1 (a) $\begin{cases} x_1 - x_2 - x_3 = 0 \\ 3x_1 + x_2 + 2x_3 = 0 \\ 4x_1 + x_3 = 0 \end{cases}$

ROUCHÉ - CAPELLI:

$$A|b = \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 4 & 0 & 1 & 0 \end{array}\right)$$

$rk\,A = rk\,A|b \Rightarrow \{$ ADMITS SOLUTIONS.

$$\det A = 1 \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + 1 \det\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} - 1 \det\begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix} =$$

$$= 1 - 5 + 4 = 0$$

$\Rightarrow rk\,A < 3$, $\det\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = 4 \Rightarrow rk\,A = 2$

SO WE HAVE ∞^{3-2} SOLUTIONS
$\shortparallel$
∞^1.

WE NEED A PARAMETER TO DESCRIBE THE SOLUTIONS

$\Rightarrow$ $\begin{cases} x_1 - x_2 = x_3 \\ 3x_1 + x_2 = -2x_3 \end{cases}$ GET $x_3 = t$, $t \in \mathbb{R}$

$\begin{cases} x_1 - x_2 = t \\ 3x_1 + x_2 = -2t \end{cases}$

USING CRAMER'S METHOD WE OBTAIN

$\Delta x = \begin{vmatrix} t & -1 \\ -2t & 1 \end{vmatrix}$ $\Delta y = \begin{vmatrix} 1 & t \\ 3 & -2t \end{vmatrix}$

$$\Delta = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad x = \frac{\Delta x}{\Delta} = \frac{t - 2t}{4} \; ; \quad y = \frac{-2t - 3t}{4} \; ; \quad z = t$$

$$\begin{cases} x = -t/4 \\ y = -5/4 \, t \\ z = t \end{cases} \quad t \in \mathbb{R} \qquad \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ -5/4 \\ 1 \end{bmatrix}}$$

(b)

$$\begin{cases} -x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = -1 \\ x_1 - 2x_2 - 2x_3 = 0 \end{cases}$$

$$A|b = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 1 & -1 & 0 & -1 \\ 1 & -2 & -2 & 0 \end{array}\right)$$

$$\det A = -1 \det\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} + 1 \det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} =$$

$$= -2 + 2 - 1 \neq 0$$

ROUCHÉ - CAPELLI:

$$\Rightarrow \quad r_k A = 3 = r_k A|b = 3 = n = \# \text{ OF VARIABLES}$$

$\Rightarrow$ { ADMITS A UNIQUE SOLUTION

THANKS TO CRAMER WE HAVE:

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & -2 & -2 \end{vmatrix} =$$

$$= 2\det\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} - 1\det\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \det\begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix} =$$

$$= 4 - 2 + 2 = 4$$

$$\Delta_y = \begin{vmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1\det\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} + \det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} =$$

$$= -2 + 4 + 1 = 3$$

$$\Delta_z = \begin{vmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & -2 & 0 \end{vmatrix} = -1\det\begin{pmatrix} -1 & -1 \\ -2 & 0 \end{pmatrix} - 1\det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + 2\det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

$$= 2 - 1 - 2 = -1$$

$$\Delta = -1$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix}}$$

(c)

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 0 \\ -x_1 + x_2 + 2x_3 + x_4 = 0 \\ 3x_2 + 5x_3 + 2x_4 = 0 \end{cases}$$

$$A|b = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ -1 & 1 & 2 & 1 & 0 \\ 0 & 3 & 5 & 2 & 0 \end{array}\right)$$

ROUCHÉ - CAPELLI:

$$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 0 & 3 & 5 \end{bmatrix} \Rightarrow \det M_1 = \det\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} + \det\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = -1+1=0$$

$$M_2 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix} \Rightarrow \det M_2 = 0 \quad (\text{BECAUSE } r_1 + r_2 = r_3)$$

$$M_3 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \Rightarrow \det M_3 = 0 \quad (\text{BECAUSE } r_1 + r_2 = r_3)$$

$$M_4 = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 1 \\ 0 & 5 & 2 \end{bmatrix} \Rightarrow \det M_4 = 0 \quad (\text{BECAUSE } r_1 + r_2 = r_3)$$

$$\Rightarrow rk\, A < 3$$

$$\tilde{M} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow \det \tilde{M} = 3$$

$$\Rightarrow rk\, A = 2 = rk\, A|b$$

$\Rightarrow$ ROUCHÉ CAPELLI: $\{$ ADMITS ∞^{4-2} SOLUTIONS $\Rightarrow \infty^2$ SOLUTIONS

$$\begin{cases} x_1 + 2x_2 = -3k_1 - k_2 \\ -x_1 + x_2 = -2k_1 - k_2 \end{cases}$$

GETTING $x_3 = k_1$, $x_4 = k_2$ $\quad k_1, k_2 \in \mathbb{R}$

THANKS TO CRAMER, WE HAVE:

$$\Delta x_1 = \begin{vmatrix} -3k_1 - k_2 & 2 \\ -2k_1 - k_2 & 1 \end{vmatrix} = -3k_1 - k_2 + 4k_1 + 2k_2 = k_1 + k_2$$

$$\Delta x_2 = \begin{vmatrix} 1 & -3k_1 - k_2 \\ -1 & -2k_1 - k_2 \end{vmatrix} = -2k_1 - k_2 - 3k_1 - k_2 = -5k_1 - 2k_2$$

$$x_1 = \frac{k_1}{3} + \frac{k_2}{3}$$

$$x_2 = -\frac{5}{3} k_1 - \frac{2}{3} k_2 \quad \Rightarrow \quad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}}$$

$$x_3 = k_1$$

$$x_4 = k_2$$

$\boxed{2}$ (a)

$$\begin{cases} x - y - z = 0 \\ 3x + y + 2z = 0 \\ 4x + hz = 0 \end{cases} \qquad A|b = \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 4 & 0 & h & 0 \end{array}\right)$$

$$\det A = 4 \det \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} + h \det \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} =$$

$$= -4 + h(1+3) = 4h - 4$$

IF $h \neq 1 \Rightarrow r_k A = r_k A|b = 3 \Rightarrow$ { ADMITS A UNIQUE SOLUTION

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

IF $h = 1 \Rightarrow r_k A = 2 = r_k A|b = 2 \Rightarrow$ { ADMITS ∞^{3-2} SOLUTIONS

$\rightsquigarrow \infty^1$ SOLUTIONS

SO WE NEED 1 PARAMETER t $(z = t)$

$$\begin{cases} x - y = t \\ 3x + y = -2t \end{cases} \Rightarrow \begin{cases} x = t + y \\ 3(t+y) + y = -2t \end{cases}$$

$$\Rightarrow \begin{matrix} y = -\frac{5}{4} t \\ x = -\frac{t}{4} \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

(b)

$$\begin{cases} -x+y+z=2 \\ x-y=1 \\ x-2y-2z=h \end{cases}$$

$$A|b = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 1 & -2 & -2 & h \end{array}\right)$$

$$\det A = \det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} - 2\det\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = -1 \neq 0$$

$\Rightarrow \text{rk}\, A = \text{rk}\, A|b = 3 \underset{\text{ROUCHÉ-CAPELLI}}{\Rightarrow}$ $\Big\{$ ADMITS A UNIQUE SOLUTION

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ h & -2 & -2 \end{vmatrix} = \det\begin{pmatrix} 1 & -1 \\ h & -2 \end{pmatrix} - 2\det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -2+h+6 = h+4$$

$$\Delta_y = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & h & -2 \end{vmatrix} = \det\begin{pmatrix} 1 & 1 \\ 1 & h \end{pmatrix} - 2\det\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} = h-1+6 = h+5$$

$$\Delta_z = \begin{vmatrix} -1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & -2 & h \end{vmatrix} = 2\det\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} - \det\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} + h\det\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =$$

$$= -2-1 = -3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -h-4 \\ -h-5 \\ 3 \end{bmatrix}$$

$\boxed{3}$ (a) $A-\lambda I = \begin{pmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = (5-\lambda)^2 - 16$

$$\Rightarrow \begin{array}{l} 5-\lambda_1 = 4 \\ 5-\lambda_2 = -4 \end{array} \rightsquigarrow \boxed{\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 9 \end{array}}$$

(b) $A-\lambda I = \begin{pmatrix} 3-\lambda & 0 & 0 \\ -4 & -1-\lambda & -8 \\ 0 & 0 & -3-\lambda \end{pmatrix}$

$\Rightarrow \det(A-\lambda I) = (3-\lambda)(1+\lambda)(3+\lambda) = 0$

$$\boxed{\begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = -1 \\ \lambda_3 = -3 \end{array}}$$

(c) $A-\lambda I = \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$\det(A-\lambda I) = (1-\lambda)^3 + (1-\lambda) = (1-\lambda)(\lambda^2-2\lambda+2)$

$$\boxed{\lambda_1 = 1}$$

(d) $A-\lambda I = \begin{pmatrix} 2-\lambda & 1 & 1 & 0 \\ 0 & 3-\lambda & 4 & 0 \\ 0 & 0 & 5-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}$

$\det(A-\lambda I) = (2-\lambda)^2(3-\lambda)(5-\lambda)$

$$\boxed{\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \\ \lambda_4 = 5 \end{array}}$$

(e) $A-\lambda I = \begin{pmatrix} 1-\lambda & 2 & 2 & 4 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & -1 & -\lambda & -2 \\ 0 & 1 & 0 & 2-\lambda \end{pmatrix}$

$\det(A-\lambda I) = (1-\lambda)\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & -\lambda & -2 \\ 1 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda)\lambda(2-\lambda)$

$$\boxed{\begin{array}{ll} \lambda_1 = 1 & \lambda_2 = 1 \\ \lambda_3 = 0 & \lambda_4 = 2 \end{array}}$$

$\boxed{4}$ (a) $\lambda = 1 \Rightarrow A - I = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$

$$\Rightarrow \begin{cases} 4x_1 + 4x_2 = 0 \\ 4x_1 + 4x_2 = 0 \end{cases} \qquad \begin{aligned} x_1 &= -t \\ x_2 &= t \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

$\lambda = 9 \qquad A - 9I = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$

$$\Rightarrow \begin{cases} -4x_1 + 4x_2 = 0 \\ 4x_1 - 4x_2 = 0 \end{cases} \qquad \begin{aligned} x_1 &= t \\ x_2 &= t \end{aligned}$$

$$\vec{v}_9 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

$\Rightarrow$ A IS DIAGONALIZABLE.

(b) $A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ -4 & -4 & -8 \\ 0 & 0 & -6 \end{pmatrix} \qquad \begin{aligned} x_3 &= 0 \\ x_2 &= t \\ x_1 &= -t \end{aligned}$

$$\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

$A + I = \begin{pmatrix} 4 & 0 & 0 \\ -4 & 0 & 8 \\ 0 & 0 & -2 \end{pmatrix} \qquad \begin{aligned} x_3 &= 0 \\ x_1 &= 0 \\ x_2 &= t \end{aligned}$

$$\vec{v}_{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

$A = \begin{pmatrix} 6 & 0 & 0 \\ -4 & 2 & -8 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= t \\ x_3 &= t/4 \end{aligned}$

$$\vec{v}_{-3} = \begin{bmatrix} 0 \\ 1 \\ 1/4 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

(c) $A - I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = t \\ x_1 = 0 \\ x_3 = 0 \end{array}$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad m_g = 1 = m_a$

BUT WE HAVE ONLY ONE EIGENVALUE, SO A IS NOT DIAGONALIZABLE.

(d)

$A - 2I = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_3 = 0 \\ x_2 = 0 \\ x_1 = t_1 \\ x_4 = t_2 \end{array}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow m_g = 2 = m_a$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$A - 3I = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \begin{array}{l} x_4 = 0 \\ x_3 = 0 \\ x_2 = t \\ x_1 = t \end{array}$

$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad m_g = 1 = m_a$

$A - 5I = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \qquad \begin{array}{l} x_4 = 0 \\ x_3 = t \\ x_2 = 2t \\ -3x_1 = -3t \rightarrow x_1 = t \end{array}$

$\vec{v}_5 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \qquad m_g = m_a = 1$

SO A IS DIAGONALIZABLE.

(e) $\begin{pmatrix} 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \Rightarrow$

$x_4 = t_1$
$x_2 = -t_1$
$-x_3 = 2t_1 - t_1 = t_1 \rightsquigarrow x_3 = -t_1$
$x_1 = t_2$

$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad m_g = 2 = m_a$

$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$

$x_2 = 0$
$x_4 = 0$
$x_3 = t$
$x_1 = -2t$

$\vec{v}_0 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad m_g = m_a = 1$

$\begin{pmatrix} -1 & 2 & 2 & 4 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

$x_2 = 0$
$x_4 = t$
$x_3 = -t$
$x_1 = -2t + 4t = 2t$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \qquad m_g = m_a = 1$

SO A IS DIAGONALIZABLE.

$\boxed{5}$ $A = \begin{pmatrix} 1 & 0 & h^2 \\ 0 & h & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & h^2 \\ 0 & h-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$

$\Rightarrow \det(A - \lambda I) = (1-\lambda)(h-\lambda)(1-\lambda) + (\lambda - h)h^2 =$

$= (\lambda - h)\left[h^2 - (1-\lambda)^2\right] = (\lambda - h)(h - 1 + \lambda)(h + 1 - \lambda)$

$$\boxed{\begin{aligned} \lambda &= h \\ \lambda &= 1-h \\ \lambda &= h+1 \end{aligned}}$$

$\Rightarrow$ A IS DIAGONALIZABLE BECAUSE THE EIGENVALUES ARE DISTINCT FOR EVERY h.