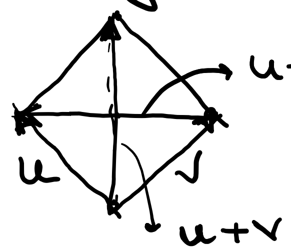
are orthogonal to each other
(HINT:



hence u+v and u-v are the diagonals of the rhombus)

4) Prove that in $\mathbb{R}^2$

$$\|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}$$

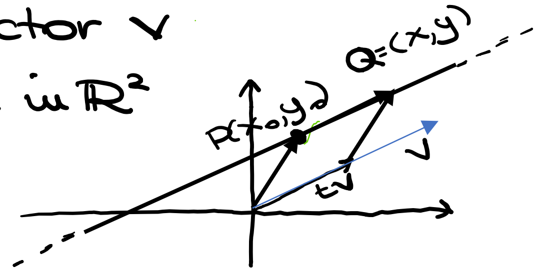
is a norm

LINES AND PLANES

How can we use vectors to describe Lines and Planes?

A LINE is identified by a point P and by a direction / vector v

for example in $\mathbb{R}^2$



all points Q(x,y) on the line can be obtained by adding the vector that has initial point at the origin and terminal point in P, and a modification of v by a scalar multiplication (parallelogram rule).

Hence we obtain the PARAMETRIC REPRESENTATION of the line

$$Q(t) = P + tv$$

extensively if $Q(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n$ and $P = (x_1^0, \dots, x_n^0)$ and $v = (v_1, \dots, v_n)$

$$\begin{cases} x_1(t) = x_1^0 + t v_1 \\ \vdots \\ x_n(t) = x_n^0 + t v_n \end{cases}$$

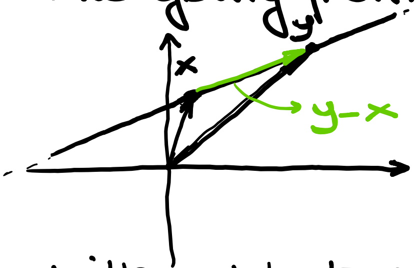
Exercise: Write the parametric representation of a line going through the point $(3,2)$ in the direction $(1,2)$

$$x(t) = (x_1(t), x_2(t)) = (3,2) + t(1,2)$$
$$= (3 + 1\cdot t, 2 + 2\cdot t)$$

in terms of a system

$$\begin{cases} x_1(t) = 3 + t \\ x_2(t) = 2 + 2t \end{cases}$$

Another way of representing a line is to identify two points on the line. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two points on the line r, then r can be seen as the line going from x in the direction $y - x$



Hence we obtain the following parametric equation

$$r(t) = x + t(y-x) =$$
$$= (1-t)x + ty$$

written at last as a combination with coefficients summing to 1 (CONVEX COMBINATION $(1-t) + t = 1$)

EXERCISE: Find the parametric equation of the line through $x = (1,0,1)$ and $y = (-1,2,1)$

$$r(t) = x + t(y-x) = (1,0,1) + t[(-1,2,1) - (1,0,1)]$$
$$= (1,0,1) + t(-1-1, 2-0, 1-1) = (1,0,1) + t(-2,2,0)$$

$$\Longrightarrow$$

$$\begin{cases} x_1(t) = 1 - 2t \\ x_2(t) = 2t \\ x_3(t) = 1 \end{cases}$$

An alternative way of identifying a line is through an orthogonal direction. Consider a point $P(x_0, y_0)$ and a vector $u = (u_1, u_2)$. The parametric equation is

$$\begin{cases} x(t) = x_0 + t u_1 \\ y(t) = y_0 + t u_2 \end{cases}$$

Solving for t the first equation and substituting it in the second equation we obtain

$$t = \frac{x-x_0}{u_1} \Longrightarrow y = y_0 + \frac{x-x_0}{u_1} u_2 \Longrightarrow (y-y_0)u_1 = (x-x_0)u_2$$

$$\Longrightarrow u_1 y - u_1 y_0 = u_2 x - u_2 x_0 \Longrightarrow \boxed{-u_2 x + u_1 y = -u_2 x_0 + u_1 y_0}$$

By setting $n = (-u_2, u_1)$, notice that the vector n is orthogonal to u:

$$n \cdot u = (-u_2, u_1)\cdot(u_1, u_2) = -u_2u_1 + u_1u_2 = 0$$

hence, denoting $p = (x, y)$ and $p_0 = (x_0, y_0)$ we can write the non parametric equation

$$n \cdot p = n \cdot p_0 \Longleftrightarrow -u_2 x + u_1 y = -u_2 x_0 + u_1 y_0$$

known as POINT-NORMAL EQUATION or as CARTESIAN EQUATION

EXERCISE: Transform the following parametrized equation into cartesian equation

$$\begin{cases} x_1(t) = 4 + 2t \\ x_2(t) = 2 - t \end{cases}$$

$$\Longrightarrow t = 2 - x_2 \Longrightarrow x_1 = 4 + 2(2 - x_2) \Longrightarrow x_1 + 2x_2 = 8$$

and if we set $p = (x_1, x_2)$, $p_0 = (4, 2)$ and $n = (1, 2)$

$\Longrightarrow$ the equation can be written as

$$n \cdot p = n \cdot p_0 \Longleftrightarrow (1,2)\cdot(x_1, x_2) = (1,2)\cdot(4\cdot 2)$$

$$\Longrightarrow x_1 + 2x_2 = 4 + 4 \Longrightarrow x_1 + 2x_2 = 8$$

HOMEWORK:

1) Is the point $\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$ on the line $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$? If it is find the value of t.

2) Find parametric and cartesian equations for the line that passes through $(2, -3, 5)$ and parallel to the line of parametric equation
$x = 5 + 2t$, $y = 7 - 3t$, $z = -2 + t$

3) Write the parametric and Cartesian equations of the line passing through the point $A(3, -5)$ and parallel

to the line $x+2y-4=0$

4) Given the lines $r: x+ky+k=0$ and $s: kx+y+k=0$ determine the values of k such that

i) the lines are parallel

ii) their common point belongs to the line $t: x+y+3=0$

A PLANE is determined by two vectors v and w which have different directions (linearly independent). For any scalar $s,t \in \mathbb{R}$ the vector is called a LINEAR COMBINATION of v and w. All the linear combinations of v and w lie on the plane determined by the two vectors. Hence the following parametric equation

$$x = sv + tw \quad \text{where } x=(x_1, x_2, \dots, x_n)$$

$s,t \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, represents a plane that passes through the origin

More generally if the plane passes through the point $p \neq 0 \in \mathbb{R}^n$, and v and w are linear independent direction vectors from p, then the plane has equation

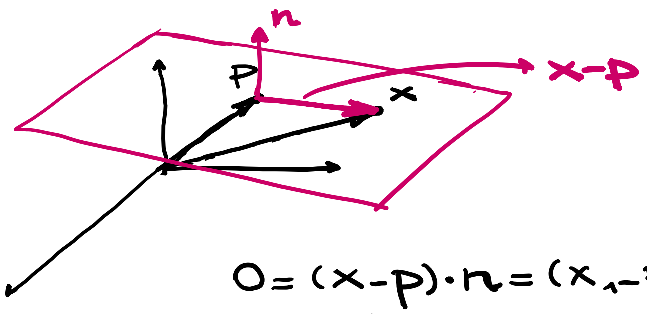
$$x = p + sv + tw, \quad s,t \in \mathbb{R}$$

Also three non collinear points determine a plane. Given the points p, q and r, we can take $q-p$ and $r-p$ as displacement vectors from p, so we get the parametrized equation

$$x = p + s(q-p) + t(r-p) = (1-s-t)p + sq + tr, \quad s,t \in \mathbb{R}$$

the last being a linear combination of the position vectors p, q and r whose coefficients sum to 1

A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. So for example in $\mathbb{R}^3$, given a position vector $p=(p_1, p_2, p_3)$ and a vector $n=(a,b,c)$ orthogonal to the plane, called NORMAL VECTOR, if $x=(x_1, x_2, x_3)$ is a point on the plane, hence n will be orthogonal to the vector of the plane $x-p$, and their inner product must be null



$$0=(x-p)\cdot n=(x_1-p_1, x_2-p_2, x_3-p_3)\cdot(a,b,c)$$
$$=a(x_1-p_1)+b(x_2-p_2)+c(x_3-p_3)$$

and hence we get the POINT-NORMAL EQUATION of the plane

$$ax+by+cz=d$$

EXERCISES:

1) Find the parametric equations and the point-normal equation of the P which contains the points

$$p=(2,1,1),\ q=(1,0,-3),\ r=(0,1,7)$$

Solution:

$$(x(s,t), y(s,t), z(s,t))=(2,1,1)+s(1-2, 0-1, -3-1)+t(0-2, 1-1, 7-1)$$

$$\begin{cases} x(s,t)=2-s-2t \\ y(s,t)=1-s \\ z(s,t)=1-4s+6t \end{cases} \Longrightarrow$$ solve the 2^{nd} for s: $s=1-y$ and plug in the 1^{st} equation

$x=2-(1-y)-2t$ ← solve it for $2t$

$2t=2-1+y-x$ and substitute s and $2t$ written in terms of x and y in the 3^{rd} equation

$$z=1-4\underbrace{(1-y)}_{s}+3\cdot\underbrace{(1+y-x)}_{2t}$$

$$\Longrightarrow z=\cancel{1}-\cancel{4}+4y+\cancel{3}+3y-3x$$

$3x-7y+z=0$ ← PLANE THAT PASSES THROUGH THE ORIGIN

2) Find the point normal equation of the plane through the point $(1,2,3)$ and with normal vector $(4,5,6)$:

Solution: $4(x-1)+5(y-2)+6(z-3)=0$

$$\Longrightarrow 4x+5y+6z=32$$

3) Determine if the following pairs of planes intersect

i) $x+2y-3z=6$ and $x+3y-2z=6$

Solution:

$$\begin{cases} x+2y-3z=6 \\ x+3y-2z=6 \end{cases}$$ EQUATE FIRST MEMBERS AS THEY ARE BOTH EQUAL TO 6

$$\cancel{x}+2y-3z=\cancel{x}+3y-2z \iff y=-z$$

hence if we set $z=t\in\mathbb{R}$ and substitute in any of the two equations of the planes

$$\begin{cases} x=6-2(-t)-3t \\ y=-t \\ z=t \end{cases} \iff \begin{cases} x=6-5t \\ y=-t \\ z=t \end{cases}$$

$$(x(t),y(t),z(t))=(6,0,0)+t(-5,-1,1)$$

that represents a line that passes through the point $(6,0,0)$ and that has direction $(-5,-1,1)$

ii) $x+2y-3z=6$ and $-2x-4y+6z=10$

Solution

$$\begin{cases} x+2y-3z=6 \\ -2x-4y+6z=10 \end{cases}$$ let us solve the 1st equation for x

$x=6-2y+3z$ and substitute it in the 2nd equation

$$-2(6-2y+3z)-4y+6z=10$$

$$-12+\cancel{4y}-\cancel{6z}-\cancel{4y}+\cancel{6z}=10$$

$-12=10$ that is not an identity. Hence the two planes do not intersects.

HOMEWORK

1) Write parametric and non parametric equations for the plane through each of the following triplets of points

   i) $(6,0,0)$ $(0,-6,0)$ $(0,0,3)$

   ii) $(0,3,2)$ $(3,3,1)$ $(2,5,0)$

2) Write the parametric equations of the line of cartesian equation

$$\begin{cases} y = 3x + 1 \\ y - x + z = 0 \end{cases}$$

3) Being r the line in $\mathbb{R}^3$ that passes through the points $A(1,-1,2)$ and $B(-2,0,1)$, and being s the line that passes through $C(1,3,-3)$ and parallel to the direction vector $OD = (2,-2,3)$

   i) determine the reciprocal position of the two lines (intersecting, parallel or skew)
   (DEF: Skew means that lines are nor intersecting nor parallel)

   ii) if they are intersecting, determine the intersection point.

(Hint: when writing the paramtric equations of the two lines, notide that their direction vectors are not proportional; and to check if they are intersecting, solve the system with their equations)

4) Determine the reciprocal position (intersecting, parallel or skew) of the lines

$$r: \begin{cases} x = 2t \\ y = t + 1 \\ z = t + 3 \end{cases} \qquad r': \begin{cases} x = s \\ y = 2 \\ z = s + 2 \end{cases}$$

if the lines are intersecting determine the angle between them

(HINT: the angle ϑ between r and r' is the angle between the direction vectors

$u = (2,1,1)$ and $v = (1,0,1)$

hence exploit the formula

$$u \cdot v = \|u\| \, \|v\| \cdot \cos\vartheta$$

5) Determine the reciprocal position of the lines

$$r: \begin{cases} x = 2t \\ y = t+1 \\ z = t \end{cases} \qquad r': \begin{cases} x = s \\ y = 1 \\ z = 2s+1 \end{cases}$$

6) Identify the parametric equations of the line r passing through $A=(2,3,1)$ and $B=(0,0,1)$ and of the line s passing through the points $C=(0,0,0)$ and $D=(4,6,0)$.

Establish if r and s are coplanar (DEF: two lines are coplanar if they lie on the same plane) and if so, write an equation of the plane that contains r and s

(Hint: identify eventual proportionality between the direction vectors of the two lines

To write the equation of the plane that contains them, you need a further direction, that could be $\vec{AC} = \vec{OA} - \vec{OC} = (2,3,1) - (0,0,0) = (2,3,1)$ different from the direction vectors of r and s.

7) Given the lines

$$r_1: \begin{cases} x = 1+t \\ y = 2t \\ z = 1+t \end{cases} \qquad r_2: \begin{cases} x+y=1 \\ x-y+z=2 \end{cases}$$

i) show that the two lines are intersecting.

ii) determine the equation of the line

orthogonal to r_1 and r_2 and passing through their intersection point.

(Hint: to find intersection, solve the system made by the equations of the two lines. Then by writing both r_1 and r_2 equations in a parametric form, identify the direction vectors of r_1 and r_2 and write the point-normal equation of the plane that passes through the intersection point P between r_1 and r_2 and that has as generating directions, the directions of r_1 and r_2. Hence, identify from it, the normal direction)

8) Consider the following planes

$$\pi : x - y + z = 0 \quad \text{and} \quad \pi' : 8x + y - z = 0$$

i) establish their reciprocal position

ii) find the cartesian equation of the plane passing through $P = (1,1,1)$ and orthogonal to π and π'

(Hint: ii) extrapulate normal directions to π and π' from their point-normal equation)

We have seen LINES in $\mathbb{R}^2$ whose equations are

$$a x_1 + b x_2 = c$$

PLANES in $\mathbb{R}^3$, whose equations are

$$a x_1 + b x_2 + c x_3 = d$$

Generally a HYPERPLANE in $\mathbb{R}^n$ has a POINT-NORMAL EQUATION of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = d$$

that can be viewed as a subspace of $\mathbb{R}^n$ whose dimension is one less than that of $\mathbb{R}^n$, $n-1$.

LINEAR SUBSPACES

We have seen how scalar multiples of a non null vector v in $\mathbb{R}^n$ generate a straight line that passes through the origin

$$\mathcal{L}(v) = \{ rv : r \in \mathbb{R} \}$$

We can call this set as the LINE SPANNED (or GENERATED) by v.

Moreover, two vectors v_1 and v_2 that are not multiple of the other, identify, through all their possible linear combinations, a plane that passes through the origin, and denoted as

$$\mathcal{L}(v_1, v_2) = \{ r_1 v_1 + r_2 v_2 : r_1, r_2 \in \mathbb{R} \}$$

that we can call PLANE SPANNED (or GENE-RATED) by v_1 and v_2.

These two examples lead us to the concept of LINEAR SUBSPACES.
More generally given a subset $A \subseteq \mathbb{R}^n$, it is quite evident that A is endowed with $\mathbb{R}^n$ linear space operations. Although there is no warranty that

$\forall\, u, v \in A$ and $k \in \mathbb{R}$ hence $u + v \in A$ and $ku \in A$

If this happens, means if A contains all possible linear combinations of its elements

$$hu + kv \in A \quad \forall u, v \in A \text{ and } \forall h, k \in \mathbb{R}$$

A is said to be a LINEAR SUBSPACE of $\mathbb{R}^n$.
$\mathcal{L}(v)$ and $\mathcal{L}(v_1, v_2)$ are LINEAR SUBSPACES of $\mathbb{R}^n$ (HOMEWORK: Prove it)

EXAMPLE: The subset

$$A=\{x=(x,0,0,\dots,0)\in\mathbb{R}^n,\ x\in\mathbb{R}\}$$

of vectors in $\mathbb{R}^n$ with all null components, except at the most for the first, is a LINEAR SUBSPACE of $\mathbb{R}^n$, as

$$hx+ky=h(x,0,0,\dots,0)+k(y,0,0,\dots,0)$$
$$=(hx,0,0,\dots,0)+(ky,0,0,\dots,0)$$
$$=(hx+ky,0,0,\dots,0)\in A$$

HOMEWORK: Prove that all proper linear subspaces in $\mathbb{R}^2$ are only lines passing through the origin.

The essential idea beneath the concept of LINEAR SUBSPACES is that there are some vectors that SPAN (GENERATE) the whole subspace, through their linear combinations. Hence, let us recall the notion of LINEAR COMBINATIONS, and let us introduce the notions of INDEPENDENT and DEPENDENT vectors, and of DIMENSION of a LINEAR SPACE

DEFINITION: Given m vectors $x^1, x^2, \dots, x^m \in\mathbb{R}^n$ a LINEAR COMBINATION of them is a vector $x\in\mathbb{R}^n$ of the following type

$$x=k_1x^1+k_2x^2+\dots+k_mx^m=\sum_{j=1}^m k_jx^j$$

where $k_j\in\mathbb{R}$, $j=1,\dots,m$, and they are called COEFFICIENTS.

DEFINITION: vectors that have null components except for the j-th:

$$e^1=(1,0,\dots,0)$$
$$e^2=(0,1,0,\dots,0)$$
$$\vdots$$
$$e^n=(0,0,\dots,0,1)$$

are called STANDARD (or CANONICAL or NATURAL) vectors.

We can prove that any vector $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ can be written as a linear combination of these vectors

$$x=(x_1,x_2,\dots,x_n)=$$
$$=(x_1,0,\dots,0)+(0,x_2,0,\dots,0)+\dots+(0,\dots,0,x_n)$$
$$=x_1(1,0,\dots,0)+x_2(0,1,0,\dots,0)+\dots+x_n(0,\dots,0,1)$$
$$=x_1e^1+x_2e^2+\dots+x_ne^n=\sum_{i=1}^{n}x_ie^i$$

DEFINITION vectors $x^1,x^2,\dots,x^m\in\mathbb{R}^n$ are LINEARLY DEPENDENT if it is possible to express one of them as linear combination of the remaining, say for example x^m

$$x^m=\sum_{i=1}^{m-1}k_ix^i \quad , k_i\in\mathbb{R}, i=1,\dots,m-1$$

the previous equality can be rewriten as

$$\sum_{i=1}^{m-1}k_ix^i-x^m=\underset{\text{NULL VECTOR}}{0}$$

and by setting $k_m=-1 \Longrightarrow$

$$\sum_{i=1}^{m}k_ix^i=0$$

Hence we can reformulate the definition of linearly dependance through the equality $\sum_{i=1}^{m}k_ix^i=0$ in which at least one coefficient is different from zero

DEFINITION The vectors $x^1,\dots,x^m\in\mathbb{R}^n$ are LINEARLY DEPENDENT if there is a linear combination of them, where at least one coefficient is different from zero, and that is equal to the null vector

$$\sum_{i=1}^{m} k_i x^i = 0 \quad \text{if } x^i \in \mathbb{R}^n \text{ are}$$

LINEARLY DEPENDENT $\Rightarrow \exists\, k_j \neq 0$

CONVERSELY

DEFINITION: Vectors $x^1, x^2, \dots, x^m \in \mathbb{R}^n$ are LINEARLY INDEPENDENT if the only linear combination of them that is equal to the null vector is the one whose coefficients are all null:

$$k_1 x^1 + k_2 x^2 + \dots + k_m x^m = 0 \iff k_1 = k_2 = \dots = k_m = 0$$

EXAMPLE Let us prove that the whole set of standard vectors are linearly independent

$$\sum_{i=1}^{n} k_i e^i = (k_1, 0, \dots, 0) + (0, k_2, 0, \dots, 0) + \dots + (0, \dots, 0, k_n)$$
$$= (k_1, k_2, \dots, k_n)$$

is the null vector if and only if $k_i = 0 \ \forall\, i = 1, \dots, n$

EXAMPLE Verify that the vectors

$$x^1 = (3,2,1) \quad x^2 = (4,1,3) \quad x^3 = (3,-3,6)$$

are linearly dependent

Let us look for a linear combination of these three vectors that is equal to the null vector

$$\sum_{i=1}^{3} k_i x^i = 0 \iff k_1(3,2,1) + k_2(4,1,3) + k_3(3,-3,6) = (0,0,0)$$

hence we have to solve the following system

$$\begin{cases} 3k_1 + 4k_2 + 3k_3 = 0 \\ 2k_1 + k_2 - 3k_3 = 0 \\ k_1 + 3k_2 + 6k_3 = 0 \end{cases}$$

surely $k_1 = k_2 = k_3 = 0$ is a solution, but for example also $k = -1, k_2 = 1, k_3 = -1/3$ is a solution. Hence x^1, x^2, x^3 are DEPENDENT

Fundamentally the verification of the conditions of dependance or independance passes through resolutions of linear systems, that could result quite cumbersome.
The notion of rank of a matrix helps us solve, in a more rapid way, the problem of counting the number of independent vectors in a given set of vectors:

<u>THEOREM</u> The rank of a matrix identifies the maximum number of column (or row) vectors that form the matrix, and that are linearly independent

<u>EXAMPLE</u>:

1) Establish if the following set of vectors in $\mathbb{R}^4$ are LINEARLY INDEPENDENT:

$u=(0,-1,-1,-2)$ $v=(2,5,-1,2)$ $w=(1,2,-1,0)$

<u>Solution</u>: display these vectors as columns of a matrix

$$A=\begin{pmatrix} 0 & 2 & 1 \\ -1 & 5 & 2 \\ -1 & -1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$ and evaluate its rank

$A \in M_{4\times 3} \Rightarrow 1 \leq rk(A) \leq 3$

As the highlighted minor $\begin{vmatrix} 0 & 2 \\ -1 & 5 \end{vmatrix} \neq 0 \Rightarrow$

$2 \leq rk(A) \leq 3$. The determinants of the bordered submatrices are

$$\left|\begin{pmatrix} 0 & 2 & 1 \\ -1 & 5 & 2 \\ -1 & -1 & -1 \end{pmatrix}\right| = 0+\cancel{4}+\cancel{1}-\cancel{5}-2=-2\neq 0$$

$\Rightarrow rk(A)=3$ hence the three columns of A are linearly independent. Hence the given vectors are independent

2) Establish the number of linearly independent vectors as the parameter α changes

$u = (\alpha, 4, -1) \quad v = (-1, -\alpha, 1) \quad w = (3, -1, -3)$

<u>Solution</u>:

$$A = \begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{pmatrix} \alpha & -1 & 3 \\ 4 & -\alpha & -1 \\ -1 & 1 & -3 \end{pmatrix} \end{array} \in M_3 \Rightarrow \text{if } \det A \neq 0 \Rightarrow$$

$rk(A) = 3$ and the 3 vectors are independent

stands for "if and only if"

$\det A = 3\alpha^2 - 1 + \cancel{12} - \cancel{3}\alpha + \alpha - \cancel{12} = 3\alpha^2 - 2\alpha - 1$

hence $\det A = 0$ iff $3\alpha^2 - 2\alpha - 1 = 0 \Rightarrow$ iff

$\alpha = 1$ or $\alpha = -\frac{1}{3}$

Hence if $\alpha \neq 1$ or $\alpha \neq -\frac{1}{3}$ the 3 vectors are independent.

If $\alpha = 1 \Rightarrow A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 4 & -1 \end{matrix}} & \begin{matrix} 3 \\ -1 \end{matrix} \\ \begin{matrix} -1 & 1 \end{matrix} & -3 \end{pmatrix}$ the highlighted minor of order 2 is different from zero, hence $rk(A) = 2 \Rightarrow u$ and v (that correspond to the columns of the highlighted minor) are independent, whereas w can be written as linear combination of u and v

if $\alpha = -\frac{1}{3}$

$$A = \begin{pmatrix} \boxed{\begin{matrix} -\frac{1}{3} & -1 \\ 4 & \frac{1}{3} \end{matrix}} & \begin{matrix} 3 \\ -1 \end{matrix} \\ \begin{matrix} -1 & 1 \end{matrix} & 3 \end{pmatrix} \quad \begin{vmatrix} -1/3 & -1 \\ 4 & 1/3 \end{vmatrix} = -\frac{1}{9} + 4 \neq 0$$

hence $rk(A) = 2$

and hence the 3 vectors are linearly dependent and the number of linearly independent vectors are u and v, whereas w can be written as a linear combination of u and v

<u>REMARK</u>: The choice of the MINOR to determine the rank is arbitrary. I could have chosen for example

$$\begin{vmatrix} \frac{1}{3} & -1 \\ 1 & 3 \end{vmatrix} = \frac{1}{3} \cdot 3 - (-1) \cdot 1 = 1 + 1 \neq 2$$

hence this choice tells me always that u, v and w are linearly dependent, but with this choice v and w are independent, and u can be written as their linear combination.

<u>HOMEWORK</u>:

1) Determine if the following sets of vectors are independent. If not determine the number of linearly independent vectors.

i) $u = (-1, 0, -1, -2)$ $v = (-1, 2, 5, 2)$
$w = (-1, 1, 2, 0)$

iii) $u = (0, 1, -1)$ $v = (1, 2, 0)$ $w = (1, 0, 2$

2) Discuss the independence of the following set of vectors as α changes

i) $u = (0, 2, \alpha)$ $v = (3, 1, 1)$ $w = (\alpha, 3, 0)$

ii) $u = (4, \alpha, 1)$ $v = (-\alpha, -1, 1)$ $w = (-1, 3, -3)$

iii) $u = (\alpha + 3, \alpha + 3, 0)$ $v = (\alpha, 1, \alpha)$

iv) $u = (1, 0, 1)$ $v = (0, 3, 0)$ $w = (2 + \alpha, \alpha, 2)$
$z = (3, 3, 3)$

DEFINITION: Let $A \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. The subspace generated by A, denoted by $\text{Span}(A)$ (all possible linear combinations of vectors of A), is the smallest linear space of $\mathbb{R}^n$ that contains A

DEFINITION: Given a linear subspace $A \subseteq \mathbb{R}^n$ $x^1, x^2, \dots, x^m$ are a BASIS of A if they are linearly independent and if any vector $x \in A$ can be written as a unique linear combination of $x^1, x^2, \dots, x^m$ (UNIQUENESS is PROVABLE $\to$ HOMEWORK)

REMARK: A subset $S = \{x^1, x^2, \dots, x^m\}$ basis for a linear subspace hence is formed by essential vectors, none of them is redundant

EXAMPLE: we have already seen that the set of standard vectors $S = \{e^j, j = 1, \dots, n\}$ form a BASIS for $\mathbb{R}^n$

EXAMPLE: Verify that the set of vectors

$x^1 = (2,0,0)$ $x^2 = (0,2,0)$ $x^3 = (1,0,2)$

form a basis of $\mathbb{R}^3$

Solution: the 3 vectors are independent as

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \det A = 8 \quad \text{and } rk(A) = 3$$

The set of the 3 vectors moreover generates $\mathbb{R}^3$. Indeed $\forall\, y = (y_1, y_2, y_3)$ can be expressed as a linear combination of x^1, x^2, x^3

$$y = \sum_{i=1}^{3} k_i x^i \iff \begin{cases} 2k_1 + k_3 = y_1 \\ 2k_2 = y_2 \\ 2k_3 = y_3 \end{cases}$$

that is satisfied for

$$\begin{cases} k_1 = \frac{y_1}{2} - \frac{y_3}{4} \\ k_2 = \frac{y_2}{2} \\ k_3 = \frac{y_3}{2} \end{cases}$$

REMARK: These two examples show that a single LINEAR SPACE has more than one base, but all the bases of a same linear space (subspace) are formed by the same number of vectors

THEOREM if $A \subseteq \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$ and if A has a basis formed by m vectors then each linearly independent set of A has at the most m elements

m is called the DIMENSION of A.

HOMEWORK:

1) Establish which of the following subsets of $\mathbb{R}^3$ represent a linear subspace

i) $V_1 = \{(a,a,a) \in \mathbb{R}^3;\ a \in \mathbb{R}\}$

ii) $V_2 = \{(a,b,a) \in \mathbb{R}^3;\ a,b \in \mathbb{R}\}$

iii) $V_3 = \{(a, 2a, a+b) \in \mathbb{R}^3,\ a,b \in \mathbb{R}\}$

iv) $V_4 = \{(a,b,c) \in \mathbb{R}^3, a,b,c \in \mathbb{N}\}$

NATURAL NUMBERS

2) Given the vectors

$v_1 = (1,0,1,0)$ $v_2 = (2,h,2,h)$ $v_3 = (1,1+h,1,2h)$

consider $W = \mathcal{L}(v_1, v_2, v_3) = \text{span}(v_1, v_2, v_3)$
Determine dim W and a base for W as h changes

(Hint: Determine the rank of the matrix formed by displaying components of v_1, v_2, v_3 as columns. The minor that defines the rank will identify the vectors that form a basis)