate 16 minors of order 3.

FORMALIZATION

Given $A \in M_{m \times n}$, $r \in \mathbb{Z}_+$ (positive integer), let us denote with M_r a square submatrix. By BORDERED SUBMATRICES of M_r in A, all the $(r+1)$ order submatrices of A obtained by adding to M_r one of the remaining $(m-r)$ rows and one of the remaining $(n-r)$ columns of A.

Hence M_r has $(m-r) \times (n-r)$ bordered submatrices

EXERCISE:

Given

$$A_{3 \times 4} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

and
$$M_2=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

a order 2 submatrix of A.
Build all of the submatrices of order $2+1=3$ bordering M_2

Solution: the number of bordered matrices is $(m-r)(n-r)=(3-2)(5-2)=$
$=1\cdot 3=3$, and they are the following

$$\underset{\substack{\downarrow \\ \text{ADDED } 3^{rd} \text{ ROW} \\ 3^{rd} \text{ COLUMN}}}{B_1^3}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \quad \underset{\substack{\downarrow \\ \text{ADDED } 3^{rd} \text{ ROW} \\ 4^{th} \text{ COLUMN}}}{B_2^3}=\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{pmatrix}$$

$$\underset{\substack{\downarrow \\ \text{ADDED } 3^{rd} \text{ ROW} \\ \text{and } 5^{th} \text{ COLUMN}}}{B_3^3}=\begin{pmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{pmatrix}$$

EXERCISE: Evaluate the rank of the following matrix

$$B=\begin{pmatrix} 2 & 1 & -2 & 3 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & -3 & -3 \\ 3 & 1 & -4 & 3 \\ 7 & 0 & -5 & 1 \end{pmatrix}$$

Solution; Consider the submatrix of order 2 obtained by intersecting 1st and 2nd rows with 2nd and 3rd columns

$$M_2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow |M_2| = -1+2 = 1 \neq 0$$

hence $rk(B) \geq 2$.

It is possible to border M_2 in $(5-2)(4-2) = 6$ ways, the following

$$B_3^1 = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 1 & -1 \\ 4 & 2 & -3 \end{pmatrix} \quad B_3^2 = \begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 1 \\ 2 & -3 & -3 \end{pmatrix}$$

$$B_3^3 = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & 1 & -4 \end{pmatrix} \quad B_3^4 = \begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 1 \\ 1 & -4 & 3 \end{pmatrix}$$

$$B_3^5 = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 1 & -1 \\ 7 & 0 & -5 \end{pmatrix} \quad B_3^6 = \begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 1 \\ 0 & -5 & 1 \end{pmatrix}$$

Among them $\det B_3^2 = -7 \neq 0$

Hence $3 \leq rk(B) \leq 4$.

Let us apply Kronecker again to B_3^2, that can be bordered in $(5-3)\cdot(4-3) = 2$ ways, the following

$$B_4^1 = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & -3 & -3 \\ 3 & 1 & -4 & 3 \end{pmatrix}, \quad B_4^2 = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & -3 & -3 \\ 7 & 0 & -5 & 1 \end{pmatrix}$$

and $\det B'_4 = 7 \neq 0$. Hence $rk(B) = 4$.

EXERCISE: Discuss the rank of A, as $k \in \mathbb{R}$ changes

$$A = \begin{pmatrix} -1 & 1 & 1 \\ k & 2 & -1 \\ 4 & 1 & 2 \end{pmatrix}$$

Solution: A is a 3×3 non null matrix, hence $1 \leq rk A \leq 3$.

Let us evaluate $\det A$:

$$\det A = -4 - 4 + k - 8 - 1 - 2k$$
$$= -k - 17$$

and $\det A \neq 0$ if $-k - 17 \neq 0 \longrightarrow$ if $k \neq -17$. Hence $\forall\, k \in \mathbb{R} \setminus \{-17\}$ $rk(A) = 3$.

If $k = -17$ A becomes

$$A = \begin{pmatrix} -1 & 1 & 1 \\ -17 & 2 & -2 \\ 4 & 1 & 1 \end{pmatrix}$$

and as $\det A = 0$

$$1 \leq rk(A) \leq 2$$

and as the highlighted submatrix of order 2:

$$\begin{pmatrix} -1 & 1 \\ -17 & 2 \end{pmatrix}$$

has determinant equal to $(-1) \cdot 2 - (-17) \cdot 1 = 15 \neq 0$, hence $rk(A) = 2$

HOMEWORK

1. Given the matrices

$$A = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 2 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -3 & 5 \\ 1 & 1 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

calculate

a) $A+B-C$

b) $2A+2B-3C$

2. Given the matrices

$$A = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 2 & 3/2 \\ -\frac{1}{2} & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -3 & 5 \\ 1 & 1 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 1 \\ \frac{1}{3} & -\frac{13}{2} & 2 \end{pmatrix}$$

Calculate $AB+C$, $A-BC$, $B-3AC$.
Calculate the determinant of each matrix and of AB, AC, BC.

3. Calculate the determinant of the following matrices

i)

$$A=\begin{pmatrix} -3 & 14 & 7 & 1 \\ 12 & 2 & 1 & 3 \\ \frac{1}{3} & -\frac{13}{2} & 2 & -1 \\ 0 & 1 & 0 & 9 \end{pmatrix}$$

$$B=\begin{pmatrix} 1 & 0 & -5 & 1 & 0 \\ 3 & 0 & -\frac{1}{3} & 0 & 2 \\ \frac{1}{4} & 3 & \frac{1}{2} & -1 & 0 \\ 9 & 4 & 1 & 0 & 2 \\ 1 & 0 & -1 & 3 & 2 \end{pmatrix}$$

4. Given the matrix

$$D=\begin{pmatrix} t & 0 & 4 & -1 \\ 0 & 2 & 1 & 3 \\ -2 & 1 & 0 & t-1 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

determine t such that $\det D = 0$.

5. Given

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

determine X such that $AX = B$

6. Evaluate the determinant of

i) $A = \begin{pmatrix} k & -1 & 2 \\ k & -2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$

ii) $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & k \end{pmatrix}$

for which values of k the matrices are invertible? Calculate the inverse matrices.

7. Establish the rank of the follo= wing matrices

i) $A = \begin{pmatrix} -1 & -2 \\ 0 & \frac{5}{2} \\ -\frac{2}{3} & 3 \end{pmatrix}$ ii) $B = \begin{pmatrix} 5 & -3 & 2 \\ 10 & -6 & 4 \\ 1 & 4 & 3 \\ 4 & -7 & -1 \end{pmatrix}$

iii)
$$C=\begin{pmatrix} 1 & 2 & -1 & 0 & 5 \\ 0 & 2 & -\frac{1}{2} & 4 & -3 \\ -1 & -3 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 6 \\ 2 & 0 & 1 & 3 & 2 \end{pmatrix}$$

8. Discuss the rank of the following matrices as k changes

i)
$$A=\begin{pmatrix} 1 & 2 & -k & 6 \\ 0 & 2k & -\frac{1}{2} & 5 \\ -1 & -3 & 0 & 3 \end{pmatrix}$$

ii)
$$B=\begin{pmatrix} k & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & k & -2 & 0 \end{pmatrix}$$