

LINEAR SPACES

The aim of this part is to introduce you to the notion of LINEAR SPACE. Roughly speaking a linear space is a set of objects, called vectors for which it is possible to define a sum and a multiplication by scalars. Linear Spaces occur in numerous branches of applied mathematics.

DEFINITION: Let V be a set on which two operations are defined sum and scalar multiplication. The sum associates to each pair $v, w \in V$ the element $v + w \in V$ the scalar multiplication associates to each $\alpha \in \mathbb{R}$ and $v \in V$ the element $\alpha v \in V$. The set V is said LINEAR SPACE (on \mathbb{R}) if, for every $v, w, z \in V$ and every $\alpha, \beta \in \mathbb{R}$ these operations satisfy the following properties:

i) $v + w = w + v$ (COMMUTATIVITY)

ii) $(v + w) + z = v + (w + z)$ (ASSOCIATIVITY)

- iii) there exists an element $0 \in V$ such that $v + 0 = v \quad \forall v \in V$ (EXISTENCE OF A NEUTRAL ELEMENT FOR THE SUM)
- iv) $\forall v \in V$ there exists an element $-v \in V$ such that $v + (-v) = 0$ (EXISTENCE OF THE OPPOSITE OF EACH $v \in V$)
- v) $1v = v, \forall v \in V$ (EXISTENCE OF A NEUTRAL ELEMENT FOR THE SCALAR MULTIPLICATION)
- vi) $\alpha(v+w) = \alpha v + \alpha w, \forall \alpha \in \mathbb{R}, \forall v, w \in V$ (DISTRIBUTIVE PROPERTY)
- vii) $(\alpha + \beta)v = \alpha v + \beta v, \forall \alpha, \beta \in \mathbb{R}, \forall v \in V$ (DISTRIBUTIVE PROPERTY)
- viii) $\alpha(\beta v) = (\alpha\beta)v, \forall \alpha, \beta \in \mathbb{R}, \forall v \in V$ (ASSOCIATIVE PROPERTY)

EXAMPLES:

a) Let P_n be the set of all the polynomials of degree less than or equal to n , with in addition the degenerated polynomial 0 whose coefficients are all zero. Let us define as follows the sum of polynomials and the multiplication of a polynomial by a scalar, as follows

SUM: Let $f(x) = a_0 + a_1x + \dots + a_nx^n$

and $g(x) = b_0 + b_1x + \dots + b_nx^n$ be two elements of P_n . Hence the sum of $f(x)$ and $g(x)$ is

$$(f+g)(x) = (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n \in P_n$$

MULTIPLICATION BY A SCALAR $\alpha \in \mathbb{R}$

Let $\alpha \in \mathbb{R}$, $f(x) = a_0 + a_1x + \dots + a_nx^n \in P_n$
hence

$$(\alpha f)(x) = \alpha a_0 + \alpha a_1x + \dots + \alpha a_nx^n \in P_n$$

HOMEWORK: Prove that P_n endowed with these operations is a LINEAR SPACE

b) The set of matrices $M_{m \times n}$ endowed with the sum between matrices with same dimension and with the product by a scalar $\alpha \in \mathbb{R}$, is a linear space
(HOMEWORK: Show it)

c) The set of all the ordered n -tuples of real numbers, \mathbb{R}^n is a linear space with the

following operations

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n \\ y = (y_1, \dots, y_n) \in \mathbb{R}^n \Rightarrow$$

$$x + y = (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{R}^n$$

$$\alpha x = (\alpha x_1, \dots, \alpha x_n) \in \mathbb{R}^n, \alpha \in \mathbb{R}$$

Later on we will be analyzing in depth this example

FIRST PROPERTIES OF VECTOR SPACES

Based on the definition the following properties hold

PROPOSITION: In a vector space V the neutral element 0 is unique
proof Suppose by contradiction that there exists $0' \in V$ such that $v + 0' = v, \forall v \in V$. In particular $0 + 0' = 0$. The same holds for $0'$
 $0' + 0 = 0'$

By the commutative property the 1st members of the previous equalities are equal:

$$0 + 0' = 0' + 0$$

Hence $0 = 0'$.

PROPOSITION If V is a vector space, all elements $v \in V$ have a unique inverse $-v$

proof suppose there exists $w \in V$ such that $v + w = 0$. Hence

$$\begin{aligned} -v &= (-v) + 0 = (-v) + (v + w) \\ &= (-v + v) + w = 0 + w \\ &= w \end{aligned}$$

Hence $-v = w$.

PROPOSITION: Let v and w be any two vectors of a vector space V . There exists a unique vector $x \in V$ such that $v + x = w$

proof: Define $x = w + (-v)$. We have

$$\begin{aligned} v + x &= v + (w + (-v)) \\ &= (w + (-v)) + v \text{ COMMUTATIVITY} \\ &= w + (-v + v) \text{ ASSOCIATIVITY} \\ &= w + 0 = w \end{aligned}$$

Therefore $x = w + (-v)$ is such that $v + x = w$. Let us pro

that $w + (-v)$ is the unique vector of this type. Let x be whatever vector of V such that $v + x = w$

$$\begin{aligned} x &= x + 0 \\ &= x + (v + (-v)) \\ &= (x + v) + (-v) \text{ ASSOCIATIVITY} \\ &= w + (-v) \end{aligned}$$

PROPOSITION: Let V be a vector space. For every $v \in V$ we have $0v = 0$, $(-1)v = -v$ and $\alpha 0 = 0 \forall \alpha \in \mathbb{R}$

proof: We start by proving that

$0v = 0$. By definition 0 solves $x + v = v$. On the other hand

$$v + 0v = 1v + 0v \overset{\text{DISTRIBUTIVE PROPERTY}}{=} (1+0)v = 1v = v$$

Therefore also $0v$ solves $v + x = v$. Hence by uniqueness $0v = 0$

To prove $(-1)v = -v$, we observe that for every $v \in V$ we have

$$\begin{aligned} (-1)v + v &= (-1)v + 1v \overset{\text{DISTRIBUTIVE PROPERTY}}{=} (-1+1)v \\ &= 0v = 0 \end{aligned}$$

Hence $(-1)v + v = 0$ and

both $-v$ and $(-1)v$ solve $x + v = 0$. By uniqueness, this implies that $(-1)v = -v$.
 To prove that $\alpha \cdot 0 = 0$, observe that $\alpha \cdot 0 = \alpha(v - v) = \alpha v + \alpha(-v) = \alpha v - \alpha v = 0$

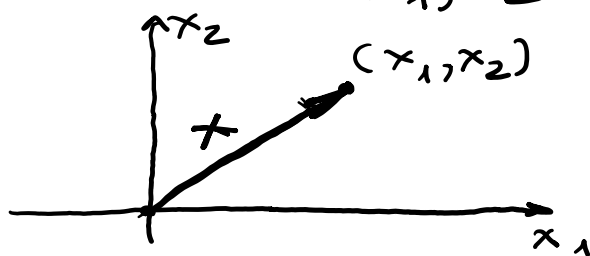
\mathbb{R}^n INNER PRODUCT LINEAR SPACE

\mathbb{R}^n is the collection of n -tuples of real ordered numbers, and we will see later on that it is among the most common (VECTOR) LINEAR SPACE. Through n -tuples $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ real case situations can be modeled such as consumption models or portfolio allocations over n assets.

We surely know how to visualize the spaces \mathbb{R}^2 and \mathbb{R}^3 .

For example \mathbb{R}^2 is the set of ordered couples of numbers, that can be viewed as points in the Cartesian coordinate plane. We can also think of vectors in \mathbb{R}^2 (same in \mathbb{R}^3) as arrows with initial point at the

origin and final point on the point of coordinates (x_1, x_2)



As we said previously, we can add vectors simply adding correspondent coordinates. Thus, having $u = (u_1, u_2)$ and $v = (v_1, v_2) \in \mathbb{R}^2$ their sum is

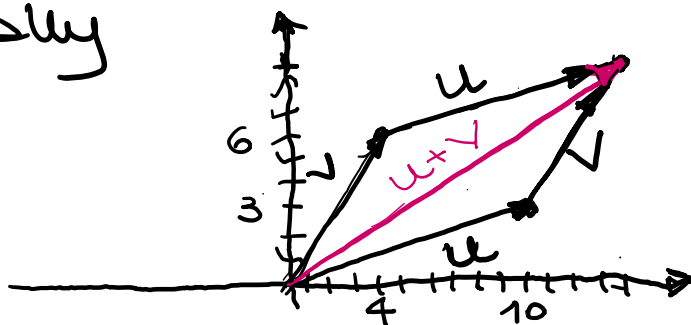
$$u + v = (u_1 + v_1, u_2 + v_2)$$

for example:

$$u = (10, 3) \text{ and } v = (4, 6) \Rightarrow$$

$$u + v = (10 + 4, 3 + 6) = (14, 9)$$

Graphically



The sum of u and v can be identified as the diagonal of the parallelogram that has as adjacent sides the directions of the vectors u and v (as shown in the previous figure). Note that u and v are identified

with their opposite parallel sides. The rule of parallelogram holds for all couple of vectors in \mathbb{R}^2 , u and v , that add up.

Also more generally a Euclidean vector is frequently represented by a ray (a directed line segment) connecting an initial point A with a terminal point B . This geometric object has a magnitude (or length) (the distance between A and B) a direction (the line that contains it) and a verse (that defines initial point and terminal point). All parallel vectors with same length and same verse are identified.

Let us see how to give a geometrical representation to the SUBTRACTION of vectors.

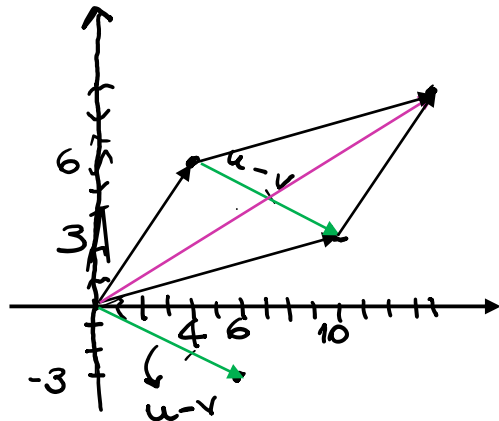
We subtract vectors simply by subtracting corresponding coordinates. Thus, having

$$u = (u_1, u_2) \text{ and } v = (v_1, v_2) \\ \Rightarrow u - v = (u_1 - v_1, u_2 - v_2)$$

Let us resume the previous example

$$u = (10, 3) \text{ and } v = (4, 6) \Rightarrow$$

$$u - v = (6, -3)$$



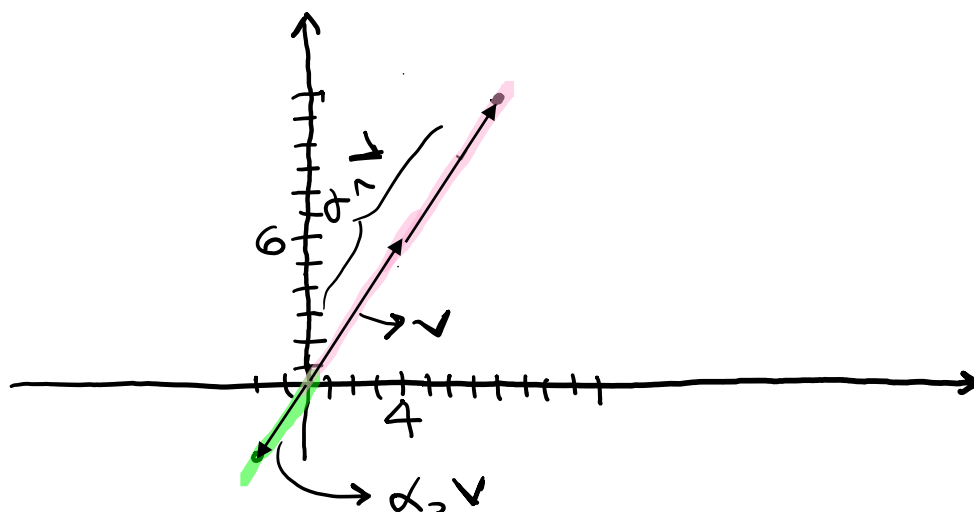
hence $u - v$ can be identified with the other diagonal of the parallelogram.

What happens if we multiply a vector $v \in \mathbb{R}^2$ by a scalar $\alpha \in \mathbb{R}$? We obtain a vector that has same direction as v , same verse if $\alpha > 0$, opposite verse if $\alpha < 0$, and length $|\alpha|$ times the length of v .

$$\text{Example: } \alpha_1 = 2, \alpha_2 = -\frac{1}{2} \quad v = (4, 6)$$

$$\Rightarrow \alpha_1 v = 2(4, 6) = (8, 12)$$

$$\alpha_2 v = -\frac{1}{2}(4, 6) = (-2, -3)$$



if $|\alpha| > 1$ the vector stretches, if $|\alpha| < 1$ the vector shortens.

We can generalize these operations to \mathbb{R}^n and define, given $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

- THE SUM of x and y is given by adding the correspondent components of x and y

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$$

- THE MULTIPLICATION BY A SCALAR $\alpha \in \mathbb{R}$ as given by multiplying each component by α :

$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in \mathbb{R}^n$$

it is easy to verify that these sum and multiplication by a scalar $\alpha \in \mathbb{R}$ verify all the properties for \mathbb{R}^n to be

2 LINEAR SPACE:

- $x + (y + z) = (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n))$
 $= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n)$
 $= (x + y) + z \quad \forall x, y, z \in \mathbb{R}^n$ (ASSOCIATIVE PROPERTY derives from the associativity of the sum between real numbers)
- $x + y = (x_1 + y_1, \dots, x_n + y_n) =$
 $= (y_1 + x_1, \dots, y_n + x_n) = y + x \quad \forall x, y \in \mathbb{R}^n$
COMMUTATIVE PROPERTY derives from the commutativity of the sum of numbers)
- The null vector $0 = (0, \dots, 0)$ is the NEUTRAL element for the sum:
$$x + 0 = (x_1 + 0, \dots, x_n + 0) = x \quad \forall x \in \mathbb{R}^n$$
- $\forall x \in \mathbb{R}^n$ the element
$$-x = (-x_1, -x_2, \dots, -x_n)$$

is the OPPOSITE of x , means it is such that
$$x + (-x) = (x_1 - x_1, \dots, x_n - x_n)$$
$$= (0, \dots, 0) = 0$$
- $\forall h, k \in \mathbb{R}, \forall x \in \mathbb{R}^n$
$$(hk)x = ((hk)x_1, \dots, (hk)x_n) = h(kx_1, \dots, kx_n)$$
$$= h(kx)$$

- $1 \cdot \mathbf{x} = (1 \cdot x_1, \dots, 1 \cdot x_n) = (x_1 \cdot 1, \dots, x_n \cdot 1)$
 $= \mathbf{x} \cdot 1 = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$
- DISTRIBUTIVE PROPERTIES: $\forall h, k \in \mathbb{R}$
 and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\begin{aligned}
 k(\mathbf{x} + \mathbf{y}) &= k(x_1 + y_1, \dots, x_n + y_n) \\
 &= (k(x_1 + y_1), \dots, k(x_n + y_n)) \\
 &= (kx_1 + ky_1, \dots, kx_n + ky_n) \\
 &= (kx_1, \dots, kx_n) + (ky_1, \dots, ky_n) \\
 &= k(x_1, \dots, x_n) + k(y_1, \dots, y_n) \\
 &= k\mathbf{x} + k\mathbf{y}
 \end{aligned}$$

$$\begin{aligned}
 (h + k)\mathbf{x} &= ((h + k)x_1, \dots, (h + k)x_n) \\
 &= (hx_1 + kx_1, \dots, hx_n + kx_n) \\
 &= (hx_1, \dots, hx_n) + (kx_1, \dots, kx_n) \\
 &= h(x_1, \dots, x_n) + k(x_1, \dots, x_n) \\
 &= h\mathbf{x} + k\mathbf{x}
 \end{aligned}$$

Hence \mathbb{R}^n is a LINEAR SPACE, and its elements will be called VECTORS. There are some other important

features that are typical of the Euclidean structure of \mathbb{R}^2 and \mathbb{R}^3 , such as LENGTH and ANGLE, and that

can be extended onto \mathbb{R}^n . These ideas are embedded in the concept we now investigate, the INNER PRODUCT (or DOT PRODUCT)

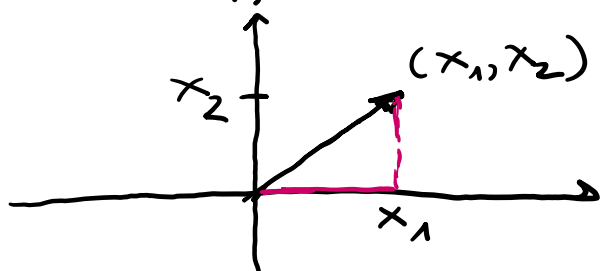
DEFINITION: For $x, y \in \mathbb{R}^n$, the INNER PRODUCT or DOT PRODUCT of x and y , denoted $x \cdot y$ (also denoted $\langle x, y \rangle$) is defined as follows

$$x \cdot y = x_1 y_1 + \dots + x_n y_n$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$

REMARK: Note that the inner product of two vectors in \mathbb{R}^n is a number, not a vector. (Example $u = (2, -2, 3)$
 $v = (1, -1, 2) \Rightarrow u \cdot v = 2 \cdot 1 + (-2) \cdot (-1) + 3 \cdot 2 = 10$)

To motivate the concept of inner product, viewing vectors in \mathbb{R}^2 as arrows with initial point at the origin, the length of a vector $x = (x_1, x_2)$ in \mathbb{R}^2 , that will be



denominated NORM of x , and denoted with the symbol $\|x\|$ is

$$\|x\| = \sqrt{x_1^2 + x_2^2} \quad (\leftarrow \text{GEOMETRIC CONSTRUCTION USING PYTHAGORAS THEOREM})$$

Similarly if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, then

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Just notice that, if $x \in \mathbb{R}^2$

$$x \cdot x = x_1 \cdot x_1 + x_2 \cdot x_2 = x_1^2 + x_2^2$$

hence

$$\|x\| = \sqrt{x \cdot x}$$

Same thing if $x \in \mathbb{R}^3$,

we can generalize the concept of NORM to \mathbb{R}^n

DEFINITION: Given $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ the NORM of x , denoted by the symbol $\|x\|$ is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

that written in terms of inner product is given by

$$\|x\| = \sqrt{x \cdot x}$$

By extending the classical definition of distance between two points

$$P = (x_1, x_2) \quad Q = (y_1, y_2) \in \mathbb{R}^2$$

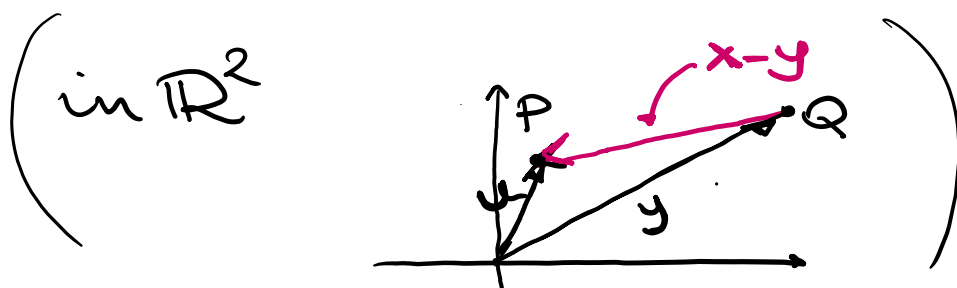
$$\|PQ\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

we can define the DISTANCE between $P=(x_1, \dots, x_n)$ and $Q=(x_1, \dots, x_n) \in \mathbb{R}^n$ as

$$\|PQ\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

and $\|PQ\|$ can be viewed also as the length of the vector obtained by subtracting the vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

$$\begin{aligned}\|x - y\| &= \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \\ &= \sqrt{(x - y) \cdot (x - y)}\end{aligned}$$



so if $y=0$ we get back the length/norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

THEOREM: For all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$

$$\|\alpha x\| = |\alpha| \|x\|$$

proof:

$$\|\alpha x\| = \sqrt{(\alpha x_1)^2 + \dots + (\alpha x_n)^2}$$

$$= \sqrt{\alpha^2 (x_1^2 + \dots + x_n^2)} = |\alpha| \sqrt{x_1^2 + \dots + x_n^2}$$

$$= |\alpha| \|x\|$$

DEFINITION: A vector $w \in \mathbb{R}^n$ whose norm is equal to 1 is called UNIT VECTOR or VERSOR

Given a non null vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ we can obtain a versor that has same direction of v by multiplying v by the scalar $\frac{1}{\|v\|}$

$$\Rightarrow w = \frac{v}{\|v\|} \text{ has norm 1}$$

proof: $\|w\| = \left\| \frac{v}{\|v\|} \right\| = \left\| \frac{1}{\|v\|} \cdot v \right\| = \frac{1}{\|v\|} \cdot \|v\| = 1$

Example: Given the vector $u = (1, 3, -2) \in \mathbb{R}^3$ find its corresponding versor:

$$\|u\| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

\Rightarrow its versor is:

$$w = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right)$$

PROPERTIES OF INNER PRODUCT

THEOREM: Let u, v and w be arbitrary vectors in \mathbb{R}^n and let $\alpha \in \mathbb{R}$ be a scalar. Then

i) $u \cdot v = v \cdot u$ (COMMUTATIVE PROPERTY)

proof: $u \cdot v = u_1 v_1 + \dots + u_n v_n =$
 $= v_1 u_1 + \dots + v_n u_n = v \cdot u$

ii) $u \cdot (v + w) = u \cdot v + u \cdot w$ (DISTRIBUTIVE PROPERTY)

proof: $u \cdot (v + w) = u_1 (v_1 + w_1) + \dots + u_n (v_n + w_n)$
 $= u_1 v_1 + u_1 w_1 + \dots + u_n v_n + u_n w_n$
 $= (u_1 v_1 + \dots + u_n v_n) + (u_1 w_1 + \dots + u_n w_n)$
 $= u \cdot v + u \cdot w$

iii) $u \cdot (\alpha v) = \alpha (u \cdot v) = (\alpha u) \cdot v$

proof:

$$\begin{aligned} u \cdot (\alpha v) &= u_1 \alpha v_1 + \dots + u_n \alpha v_n \\ &= \alpha u_1 v_1 + \dots + \alpha u_n v_n \\ &= \alpha (u_1 v_1 + \dots + u_n v_n) \\ &= \alpha (u \cdot v) \\ &= \alpha u_1 v_1 + \dots + \alpha u_n v_n \\ &= (\alpha u_1) v_1 + \dots + (\alpha u_n) v_n \\ &= (\alpha u) \cdot v \end{aligned}$$

iv) $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$

proof:

$$\begin{aligned} u \cdot u &= u_1 u_1 + u_2 u_2 + \dots + u_n u_n \\ &= u_1^2 + u_2^2 + \dots + u_n^2 \geq 0 \text{ as it} \\ &\text{is the sum of terms greater or} \end{aligned}$$

equal than zero, and it is equal to zero if and only if all terms are equal to zero, hence if and only if $u_1 = u_2 = \dots = u_n = 0$, hence if and only if $u = 0$.

$$v) (u+v) \cdot (u+v) = u \cdot u + 2(u \cdot v) + v \cdot v$$

Proof:

$$\begin{aligned} (u+v) \cdot (u+v) &= (u_1+v_1)^2 + \dots + (u_n+v_n)^2 \\ &= (u_1^2 + 2u_1v_1 + v_1^2) + \dots + (u_n^2 + 2u_nv_n + v_n^2) \\ &= (u_1^2 + \dots + u_n^2) + 2(u_1v_1 + \dots + u_nv_n) \\ &= u \cdot u + 2u \cdot v + v \cdot v \end{aligned}$$

ANGLES AND ORTHOGONALITY in \mathbb{R}^n

Any two vectors u and v in \mathbb{R}^n determine a plane, in this plane we can measure the angle between u and v , suppose it is ϑ . The inner product gives a relation between the length of u and v and the angle between them. Indeed the following result holds

THEOREM: Let u and v be two vectors in \mathbb{R}^n . Let ϑ be the angle between them. Then

$$u \cdot v = \|u\| \|v\| \cos \vartheta$$

proof: let us prove it for \mathbb{R}^2



let ϑ_1 and ϑ_2
be the angles

between the x-axis and v and u respectively. Hence the angle between u and v is given by

$$\vartheta = \vartheta_2 - \vartheta_1$$

We can obtain the components of u and v by orthogonal projection

$$u = (\|u\| \cos \vartheta_2; \|u\| \sin \vartheta_2)$$

$$v = (\|v\| \cos \vartheta_1; \|v\| \sin \vartheta_1)$$

Hence

$$u \cdot v = \|u\| \cos \vartheta_2 \|v\| \cos \vartheta_1$$

$$+ \|u\| \sin \vartheta_2 \|v\| \sin \vartheta_1$$

$$= \|u\| \|v\| (\cos \vartheta_2 \cos \vartheta_1 + \sin \vartheta_2 \sin \vartheta_1)$$

$$= \|u\| \|v\| \cos \vartheta$$

↓
RULE OF cos and

sine of difference

of angles says that

$$\cos(\vartheta_2 - \vartheta_1) = \cos \vartheta_2 \cos \vartheta_1 + \sin \vartheta_2 \sin \vartheta_1$$

REMARK: the angle between two non null vectors u and v in \mathbb{R}^n is given by

$$\theta = \arccos\left(\frac{u \cdot v}{\|u\| \|v\|}\right)$$

A further consequence of the theorem is that the angle between two nonnull vectors u and v in \mathbb{R}^n is:

- i) ACUTE if $u \cdot v > 0$
- ii) OBTUSE if $u \cdot v < 0$
- iii) RIGHT if $u \cdot v = 0$

Hence two non null vectors are ORTHOGONAL if and only if

$$u \cdot v = 0$$

We begin our study of orthogonality with an easy result

THEOREM (Orthogonality and 0)

- i) 0 is orthogonal to every vector in \mathbb{R}^n
- ii) 0 is the only vector in \mathbb{R}^n that is orthogonal to itself

proof: i) $\forall v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$$0 \cdot v = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n = 0$$

hence 0 and v are orthogonal

ii) if $v \in \mathbb{R}^n$ is such that $v \cdot v = 0$
 $\Rightarrow v = 0$ by definition of inner product.

Next result extends the PYTHAGORAS theorem to \mathbb{R}^n

PYTHAGOREAN THEOREM

Given u and v orthogonal vectors in \mathbb{R}^n . Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

proof: We have

$$\|u + v\|^2 = (u + v) \cdot (u + v) =$$

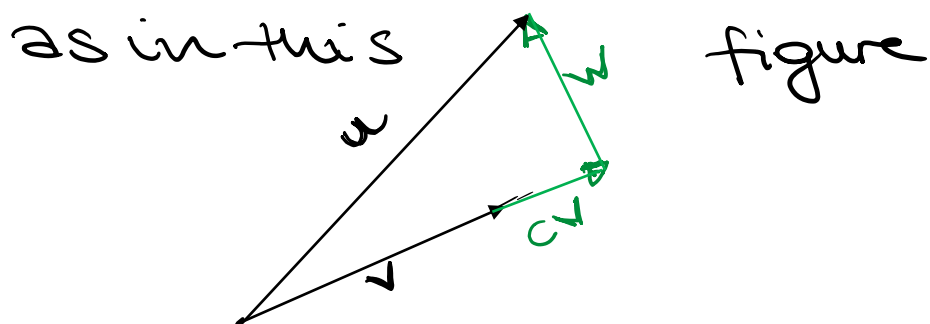
$$= u \cdot u + 2(u \cdot v) + v \cdot v =$$

$$= u \cdot u + v \cdot v$$

$$= \|u\|^2 + \|v\|^2$$

$u \cdot v = 0$
as u and v are orthogonal

The next result proves that given $u, v \in \mathbb{R}^n$, with $v \neq 0$, then u can be written as a scalar multiple of v plus a vector w orthogonal to v



THEOREM (ORTHOGONAL DECOMPOSITION)
 Suppose $u, v \in \mathbb{R}^n$, with $v \neq 0$. Set

$$c = \frac{u \cdot v}{\|v\|^2} \text{ and } w = u - \frac{u \cdot v}{\|v\|^2} v.$$

Then

$$w \cdot v = 0 \text{ and } u = cv + w$$

proof: Let $c \in \mathbb{R}$ be a scalar. Then

$$u = cv + (u - cv)$$

Thus we need to choose $c \in \mathbb{R}$ so that v is orthogonal to $u - cv$. In other words we want

$$\begin{aligned} 0 &= (u - cv) \cdot v = \\ &= u \cdot v - c\|v\|^2 \end{aligned}$$

the equation above shows that we should choose c to be

$$c = \frac{u \cdot v}{\|v\|^2}$$

Hence we can write

$$u = \frac{u \cdot v}{\|v\|^2} v + \left(u - \frac{u \cdot v}{\|v\|^2} v \right)$$

Let us use this decomposition to prove the CAUCHY-SCHWARZ inequality among the most important inequalities in mathematics

THEOREM (CAUCHY-SCHWARZ INEQUALITY)
Suppose $u, v \in \mathbb{R}^n$. Then

$$|u \cdot v| \leq \|u\| \|v\|$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other

proof: if $v = 0$ then both sides of the desired inequality equal 0. Thus we can assume $v \neq 0$. Consider the orthogonal decomposition

$$u = \frac{u \cdot v}{\|v\|^2} v + w$$

where w is orthogonal to v . By the Pythagorean theorem

$$\begin{aligned} \|u\|^2 &= \left\| \frac{u \cdot v}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \frac{|u \cdot v|^2}{\|v\|^4} + \|w\|^2 \geq \frac{|u \cdot v|^2}{\|v\|^4} \end{aligned}$$

Multiplying both sides of this inequality by $\|v\|^2$ and then taking square roots gives the desired inequality

we get an inequality if and only if $w=0$, but recalling, from the orthogonal decomposition, that $w = u - \frac{u \cdot v}{\|v\|^2} v$

then we get an equality if and only if

$$w=0 \iff u - \frac{u \cdot v}{\|v\|^2} v = 0 \iff$$

$$u = \frac{u \cdot v}{\|v\|^2} v \rightarrow \text{SCALAR}$$

hence if u is a scalar multiple of v or v is a scalar multiple of u .

The next result, called the Triangle inequality, has the geometric interpretation that the length of each side of a triangle is less than the sum of the lengths of the other two sides.

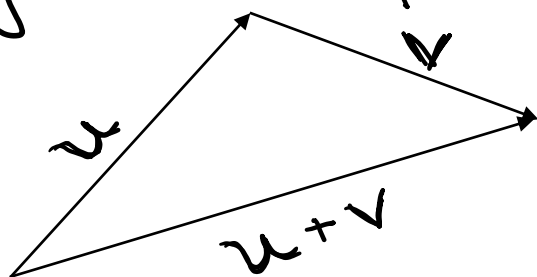
Note that the Triangle Inequality implies that the shortest path between two points is a line segment

THEOREM (TRIANGLE INEQUALITY)

Suppose $u, v \in \mathbb{R}^n$. Then

$$\|u + v\| \leq \|u\| + \|v\|$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.



proof: We have

$$\begin{aligned} \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + 2(u \cdot v) + v \cdot v \\ &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \quad \text{CAUCHY-SCHWARTZ} \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

and taking square roots on both sides gives the result.

We get equality when $u \cdot v = \|u\|\|v\|$
(If one of u, v is a nonnegative multiple of the other then this equality holds. Conversely suppose $u \cdot v = \|u\|\|v\|$ holds

Then the condition for equality in Cauchy Schwarz inequality implies that one of u, v is a scalar multiple of the other. Clearly $u \cdot v = \|u\| \|v\|$ forces the scalar to be nonnegative

DEFINITION: Any assignment of a real number to a vector as the euclidean norm $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$ is called a NORM if it satisfies the following three properties

- i) $\|u\| \geq 0$ and $\|u\| = 0$ iff $u = 0$
- ii) $\|\alpha u\| = |\alpha| \|u\|$
- iii) $\|u + v\| \leq \|u\| + \|v\|$.

HOMEWORK:

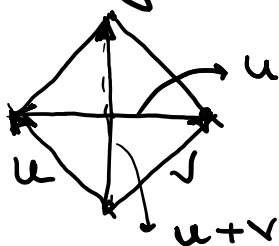
1) Find the length of the following vectors

$$(1, 2, 3) \quad (-1, 0, 3, 4) \quad \left(\frac{1}{2}, -2, 0, 1\right)$$

2) Find the distance from $P(5, 2)$ and $Q(3, 3)$

3) Use vector notation to prove that the diagonals of a rhombus

are orthogonal to each other
(HINT:



hence $u+v$ and $u-v$ are the diagonals of the rhombus)

4) Prove that in \mathbb{R}^2

$$\|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}$$

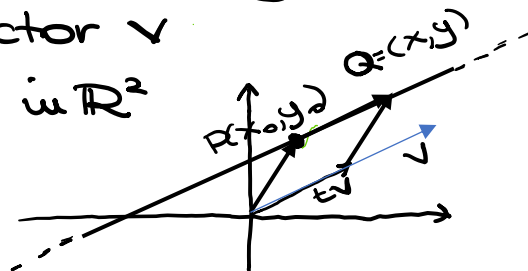
is a norm

LINES AND PLANES

How can we use vectors to describe Lines and Planes?

A LINE is identified by a point P and by a direction / vector v

for example in \mathbb{R}^2



all points $Q(x, y)$ on the line can be obtained by adding the vector that

has initial point at the origin and terminal point in P , and a modification of v by a scalar multiplication (parallelogram rule).

Hence we obtain the PARAMETRIC REPRESENTATION of the line

$$Q(t) = P + t v$$

extensively if $Q(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$ and $P = (x_1^0, \dots, x_n^0)$ and $v = (v_1, \dots, v_n)$