FUNCTIONS OF SEVERAL VARIABLES

Many familiar quantities are functions of two or more variables. For example the volume of a box:

$$V = \underset{\text{LENGTH}}{l}\,\underset{\text{WIDTH}}{w}\,\underset{\text{HEIGHT}}{h}$$

DEFINITION: Given a subset $D \subseteq \mathbb{R}^n$, an application that associates to each $x=(x_1, x_2, \dots x_n) \in D$ one and only one real number

$$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$f = f(x_1, x_2, \dots, x_n) = f(x)$$

is called REAL FUNCTION OF SEVERAL VARIABLES

As with functions of one variable, the DOMAIN of a function is the set of all points in $\mathbb{R}^n$ for which the equation that identifies the function is defined.

The subset of $\mathbb{R}$ such that its elements are image of some points in D is called RANGE

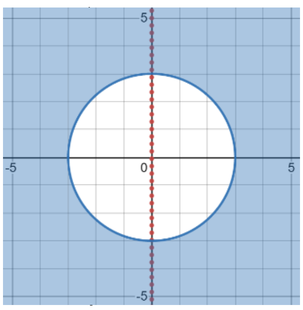
EXAMPLES:

i) $f(x,y) = x^2+y^2$ the domain is $D = \mathbb{R}^2$

ii) $f(x,y) = \ln(xy)$ the domain is the set of all points (x,y) for which $xy>0$, hence all the points of the first and third quadrants

iii) $f(x,y) = \dfrac{\sqrt{x^2+y^2-9}}{x}$ this function is defined for all points (x,y) such that

$$\begin{cases} x \neq 0 \\ x^2+y^2 \geq 9 \end{cases}$$



the domain is the set of all points lying on or outside the circle $x^2+y^2=9$ except those points on the y-axis.

iv) $$g(x,y,z)=\frac{x}{\sqrt{9-x^2-y^2-z^2}}$$

the function g is defined for all points (x,y,z) such that

$$x^2+y^2+z^2<9$$

Hence the domain is the set of all points (x,y,z) lying inside a sphere of radius 3 centered at the origin

Functions of several variables can be combined in the same way as functions of single variables. For instance, you can form the sum, difference, product and quotient of two functions of several variables

$$(f\pm g)(x)=f(x)\pm g(x)$$

$$(fg)(x)=f(x)\cdot g(x) \qquad \text{where } x=(x_1,\dots,x_n)\in D$$

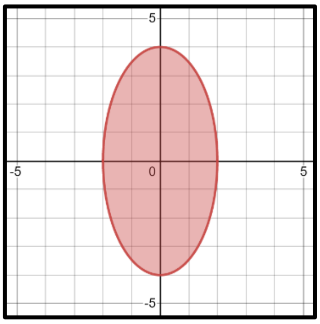
$$\frac{f}{g}(x)=\frac{f(x)}{g(x)}, \; g(x)\neq 0$$

You cannot form the composite of two functions of several variables. However you can form the composite function $(g\circ h)(x)$, where g is a function of a single variable and h is a function of several variables

$$(g\circ h)(x)=(g\circ h)(x_1,x_2,\dots,x_n)=g(h(x_1,x_2,\dots,x_n))$$

The domain of this composite function consists of all $x=(x_1,\dots,x_n)$ in the domain of h such that $h(x_1,\dots,x_n)$ is in the domain of g.

EXAMPLE:
$$f(x,y)=\sqrt{16-4x^2-y^2}$$
can be viewed as the composite of the function of two variables given by $h(x,y)=16-4x^2-y^2$ and the function of a single variable given by
$$g(u)=\sqrt{u}$$



The domain of this function is the set of all point lying on or inside the ellipse $4x^2+y^2=16$

GRAPH OF A FUNCTION OF TWO VARIABLES

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The GRAPH of a function of two variables is the set of all points (x,y,z) for which $z=f(x,y)$ and (x,y) belongs to the domain of f. This graph can be interpreted as a SURFACE IN SPACE

EXAMPLE: What is the range of

$$z=f(x,y)=\sqrt{16-4x^2-y^2}$$

Describe the graph.

Solution the domain D_f is the set of all points (x,y) such that

$$16-4x^2-y^2 \geq 0$$

rearranging terms we get

$$4x^2+y^2 \leq 16 \iff \frac{x^2}{4}+\frac{y^2}{16} \leq 1$$ ELLIPSE IN THE x-y PLANE

Hence D_f is the set of all points lying on or inside the ellipse of equation $\frac{x^2}{4}+\frac{y^2}{16} \leq 1$

The range of f is all values $z=f(x,y)$ such that $0 \leq z \leq \sqrt{16}$ or $0 \leq z \leq 4$.

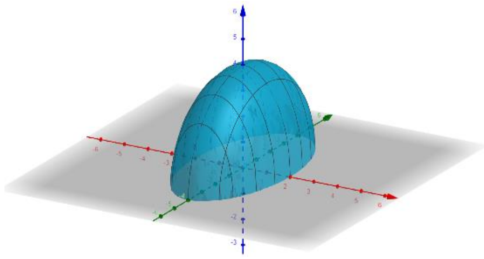
A point (x,y,z) is on the graph of f if and only if

$$z=\sqrt{16-4x^2-y^2}$$

$$z^2 = 16 - 4x^2 - y^2 \iff 4x^2 + y^2 + z^2 = 16 \iff$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \quad 0 \le z \le 4$$

hence the graph of f is the upper part of an ELLIPSOID



EXAMPLE: What is the range of

$$z = f(x,y) = x^2 + y^2$$

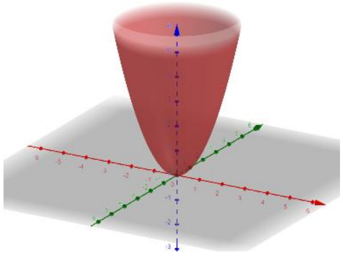
Describe the graph

Solution: the domain of f is all of $\mathbb{R}^2$.
The range of f is made by all values $z \geq 0$ as $x^2+y^2 \geq 0$ and is equal to 0 iff $(x,y)=(0,0)$
A point (x,y,z) is on the graph of f iff

$$z = x^2 + y^2$$

this equation represents an ELLIPTICAL PARABOLOID
(The generic equation of an ELLIPTICAL PARABOLOID is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{z}{c}, \quad a,b,c \in \mathbb{R} \text{ and } a \neq 0, b \neq 0, c \neq 0)$$



If $a=b$ an elliptic paraboloid is a circular paraboloid or paraboloid of revolution, and it is obtained by revolving a parabola around its axis

HOMEWORK: Find the Domain and Range of the following functions

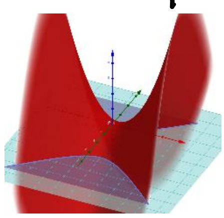
i) $z = f(x,y) = \sqrt{4 - x^2 - y^2}$

ii) $z = f(x,y) = \sqrt{4 - x^2 - 4y^2}$

iii) $z = -x^2 - y^2$

iv) $z = x^2 - 4y^2$ (HINT: This is a HYPERBOLIC PARABOLOID whose generic equations are

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$



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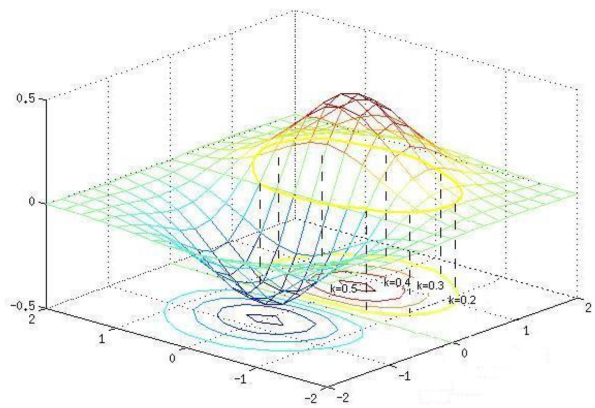
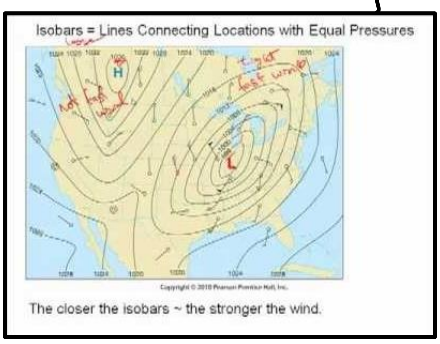
LEVEL CURVES

A second way to visualize a function of two variables is to use a SCALAR FIELD in which the function $z=f(x,y)$ is fixed to a constant value $z=c$, $c\in\mathbb{R}$.
Hence a LEVEL CURVE $f(x,y)=c$ is the set of all points in the domain of f at wich f takes on a given value k. In other words, it shows where the graph of f has height .
You can see from the picture below the relation between level curves and horizontal traces.
The level curves $f(x,y)=c$ are just traces of the graph of f in the horizontal plane $z=c$ projected down to the xy-plane

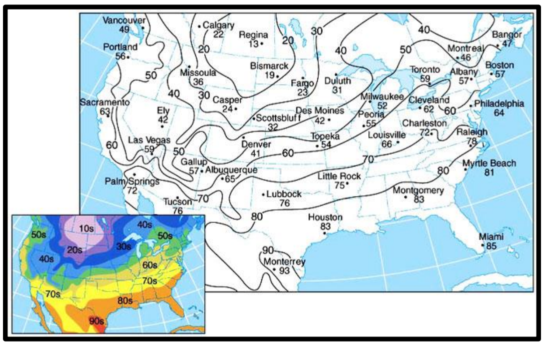


In weather maps level curves are essential tools

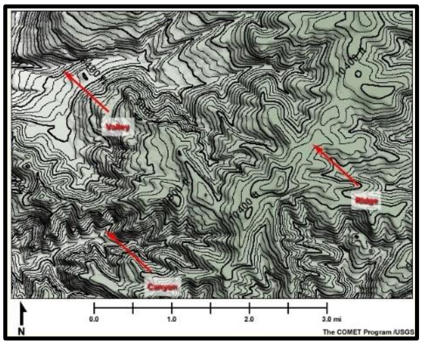


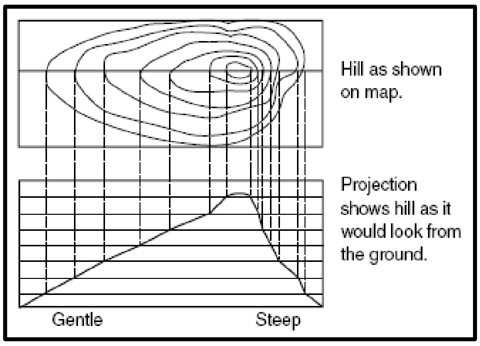
For example the figure on the left hand side shows level curves of equal pressure called ISOBARS measured in millibars.
Isobars can be used to identify regions of high pressure and low pressure. High pressure regions are usually associated with dry weather because, as the air sinks it warms and the moisture evaporates. Low pressure regions usually bring precipitation because when the air rises it cools and the water vapor condenses

In weather maps for which the level curves represent points of equal temperature, the level curves are called ISOTHERMS
An example in the figure below



Level curves are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called TOPOGRAPHIC MAP, and the lines on it are used to connect points with the same elevation. They are used to define the shape and the steepness in elevation of various landforms. Below two figures showing a typical topographic map and the way it is obtained





Level curve maps depict the variation of z with respect to x and y by the spacing between level curves. Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z. Furthermore, to produce a good three-dimensional illusion in these maps, it is important, when drawing the curves

$$f(x,y)=c$$

to choose c values that are evenly spaced

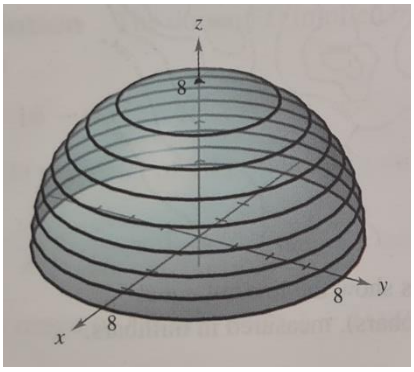
EXAMPLE 1: Sketch level curves of the hemisphere

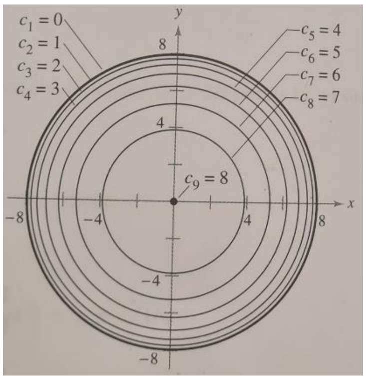
$$z=f(x,y)=\sqrt{64-x^2-y^2}$$

Solution: For each value of $c\in\mathbb{R}$, the equation $f(x,y)=c$ is a circle. For example, when $c=0$, the level curve is

$$x^2+y^2=64 \longrightarrow \text{CIRCLE OF RADIUS } 8$$

Below the hemisphere /grapf of f and 9 level curves





Example 2: Sketch level curves of the hyperbolic paraboloid

$$z = f(x,y) = y^2 - x^2$$

Solution: For each value of c, let $f(x,y) = c$, and sketch the resulting level curve in the xy-plane. For this function each of the level curves ($c \neq 0$) are hyperbolas whose asymptotes are the lines $y = \pm x$

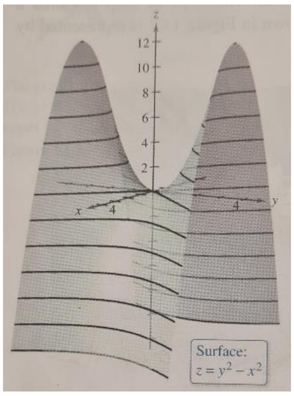
When $c < 0$, the transverse axis is horizontal. For instance the level curve for $c = -4$ is

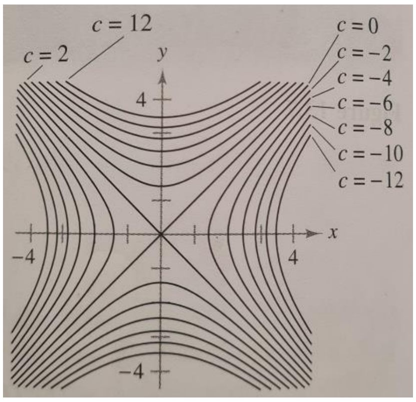
$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$$

When $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is

$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1$$

When $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes. Both graph of f and level curves (at increments of 2) below

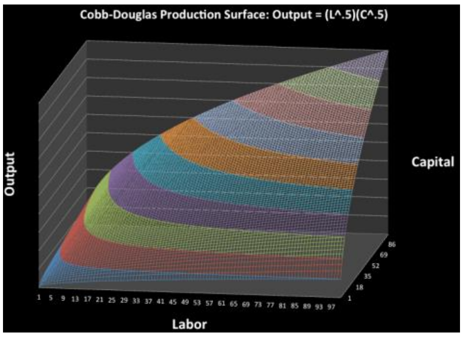


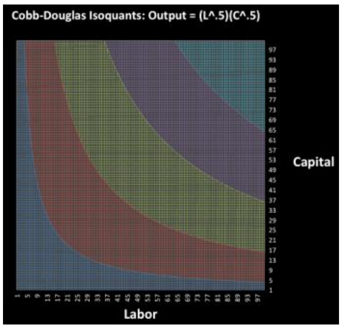


One example of a function of two variables used in economics is the COBB-DOUGLAS PRODUCTION FUNCTION. This function is used to model the number of units produced by varying amounts of labor and capital. If x measures the units of labor and y measures the unit of capital, then the number of units produced is:

$$z = f(x,y) = C x^{\alpha} y^{1-\alpha}$$

where C and α are constants with $0 < \alpha < 1$.
Below graph and level curves for the case $C = 1$, $\alpha = 0.5$

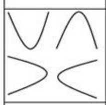
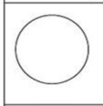
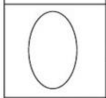
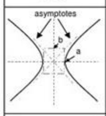
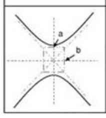




HOMEWORK: Describe the level curves of the following function. Sketch a level curve map of the surfaces using level curves for the given c-values

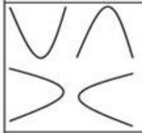
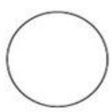
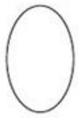
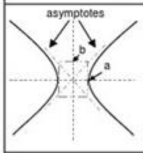
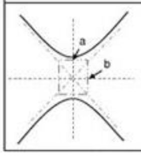
i) $z = x + y$, $c = -1, 0, 2, 4$ (HINT: LINES)
ii) $z = 6 - 2x - 3y$, $c = 0, 2, 4, 6, 8, 10$ (HINT: LINES)
iii) $z = x^2 + 4y^2$, $c = 0, 1, 2, 3, 4$ (HINT: ELLIPSE)
iv) $z = \sqrt{9 - x^2 - y^2}$, $c = 0, 1, 2, 3$ (HINT: CIRCLES)
v) $z = xy$, $c = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6$ (HINT: also $xy = k$ is the equation of a hyperbola)

General Equations of Conics:

Shape	Name	General Equation	Description	What does the equation tell us?
	**Parabola**	$y = ax^2 + bx + c$ or $y^2 = ax + b$	Only one squared term (x **or** y).	y-intercept is at c. x-intercept at $-\frac{b}{a}$
	**Circle**	$x^2 + y^2 = r^2$	x and y terms both squared, and added. Coefficients of x^2 and y^2 are equal and there is no xy term. (special ellipse)	To find the radius make the coefficients of x^2 and y^2 equal 1. Centre (0,0) radius= r
	**Ellipse**	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	x and y terms both squared, and added	Intercepts x-axis at $\pm a$ and y-axis at $\pm b$ (notice that when $a = b$ it's a circle)
	**Hyperbola**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	x and y terms both squared, then subtracted	The diagonals of the rectangle (formed with length from –a to a and height from –b to b) are asymptotes for the hyperbola.
	**Hyperbola**	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	x and y terms both squared, then subtracted	The hyperbola is still formed using the rectangle. y^2 first turns it the other way.

← SUMMARY OF EQUATIONS OF FUNDAMENTAL CONICS

General Equations of Conics:

Shape	Name	General Equation	Description	What does the equation tell us?
	**Parabola**	$y = ax^2 + bx + c$ **or** $y^2 = ax + b$	Only one squared term (x **or** y).	y-intercept is at c. x-intercept at $-\frac{b}{a}$
	**Circle**	$x^2 + y^2 = r^2$	x and y terms both squared, and added. Coefficients of x^2 and y^2 are equal and there is no xy term. (special ellipse)	To find the radius make the coefficients of x^2 and y^2 equal 1. Centre (0,0) radius= r
	**Ellipse**	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	x and y terms both squared, and added	Intercepts x-axis at $\pm a$ and y-axis at $\pm b$ (notice that when $a = b$ it's a circle)
	**Hyperbola**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	x and y terms both squared, then subtracted	The diagonals of the rectangle (formed with length from –a to a and height from –b to b) are asymptotes for the hyperbola.
	**Hyperbola**	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	x and y terms both squared, then subtracted	The hyperbola is still formed using the rectangle. y^2 first turns it the other way.