1

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 3 & 1 \\ -4 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $$A+B = \begin{pmatrix} 3+0 & 0+3 & 1+1 \\ -4+0 & 2+2 & 1-2 \\ 2+0 & -1-1 & -1+1 \\ 1+0 & 2+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 \\ -4 & 4 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

(b) $$A^T = \begin{pmatrix} 3 & 0 & 2 & 1 \\ 0 & 2 & -1 & 2 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

(c) $$(A+B)^T = \begin{pmatrix} 3 & -4 & 2 & 1 \\ 3 & 4 & -2 & 2 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 3 & 2 & -1 & 0 \\ 1 & -2 & 1 & 1 \end{pmatrix} \Rightarrow A^T + B^T = \begin{pmatrix} 3 & -4 & 2 & 1 \\ 3 & 4 & -2 & 2 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

So $(A+B)^T = A^T + B^T$

(d) $$C = 4A = 4\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 4 \\ 0 & 8 & 4 \\ 8 & -4 & -4 \\ 4 & 8 & 0 \end{pmatrix}$$

2 WE HAVE TO VERIFY THAT THE COLUMNS' NUMBER OF THE FIRST MATRIX IS EQUAL TO THE ROWS' NUMBER OF THE SECOND MATRIX.

(a) $A \rightsquigarrow 3 \times 4 \qquad B \rightsquigarrow 4 \times 2$

$$A \cdot B = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -1 & 3 & 5 \\ 1 & -3 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \\ 4 & -5 \end{pmatrix} = \begin{pmatrix} 17 & -16 \\ 17 & -26 \\ -1 & -1 \end{pmatrix}$$

(b) $A \quad 3\times 2 \qquad\qquad B \quad 2\times 3$

$$A\cdot B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 5 \\ 2 & 4 \end{pmatrix}$$

(c) $A \quad 2\times 2 \qquad\qquad B \quad 3\times 2$

IT'S NO POSSIBLE TO DO $A\times B$

(d) $A \quad 3\times 3 \qquad\qquad B \quad 2\times 2$

(e) $A \quad 2\times 2 \qquad\qquad B \quad 2\times 2$

$$AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$$

$\boxed{3}$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$\neq$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$\boxed{4}$

$$A\cdot B\cdot C = (A\cdot B)\cdot C$$

$$AB = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \beta & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2\beta - 1 & 0 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 2\beta - 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 2\beta - 1 \end{pmatrix}$$

1×1 MATRIX $\rightarrow$ IT'S A NUMBER

$$2\beta - 1 = 0 \longleftrightarrow \beta = \frac{1}{2}$$

$\boxed{5}$

$$AX = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \end{pmatrix}$$

$$AX = B \iff$$

$$\begin{pmatrix} x+y \\ x-2y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \iff \begin{cases} x+y = 1 \\ x-2y = 4 \end{cases} \iff \begin{cases} x = 1-y \\ 1-y-2y = 4 \end{cases} \to \begin{cases} x = 2 \\ y = -1 \end{cases}$$

$$X = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\boxed{6}$

$$A \cdot A = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2+2k & k^2 \end{pmatrix}$$

$$AA = A \iff \begin{pmatrix} 1 & 0 \\ 2+2k & k^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix} \iff \begin{cases} 2+2k = 2 \\ k^2 = k \end{cases}$$

$$\begin{cases} 2+2k = 2 \\ k^2 = k \end{cases} \iff \boxed{k=0}$$

$\boxed{7}$

$$A = \begin{pmatrix} 0 & 3t-2 & -1 \\ t^2 & 3 & t^2+4 \\ -1 & 4t & 1 \end{pmatrix}$$

$$\begin{cases} t^2 = 3t-2 \\ -1 = -1 \\ t^2+4 = 4t \end{cases} \to \begin{cases} t^2 - 3t + 2 = 0 \\ t^2 - 4t + 4 = 0 \end{cases} \to \begin{cases} t = 1 \lor t = 2 \\ t = 2 \end{cases}$$

$$\boxed{t=2}$$

$\boxed{8}$

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 5 & -1 & 3 \\ 4 & 3 & 2 & -2 \\ 2 & 0 & 1 & 8 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{1} \\ \cancel{-1} & 5 & -1 & 3 \\ \cancel{4} & 3 & 2 & -2 \\ \cancel{2} & 0 & 1 & 8 \end{pmatrix} \rightsquigarrow M_{11} = \begin{pmatrix} 5 & -1 & 3 \\ 3 & 2 & -2 \\ 0 & 1 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \cancel{3} & 1 \\ \cancel{-1} & \cancel{5} & \cancel{-1} & \cancel{3} \\ 4 & 3 & \cancel{2} & -2 \\ 2 & 0 & \cancel{1} & 8 \end{pmatrix} \rightsquigarrow M_{23} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -2 \\ 2 & 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \cancel{3} & 1 \\ -1 & 5 & \cancel{-1} & 3 \\ \cancel{4} & \cancel{3} & \cancel{2} & \cancel{-2} \\ 2 & 0 & \cancel{1} & 8 \end{pmatrix} \rightsquigarrow M_{33} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 5 & 3 \\ 2 & 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cancel{2} & 3 & 1 \\ -1 & \cancel{5} & -1 & 3 \\ 4 & \cancel{3} & 2 & -2 \\ \cancel{2} & \cancel{0} & \cancel{1} & \cancel{8} \end{pmatrix} \rightsquigarrow M_{42} = \begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 3 \\ 4 & 2 & -2 \end{pmatrix}$$

$\boxed{9}$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \to \det A = 2(-2) - 3 \cdot 1 = -7$$

$$B = \begin{pmatrix} -11 & 3 \\ 2 & 0 \end{pmatrix} \to \det B = -11 \cdot 0 - 3 \cdot 2 = -6$$

$$C = \begin{pmatrix} 2 & 3 & -2 \\ 1 & -2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \to$$

HINT: CHOOSE THE COLUMN OR THE ROW WITH MORE ZERO

$$\det C = \cancel{(-1)^{3+1} \cdot 0 \cdot \det \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}} + (-1)^{3+2} (-1) \det \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} + (-1)^{3+3} \cdot 2 \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} + 2 \det \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} =$$

$$= 2 \cdot 0 - (-2) 1 + 2 \left[2(-2) - 3 \right] = 2 + 2 \cdot (-7) = -12$$

$$D = \begin{pmatrix} 2 & -2 & -2 \\ 1 & 1 & 0 \\ -3 & 4 & 0 \end{pmatrix} \to$$

$$\det D = (-1)^{3+1}(-2)\det\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix} = (-2)\left[1\cdot 4 + 3\cdot 1\right] = (-2)(7) = -14$$

$$E = \begin{pmatrix} 7 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 4 & 3 \end{pmatrix} \to$$

$$\det E = (-1)^{1+1}\cdot 7\cdot \det\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix} = 7\left[3-0\right] = 21$$

FIRST ROW

$$F = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix} \to \det F = (-1)^{3+3}\cdot 5\cdot \det\begin{pmatrix} 1 & -4 \\ 0 & 2 \end{pmatrix} = 5\cdot(2-0) = 10$$

LAST ROW

$$G = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \to \det G = (-1)^{3+3}\cdot 3\cdot \det\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = 3(-2+0) = -6$$

LAST COLUMN

$$H = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 4 & 1 & -1 & 2 \\ 3 & 6 & 6 & 8 \\ 3 & 2 & 1 & 3 \end{pmatrix} \to$$

$$\det H = (-1)^{1+1}\cdot 1\cdot \det\begin{pmatrix} 1 & -1 & 2 \\ 6 & 6 & 8 \\ 2 & 1 & 3 \end{pmatrix} + (-1)^{1+2}\cdot 2\cdot \det\begin{pmatrix} 4 & -1 & 2 \\ 3 & 6 & 8 \\ 3 & 1 & 3 \end{pmatrix} +$$

FIRST ROW

$$+ (-1)^{1+3} 2\cdot\det\begin{pmatrix} 4 & 1 & 2 \\ 3 & 6 & 8 \\ 3 & 2 & 3 \end{pmatrix} + (-1)^{4+1}\cdot 3\cdot \det\begin{pmatrix} 4 & 1 & -1 \\ 3 & 6 & 6 \\ 3 & 2 & 1 \end{pmatrix} =$$

$$= \left[(-1)^{1+1}\cdot 1\,(6\cdot 3 - 8\cdot 1) + (-1)^{1+2}\cdot(-1)\,(18-16) + (-1)^{1+3}\cdot 2\,(6-12)\right]$$

$$-2\left[4(6\cdot 3 - 8\cdot 1) + 1\,(9-24) + 2\,(3-18)\right] +$$

$$2\left[4(8-16)-1(9-24)+2(6-18)\right]+$$

$$-3\left[4(6-12)-1(3-18)-1(6-18)\right]=$$

$$=(10+2-12)-2(40-15-30)+2(8+15-24)+$$

$$-3(-24+15+12)=10-2-9=-1$$

$\boxed{10}$

$$D=\begin{pmatrix} t & 0 & 4 & -1 \\ 0 & 2 & 1 & 3 \\ -2 & 1 & 0 & t-1 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

$$\det D \underset{\text{LAST ROW}}{=} (-1)^{4+1}\cdot 4\,\det\begin{pmatrix} 0 & 4 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & t-1 \end{pmatrix}+(-1)^{4+4}\cdot 1\,\det\begin{pmatrix} t & 0 & 4 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$=-4\left[-4\det\begin{pmatrix} 2 & 3 \\ 1 & t-1 \end{pmatrix}-1\det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]+t(0-1)+4(0+4)=$$

$$=-4\left[-4(2t-2-3)-1(-1)\right]-t+16=$$

$$=-4(-8t+20+1)-t+16=32t-84-t+16=$$

$$=31t-68$$

$$31t-68=0$$

$$\boxed{t=\frac{68}{31}}$$

$\boxed{11}$

$$A=\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \longrightarrow \begin{matrix} A_{11}=-1 & A_{12}=2 \\ A_{21}=2 & A_{22}=1 \end{matrix} \qquad \det A=-5$$

$$A_{cof}=\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix} \longrightarrow A_{cof}^{T}=\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{A_{cof}^{T}}{\det A} = \begin{pmatrix} +\frac{1}{5} & +\frac{2}{5} \\ +\frac{2}{5} & -\frac{1}{5} \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & 2 \\ 2 & 0 & 7 \end{pmatrix}$$

$B_{11} = \begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix}$ $B_{21} = \begin{pmatrix} -1 & 3 \\ 0 & 7 \end{pmatrix}$

$B_{12} = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$ $B_{22} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

$B_{13} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ $B_{23} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$

$B_{31} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ $B_{32} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ $B_{33} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\det B = 4$

$$B_{cof} = \begin{pmatrix} 7 & -3 & -2 \\ 7 & 1 & -2 \\ -5 & 1 & 2 \end{pmatrix} \longrightarrow B_{cof}^{T} = \begin{pmatrix} 7 & 7 & -5 \\ -3 & 1 & 1 \\ -2 & -2 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{B_{cof}^{T}}{\det B} = \begin{pmatrix} 7/4 & 7/4 & -5/4 \\ -3/4 & 1/4 & 1/4 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

12

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

A INVERTIBLE $\iff \det A \neq 0$

$$\det A = \alpha(\alpha - 1) - 1 \cdot 0 + 1(1-\alpha) = \alpha^2 - \alpha + 1 - \alpha =$$
$$= (\alpha - 1)^2 \qquad \det A = 0 \iff \boxed{\alpha = 1}$$

B INVERTIBLE $\iff \det B \neq 0$

$$\det B = \beta(\beta^2-1) - 1\cdot\beta = \beta^3 - 2\beta =$$

$$= \beta(\beta^2-2) \to \boxed{\beta=0},\ \boxed{\beta=\sqrt{2}},\ \boxed{\beta=-\sqrt{2}}$$

$\boxed{13}$ (a)

$$A = \begin{pmatrix} 1 & 0 & 1 & 5 \\ 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix} \qquad Rk\,A \le 3$$

- $\det M_1 = \det\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix} = 2 - 2 = 0$

- $\det M_2 = \det\begin{pmatrix} 0 & 1 & 5 \\ 2 & 0 & 1 \\ 2 & 1 & 6 \end{pmatrix} = -10 + 10 = 0$

- $\det M_3 = \det\begin{pmatrix} 1 & 0 & 5 \\ 1 & 2 & 1 \\ 2 & 2 & 6 \end{pmatrix} = 0$

$\Rightarrow Rk\,A \le 2$

- $\det M_4 = \det\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \neq 0$

$\Rightarrow \boxed{Rk\,A = 2}$

(b)
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad Rk\,A \le 4$$

- $\det M_1 = \det\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 0$

- $\det M_2 = \det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = -1\cdot\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} =$

$$= -1\left[(-1)\det\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} + \det\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right] =$$

$$= -1\,(1+1) = -2 \neq 0$$

$$\Rightarrow \boxed{\mathrm{Rk}\ A = 4}$$