WARNING

These slides are meant to help the teacher during the lecture, they are not substitute of the book, they may be incorrect and incomplete.

During the lecture the teacher explains and integrates what is written in the slides.

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Systems of Linear Equations

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Introduction

An equation is **linear** if it has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

Letters $a_1, a_2, .., a_n$ are real numbers called **parameters** or coefficients, while $x_1, x_2, ..., x_n$ are **variables**, called also **unknowns**.
Linear equations describe geometric objects such as lines and planes.

Linear system

A linear system of m equations and n unknowns can be written

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = & b_2 \\ \vdots \qquad \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = & b_m \end{cases}$$

In this system the a_{ij}'s and b_i's are given real numbers called coefficients. A solution of the system is a n-tuple of real numbers $x_1, x_2, .., x_n$ which satisfies each of the m equations.

Questions

We are interested in the following questions:

1. Does a solution exist?
2. How many solutions are there?
3. Is there an efficient algorithm that computes actual solutions?

Methods

There are essentially three methods of solving such systems:

1. substitution
2. elimination of variables
3. matrix methods

Substitution

1. Solve one equation of the system for one variable, say x_n, in terms of the other variables.
2. Substitute x_n expressed in terms of the other $n-1$ unknowns into the other $m-1$ equations. We have now a system of $m-1$ equations and $n-1$ unknowns $x_1, x_2, ..., x_{n-1}$.
3. Explicit x_{n-1} in one equation and substitute it in the others...
4. Continue this process until you obtain a system with only one equation. A single equation is easy to solve.
5. Solve the single equation
6. Go backward substituting outcomes from the last solution to find all the x_i's.

Example

Given the system

$$\begin{cases} x_1 & +2\,x_2 & +x_3 & = 4 \\ x_1 & -x_2 & +x_3 & = 5 \\ 2\,x_1 & +3\,x_2 & -x_3 & = 1 \end{cases}$$

find a solution by substitution. This system has 3 equations and 3 unknowns.

Example (2)

Solve equation (3) wrt x_3

$$x_3 = 2\ x_1 + 3\ x_2 - 1$$

Substitute this expression into the two equations

$$\begin{cases} x_1 & +2\ x_2 & +x_3 & = 4 \\ x_1 & -x_2 & +x_3 & = 5 \end{cases}$$

you obtain the new system of two equations in two unknowns

$$\begin{cases} 3\ x_1 & +5\ x_2 & = 5 \\ 3\ x_1 & +2\ x_2 & = 6 \end{cases}$$

Example (3)

From the first equation we have $x_1 = \frac{5-5\ x_2}{3}$
which substituted in the second leads to the equation

$$-3\ x_2 = 1$$

Hence $x_2 = -\frac{1}{3}$, and going backward
$x_1 = \frac{5-5\ x_2}{3} = \frac{5-5\ \left(-\frac{1}{3}\right)}{3} = \frac{20}{9}$ and $x_3 = 2\ \left(\frac{20}{9}\right) + 3\ \left(-\frac{1}{3}\right) - 1 = \frac{22}{9}$

The solution of the system is $\left(\frac{20}{9}, -\frac{1}{3}, \frac{22}{9}\right)$

Elimination method

This methods is based on these two facts:

- If we multiply both sides of an equation by a number (different from zero) the solution does not change.
- If we add two equations in a system and substitute one of them by the resulting equation the solution does not change.
- If we interchange equations the solution does not change.

To solve a system by elimination you must add proper multiple of the first equation to each of the succeeding equations in order to obtain a new equivalent (same solution) system without unknown x_1. Then disregard the first equation and replicate with the second equation and x_2. Continue to eliminate variables until you reach the last equation, that should be easy to solve.

Example

Given the system

$$\begin{cases} x & -3\,y & +6z & = -1 \\ 2x & -5y & +10z & = 0 \\ 3x & -8y & +17z & = 1 \end{cases}$$

find a solution by elimination.
This system has 3 equations in 3 unknowns.

Example (2)

1. Multiply equation (1) by -2 and add the first and the second equations obtaining
$$y - 2z = 2$$
2. Multiply equation (1) by -3 and add the first and the third equations obtaining
$$y - z = 4$$

We obtain the new system of 2 equations in 2 unknows

$$\begin{cases} y & -2z & = 2 \\ y & -z & = 4 \end{cases}$$

Example (3)

1. Multiply equation (1) by -1 and add the equations:

$$z = 2$$

2. Go backward substituting z in one of the last equations and get $y = 6$, finally substitute z and y in the first equation of the original system getting $x = 5$

The final solution is $(5, 6, 2)$.

Notes

Multiplying an equation by a scalar, adding two equations and interchanging equations in a linear systems are called **elementary equation operations** and since these operation are reversible, any solution of the transformed system will also be a solution of the original system.

Definition

Two systems of linear equations are **equivalent** if any solution of one system is also a solution of the other.

Exercises

Exercise 1 Solve the system

$$\begin{cases} x_1 & + \ x_2 & +x_3 & = 0 \\ 12x_1 & +2x_2 & -3x_3 & = 5 \\ 3\ x_1 & +4\ x_2 & +x_3 & = -4 \end{cases}$$

Exercise 2 Solve the system

$$\begin{cases} 2x & -4y & = 0 \\ x & -2y & = 5 \end{cases}$$

Systems of equations in matrix form

The system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = & b_2 \\ \vdots \qquad \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = & b_m \end{cases}$$

can be written in the matrix fom

$$A\mathbf{x} = \mathbf{b}$$

A is a $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$\mathbf{x}$ is a $n \times 1$ matrix (a column matrix) and $\mathbf{b}$ is a $m \times 1$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Augmented matrix

The matrix obtained by adding on a column corresponding to the vector **b**

$$(A \mid \mathbf{b}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

is called the **augmented matrix**.

The existence and the "quantity" of solutions depends on the ranks of coefficient matrix A and the augmented matrix $A \mid \mathbf{b}$.

Matrix Rank

The **rank** of a matrix A is the dimension of the space spanned by its vectors (columns or rows). It is denoted by $\mathbf{Rank}(\mathbf{A})$.

It is equivalent to say:
the rank is the number of linear independent vectors.

Note: there are many equivalent definitions of rank

Example

Consider the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

to study the rank consider the vector $\mathbf{v} = (1,1)$ and $\mathbf{w} = (-1,2)$.
Are they linearly independent?
They are linearly independent if $\alpha\mathbf{v} + \beta\mathbf{w} = \mathbf{0}$ iff $\alpha = \beta = 0$.
Let's do it!

$$\alpha\mathbf{v} + \beta\mathbf{w} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{0} \rightarrow \begin{cases} \alpha - \beta & = 0 \\ \alpha + 2\beta & = 0 \end{cases}$$

The only solution is $\alpha = \beta = 0$. The vectors $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, the rank of this matrix is 2.

Rank and minors

Can we find a simpler way to verify it?
Consider

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

hence $\mathbf{v} = (a_{11}, a_{21})$ and $\mathbf{w} = (a_{12}, a_{22})$. Thus

$$\alpha \mathbf{v} + \beta \mathbf{w} = \mathbf{0} \to \begin{cases} \alpha a_{11} + \beta a_{12} & = 0 \\ \alpha a_{21} + \beta a_{22} & = 0 \end{cases}$$

By substitution $\alpha = -\frac{a_{12}}{a_{11}}\beta$ (assuming $a_{11} \neq 0$) and hence

$$\beta \left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} \right) = 0$$

Rank and minors (2)

The equation

$$\beta \left(\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} \right) = 0$$

admits solution $\beta = 0$ if $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

But

$$a_{11}a_{22} - a_{12}a_{21}$$

is the determinant of matrix A!!
Hence $rank(A) = 2$ if $det(A) \neq 0$.

Lemma

The rank of a $m \times n$ matrix A is equal the maximal order of the non-zero minor.

Exercises

Ex 1 Find the rank of matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

Ex 2 Study the rank of matrix A in function of a?

$$A = \begin{pmatrix} a & 1 & 4 \\ 2 & 1 & a^2 \\ 1 & 0 & -3 \end{pmatrix}$$

Rouche-Capelli Theorem

Theorem

A system of linear equations with n unknowns and m equations admits a solution if and only if the rank of its coefficient matrix A is equal to the rank of its augmented matrix $(A \mid \mathbf{b})$. If there are solutions, they form an affine subspace of $\mathbb{R}^n$ of dimension $n - rank(A)$.
In particular, if $rank(A) = n$, the solution is unique, otherwise there is an infinite number of solutions.

Examples

Consider the system

$$\begin{cases} x & +y & -3z & = 1 \\ -x & +2y & +4z & = 2 \\ x & +4y & -2z & = 4 \end{cases}$$

we know that the solution is a line with parametric equation

$$\begin{cases} x & = 10 - 10t \\ y & = t \\ z & = 3 - 3t \end{cases}$$

Example (cont')

The coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & -3 \\ -1 & 2 & 4 \\ 1 & 4 & -2 \end{pmatrix}$$

the augmented matrix

$$A \mid \mathbf{b} = \begin{pmatrix} 1 & 1 & -3 & 1 \\ -1 & 2 & 4 & 2 \\ 1 & 4 & -2 & 4 \end{pmatrix}$$

What about their ranks?

Example (cont')

The rank of A is 2 since $det(A) = 0$, while $\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3$.

The rank of $A \mid \mathbf{b}$ is 2 since all the (4) minors of order 3 are zero:

$$\begin{vmatrix} 1 & -3 & 1 \\ 2 & 4 & 2 \\ 4 & -2 & 4 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & -3 & 1 \\ -1 & 4 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \\ 1 & 4 & 4 \end{vmatrix} = 0$$

Note that the minors of order 3 are 4 but one is $|A|$.
$rank(A) = rank(A \mid \mathbf{b}) = 2$ by R-C Theorem solution exists and is a space of dimension $n - rank(A) = 3 - 2 = 1$, it is a line.

Example

Consider the system

$$\begin{cases} 2x & -4y & = 0 \\ x & -2y & = 5 \end{cases}$$

we know that this system does not admit solution.
The coefficient matrix and the augmented matrix have different rank:

$$A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \qquad A \mid \mathbf{b} = \begin{pmatrix} 2 & -4 & 0 \\ 1 & -2 & 5 \end{pmatrix}$$

$rank(A) = 1 \neq 2 = rank(A \mid \mathbf{b})$.

Exercises

Solve the following systems if they admit solutions

1.
$$\begin{cases} x + 2y & = 5 \\ x + y + 3z & = 1 \end{cases}$$

2.
$$\begin{cases} x + y + z + u & = 1 \\ x + 2y + 3z + 4u & = 5 \end{cases}$$

3.
$$\begin{cases} x - y & = 6 \\ x + y & = 1 \\ 2x + y & = 9 \end{cases}$$

Exercises

Solve the following systems discussing the existence of solutions as the parameter (r, k or t) varies

1. $$\begin{cases} rz - y + 2x &= r \\ x + y &= r - 1 \\ z - ry &= 0 \end{cases}$$

2. $$\begin{cases} x - k^2 z &= 0 \\ ky + z &= 1 \\ kx - 2y &= 1 \end{cases}$$

3. $$\begin{cases} z + ty &= 3 \\ y + tz &= t \\ y + z &= 1 \end{cases}$$

A system of linear equations is homogeneous if all of the constant terms are zero:

$$\begin{array}{lllll} a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1n}x_n & =0 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots+ & a_{2n}x_n & =0 \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{mn}x_n & =0. \end{array}$$

A homogeneous system can be written in matrix form

$$A\mathbf{x} = \mathbf{0}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is a column vector with n entries, and $\mathbf{0}$ is the zero vector with m entries.

A homogeneous system admits always at least the *trivial* solution, all the unknowns equal zero.

Q: When does the system admit solution different from zero?

A: When the rank of A is lower than the number of unknows.

Example

Study the system

$$\begin{cases} x + ky - z &= 0 \\ y - kx + z &= 0 \\ 3y - x + z &= 0 \end{cases}$$

Sol: if $k \neq 3$ and $k \neq 1$ the system admits the trivial solution $(0, 0, 0)$.
If $k = 3$ or $k = 1$?

Cramer's method

Consider a square system (2 unknows and 2 equations)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

If the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible the solution of the system $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Cramer's method

Recall that $A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$ hence

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Exercises

Exercise 1 Solve the system

$$\begin{cases} 4x + y = & -7 \\ 2x - 3y = & 7 \end{cases}$$

Exercise 2 Study the solution of the following system as k varies

$$\begin{cases} kx + y = & 0 \\ x + ky = & k \end{cases}$$