EIGENVALUES AND EIGENVECTORS

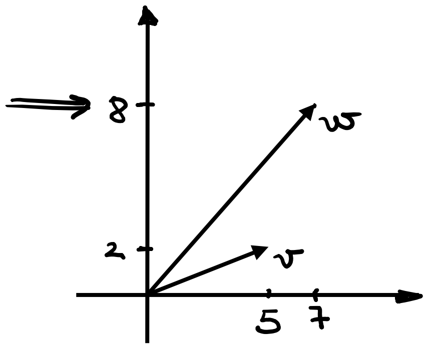
Given a square matrix $A \in M_n$, A identifies a transformation

$$A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

that transforms a vector $v \in \mathbb{R}$ in a vector $w = A \cdot v \in \mathbb{R}^n$. The consequences of this transformation may be changements in direction or/and in length and verse of the vector. For example

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \quad \text{and } v = (5, 4) \Longrightarrow$$

$$w = Av = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$



v and w are two independent vectors, hence the effect of A on v was a change of direction and a change of length

There are although vectors that, given a matrix $A \in M_n$ are resistant to direction change, and after undergoing to A-transformation, they exhibit at the most a change of lenghth and/or verse. These vectors are called EIGENVALUES

DEFINITION: Let $A \in M_n$ be a square matrix. A vector $v \in \mathbb{R}^n$ is an EIGENVECTOR of A if

i) $v \neq 0$ (different from the null vector)

ii) $Av = \lambda v$ for some scalar $\lambda \in \mathbb{R}$

and λ is called EIGENVALUE

REMARKS: 1) If v is the null vector, hence the epuality $Av = \lambda v$ is trivially verified for all $A \in M_n$ hence doesn't make to consider the case $v = 0$

2) As eigenvectors are vectors immune to direction changement, it makes sense to calculate eigenvectors only for SQUARE matrices $A \in M_n$

HOW TO FIND EIGENVALUES?

$v \in \mathbb{R}^n$ is an EIGENVECTOR if and only if

$$Av = \lambda v \iff Av = \lambda I_n v \iff Av - \lambda I_n v = 0$$
$$\iff (A - \lambda I_n) v = 0$$

the last equations represents an homogeneous system of n equations in n unknowns, that has non trivial solutions if and only if

$$\det(A - \lambda I_n) = 0$$

Hence 1st STEP:

Look for $\lambda \in \mathbb{R}$ values such that $\det(A - \lambda I_n) = 0$ and these values are called EIGENVALUES

DEFINITION: The polynomial $P(\lambda) = \det(A - \lambda I_n)$ is called the CHARACTERISTIC POLYNOMIAL. The equation $\det(A - \lambda I_n) = 0$ is called CHARACTERISTIC EQUATION

EXAMPLE: Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$$

The characteristic polynomial is given by:

$$\det(A - \lambda I_2) = \det\left[\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{pmatrix}$$
$$= -\lambda(5-\lambda) + 6 = \lambda^2 - 5\lambda + 6$$
$$= (\lambda - 2)(\lambda - 3)$$

And solving the characteristic equation we get

$$\det(A - \lambda I_2) = (\lambda - 2)(\lambda - 3) = 0 \quad \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 3 \end{cases}$$

Let us compute eigenvectors for each single eigenvalue

If $\lambda_1 = 2 \Longrightarrow$ we substitute it in the system

$(A - \lambda I_2)v = 0$, setting $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow$ the system to solve is $\begin{pmatrix} -2 & 1 \\ -6 & 5-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow$

$\begin{pmatrix} -2 & \boxed{1} \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ whose coefficient matrix has determinant equal to 0

hence the rank of $A - 2I_2$ is 1 and the system admits ∞^{2-1} solutions. The order 1 highlighted minor determines the equivalent system with which I will solve the system and find the eigenvectors

$$\begin{cases} x_1 = t \\ x_2 = 2t \end{cases}, t \in \mathbb{R}$$

hence all possible eigenvectors correspondent to $\lambda_1 = 2$ are of the form

$$v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$$

Hence the set of all the Eigenvectors related to $\lambda_1 = 2$ form an EIGENSPACE that we denote in the following way

$$E_2 = \left\{ v \in \mathbb{R}^2;\ v = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

that is a line that passes through the origin, hence it is a LINEAR SUBSPACE of dimension 1 and a base of it is $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$;

Same reasoning for $\lambda_2 = 3$, hence

$$\begin{pmatrix} -3 & 1 \\ -6 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} -3 & \boxed{1} \\ -6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x_1 + x_2 = 0 \\ -6x_1 + 2x_2 = 0 \end{cases}$$

the $rk(A - 3I_2) = 1$, hence the system admits $\infty^{2-1} = \infty^1$ solutions

and by solving according to the highlighted minor we get

$$\begin{cases} x_1 = t \in \mathbb{R} \\ x_2 = 3t \end{cases} \Longleftrightarrow$$ all possible eigenvectors correspondent to $\lambda_2 = 3$ are of the form

$$v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$

and the Eigenspace correspondent to $\lambda_2 = 3$, that is the collection of all possible eigenvectors correspondent to $\lambda_2 = 3$ is a LINE passing through the origin

$$E_3 = \left\{ v \in \mathbb{R}^2 ;\ v(t) = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}, t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

that is a LINEAR SUBSPACE of $\mathbb{R}^2$ of dimension 1 and Base given by the generator $\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

ALGEBRAIC AND GEOMETRIC MULTIPLICITY

DEFINITION The ALGEBRAIC MULTIPLICITY of an eigenvalue $\lambda = e$ is its multiplicity as a root of the characteristic equation (namely it is the power of the factor $(\lambda - e)$ in the factorization of the characteristic polynomial

$$P(\lambda) = \det(A - \lambda I_2) = (\lambda - e)^{m} \cdot \underbrace{Q(\lambda)}_{\to\ \text{QUOTIENT}} \quad \to m_A(\lambda)$$

and it will be denoted with $m_A(\lambda)$

EXAMPLE: Find eigenvalues of the following matrix, and determine their algebraic multiplicity

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{pmatrix}$$

the matrix is LOWER TRIANGULAR Hence all the eigenvalues correspond to the entries on the diagonal

$$\det(A - \lambda I_4) = \det \begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 5 & 3-\lambda & 0 & 0 \\ 9 & 1 & 3-\lambda & 0 \\ 1 & 2 & 5 & -1-\lambda \end{pmatrix} =$$

$$= (2-\lambda)(3-\lambda)^2(-1-\lambda) = (\lambda-2)(\lambda+1)(\lambda-3)$$

Hence $\lambda_1 = 2$ and $m_A(2) = 1$, $\lambda_2 = -1$ and $m_A(-1) = 1$ and $\lambda_3 = 3$ and $m_A(3) = 2$

Before going on let us formalize what we have said on Eigenspaces

DEFINITION: the set E_λ that consists of the zero vector and of all the eigenvectors corresponding to the eigenvalue λ is an LINEAR SUBSPACE and is called EIGENSPACE.
The dimension of E_λ is called GEOMETRIC MULTIPLICITY of λ and will be denoted $m_G(\lambda)$

Let us evaluate eigenvectors of the previous matrix

if $\lambda_1=2 \longrightarrow$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 3-2 & 0 & 0 \\ 9 & 1 & 3-2 & 0 \\ 1 & 2 & 5 & -1-2 \end{pmatrix}}_{A-2I_4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 9 & 1 & 1 & 0 \\ 1 & 2 & 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

as $\det(A-2I_4)=0$ its rank $\neq 4$, but the highlighted submatrix of order 3 has $\det \neq 0$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & -3 \end{vmatrix} = -3 \neq 0 \quad \text{hence } rk(A-2I_4)=3$$

and the system admits $\infty^{4-3}=\infty^1$ solutions. Also

$$m_G(2)=4-3=1 \quad \left(\text{In general } m_G(\lambda) = n - rk(A-\lambda I_n)\right)$$

and the eigenvectors corresponding to $\lambda_1=2$ are given by the resolution of the following system

$$\begin{cases} x_1 = t \in \mathbb{R} \\ x_2 = -5t \\ x_2 + x_3 = -9t \\ 2x_2 + 5x_3 - 3x_4 = -t \end{cases} \longrightarrow \begin{cases} x_1 = t \\ x_2 = -5t \\ x_3 = -9t + 5t = -4t \\ -3x_4 = -t - 2(-5t) - 5(-4t) \\ \quad = -t + 10t + 20t \end{cases}$$

$$\begin{cases} x_1 = t \\ x_2 = -5t \\ x_3 = -4t \\ x_4 = -\frac{29}{3}t \end{cases}, t \in \mathbb{R} \iff v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = t \begin{pmatrix} 1 \\ -5 \\ -4 \\ -\frac{29}{3} \end{pmatrix}$$

hence

$$E_2 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -5 \\ -4 \\ -\frac{29}{3} \end{pmatrix} \right\} = \left\{ v \in \mathbb{R}^3 ;\ v(t) = t \cdot \begin{pmatrix} 1 \\ -5 \\ -4 \\ -\frac{29}{3} \end{pmatrix}, t \in \mathbb{R} \right\}.$$

if $\lambda_2 = -1$

$$\underbrace{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 5 & 4 & 0 & 0 \\ 9 & 1 & 4 & 0 \\ 1 & 2 & 5 & 0 \end{pmatrix}}_{A-(-1)I_4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\operatorname{rk}(A+I_4) = 3$ as the highlighted submatrix of order 3 has $\det \neq 0 \Longrightarrow$

$$m_G(-1) = 4 - \operatorname{rk}(A+I_4) = 4 - 3 = 1$$

and the Eigenvectors are given by solving

$$\begin{cases} 3x_1 = 0 \\ 5x_1 + 4x_2 = 0 \\ 9x_1 + x_2 + 4x_3 = 0 \\ x_4 = t \in \mathbb{R} \end{cases} \iff \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = t \in \mathbb{R} \end{cases} \Longrightarrow$$

$$v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R} \quad \text{and}$$

$$E_{(-1)} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ v \in \mathbb{R}^4 ;\ v(t) = t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

if $\lambda_3 = 3 \longrightarrow$

$$\underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 9 & 1 & 0 & 0 \\ 1 & 2 & 5 & -4 \end{pmatrix}}_{A-3I_4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The highlighted submatrix of order 3 has determinant $\neq 0 \Rightarrow rk(A-3I_4) = 3 \Rightarrow$ the system admits $\infty^{4-3} = \infty^1$ solutions and

$$m_G(3) = 4 - rk(A-3I_4) = 4-3 = 1$$

and the eigenvectors are given by solving

$$\begin{cases} 5x_1 = 0 \\ 9x_1 + x_2 = 0 \\ x_1 + 2x_2 + 5x_3 = 4t \\ x_4 = t \in \mathbb{R} \end{cases} \quad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = \frac{4}{5}t \\ x_4 = t \end{cases} \Rightarrow v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 4/5 \\ 1 \end{pmatrix} \quad t \in \mathbb{R}$$

$$\Rightarrow E_3 = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 4/5 \\ 1 \end{pmatrix} \right\} = \left\{ v \in \mathbb{R}^4;\ v(t) = t \begin{pmatrix} 0 \\ 0 \\ 4/5 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

SUMMARY

λ	$\lambda_1 = 2$	$\lambda_2 = -1$	$\lambda_3 = 3$
E_λ	$\text{span}\left\{\begin{pmatrix} 1 \\ -5 \\ -4 \\ -\frac{29}{3} \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ \frac{4}{5} \\ 1 \end{pmatrix}\right\}$
$m_A(\lambda)$	1	1	2
$m_G(\lambda)$	1	1	1

Example: Find eigenvalues and their algebraic and geometric multiplicites, and find eigenvectors of

$$A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution: The characteristic equation is

$$\det(A-\lambda I_3)=\det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}=-\lambda^3+3\lambda+2$$

$$=-(\lambda-2)(\lambda+1)^2$$

Hence $\lambda_1=2$ and $m_A(2)=1$, and $\lambda_2=-1$ and $m_A(-1)=2$

Solving the equation $(A-\lambda_i I_2)v=0$ for $i=1,2$ we find

$\lambda_1=2$ $\quad \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}=4-1=3\neq 0$

$$\underbrace{\begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -2 \end{pmatrix}}_{A-2I_3}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$rk(A-2I_3)=2$

$\to m_G(2)=3-rk(A-2I_3)$

$=3-2=1$

and the system admits $\infty^{3-2}=\infty^1$ solutions given by

$$\begin{cases} -2x_1+x_2=-t \\ x_1-2x_2=-t \\ x_3=t\in\mathbb{R} \end{cases}$$

$$\begin{cases} x_1=\dfrac{\begin{vmatrix} -t & 1 \\ -t & -2 \end{vmatrix}}{3}=\dfrac{2t+t}{3}=\dfrac{3t}{3}=t \\ x_2=\dfrac{\begin{vmatrix} -2 & -t \\ 1 & -t \end{vmatrix}}{3}=\dfrac{2t+t}{3}=\dfrac{3t}{3}=t \\ x_3=t\in\mathbb{R} \end{cases}$$

$$v(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}=t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ t\in\mathbb{R}$$ and

$$E_2=\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

if $\lambda=-1$

$$\underbrace{\begin{pmatrix} \boxed{1} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A-(-1)I_3}\begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

clearly $rk(A+I_3)=1$

$\Rightarrow$ the system admits $\infty^{3-1} = \infty^2$ solutions and $m_G(-1) = 2$. The eigenvectors are given by

$$\begin{cases} x_1 = -s-t \\ x_2 = s \in \mathbb{R} \\ x_3 = t \in \mathbb{R} \end{cases} \longrightarrow v(t) = \begin{pmatrix} x_1(s,t) \\ x_2(s,t) \\ x_3(s,t) \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow E_{-1} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a plane passing through the origin

SUMMARY

λ	2	-1
E_λ	$\text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$	$\text{span} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
$m_A(\lambda)$	1	2
$m_G(\lambda)$	1	2

Notice from the previous examples that

$$m_G(\lambda) \leq m_A(\lambda) \quad \forall \lambda \text{ eigenvalues}$$

This matter can be generalized

MULTIPLICITY THEOREM

For any eigenvalue λ of a given matrix $A \in M_n$

$$m_G(\lambda) \leq m_A(\lambda)$$

HOMEWORK: Compute eigenvalues, eigenvectors, algebraic multiplicity and geometric multiplicity of i)

$$A=\begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

ii)

$$B=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

iii)

$$C=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

REMARKS:

1) As a consequence of the proposition we have that if $m_A(\lambda)=1 \Rightarrow$ necessarily $m_G(\lambda)=1$

2) Eigenvectors corresponding to different eigenvalues are always linearly independent

EXAMPLE: Compute all eigenvalues, eigenvectors algebraic and geometric multiplicities of

$$A=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$\det(A-\lambda I_3)=\det\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 1 & -1-\lambda \end{pmatrix}=(3-\lambda)\left[(\lambda-1)(\lambda+1)-3\right]$$

$$=(3-\lambda)(\lambda^2-1-3)=(3-\lambda)(\lambda^2-4)=(3-\lambda)(\lambda+2)(\lambda-2)$$

that cancels for $\lambda_1=3$, $\lambda_2=-2$ and $\lambda_3=2$

As $m_A(\lambda_i)=1$, $\forall i=1,2,3 \Rightarrow m_G(\lambda_i)=1$, $\forall i=1,2,3$

if $\lambda_1 = 3 \Rightarrow$

$$(A-3I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix} = 8-3 = 5$$

$\Rightarrow rk(A-3I_3) = 2 \Rightarrow$

the system admits $\infty^{3-2} = \infty^1$ solutions given by the resolution of the following system

$$\begin{cases} x_1 = t \\ -2x_2 + 3x_3 = 0 \\ x_2 - 4x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} = \left\{ v \in \mathbb{R}^3 ;\ v(t) = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}$$

if $\lambda_2 = -2 \Rightarrow$

$$(A-(-2)I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3+2 & 0 & 0 \\ 0 & 1+2 & 3 \\ 0 & 1 & -1+2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow rk(A+2I_3) = 2 \Rightarrow$$

the system admits $\infty^{3-2} = \infty^1$ solutions, given by

$$\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} = 15$$

$$\begin{cases} 5x_1 = 0 \\ 3x_2 = -3t \\ x_3 = t \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = -t \\ x_3 = t \end{cases} \Rightarrow v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$E_{-2} = \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\} = \left\{ v \in \mathbb{R}^3 ;\ v(t) = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

if $\lambda_3 = 2 \Rightarrow$

$$\underbrace{\begin{pmatrix} 3-2 & 0 & 0 \\ 0 & 1-2 & 3 \\ 0 & 1 & -1-2 \end{pmatrix}}_{A-2I_3}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$$

$\Rightarrow rk(A-2I_3) = 2 \Rightarrow$ the system admits $\infty^{3-2} = \infty^1$

solutions given by

$$\begin{cases} x_1=0 \\ -x_2=-3t \\ x_3=t \end{cases} \Leftrightarrow \begin{cases} x_1=0 \\ x_2=3t \\ x_3=t \end{cases} \Rightarrow v(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}=t\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, t\in\mathbb{R}$$

$$\Rightarrow E_2=\operatorname{span}\left\{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}\right\}=\left\{v\in\mathbb{R}^3 ; v(t)=t\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, t\in\mathbb{R}\right\}$$

REMARK: The previous example shows us that if $A\in M_n$ admits n different eigenvalues λ_i, $i=1,\dots,n$

$$m_A(\lambda_i)=m_G(\lambda_i)=1$$

DIAGONALIZATION

The DIAGONALIZATION of a square matrix is the process of transforming the matrix into a diagonal matrix that shares properties with the initial matrix, but it is simpler to handle.

Before going on let us give a definition and a result

DEFINITION: A matrix $B\in M_n$ is called SIMILAR to a matrix $A\in M_n$ if it exists an invertible matrix $P\in M_n$ such that $B=PAP^{-1}$

THEOREM: If $A,B\in M_n$ are SIMILAR then they have the same characteristic polynomial and hence the same eigenvalues

proof: if $B=PAP^{-1}$ then

$$B-\lambda I_n= PAP^{-1}-\lambda PP^{-1}=P(A-\lambda I_n)P^{-1} \quad \text{hence}$$

$$\det(B-\lambda I_n)=\det\left[P(A-\lambda I_n)P^{-1}\right]=$$

$$=\det(P)\det(A-\lambda I_n)\det P^{-1} = \det(A-\lambda I_n)$$

If a matrix is diagonal it is trivial to compute D^k, its determinant ($\leftarrow$ product of the entries in the diagonal), etc.

For example, if:

$$D=\begin{pmatrix}3&0&0\\0&1&0\\0&0&2\end{pmatrix} \Rightarrow$$

$$D^2=D\cdot D=\begin{pmatrix}3&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}3&0&0\\0&1&0\\0&0&2\end{pmatrix}=\begin{pmatrix}9&0&0\\0&1&0\\0&0&4\end{pmatrix}$$

and in general, if

$$D=\begin{pmatrix}d_{11}&0&\cdots&0\\0&d_{22}&&\\\vdots&0&\ddots&0\\0&0&&d_{nn}\end{pmatrix}\Rightarrow D^k=\begin{pmatrix}d_{11}^k&0&&0\\0&d_{22}^k&&\vdots\\\vdots&&\ddots&0\\0&0&0&d_{nn}^k\end{pmatrix}$$

But if A is not DIAGONAL there is no special rule to compute A^k. If A is SIMILAR to a diagonal matrix A^k becomes quite easy to achieve, indeed:

"Let $A=PDP^{-1} \Rightarrow A^k=PD^kP^{-1}$"

proof: Let us prove this result by INDUCTION

- if $k=2$ $A^2=(PDP^{-1})(PDP^{-1})$
  $=PD(P^{-1}P)DP^{-1}=PD^2P^{-1}$

- if $A^{n-1}=PD^{n-1}P^{-1}$ holds let us prove it for $k=n$ (THIS IS THE INDUCTION HYPOTHESIS)

$$A^n=A\cdot A^{n-1}=PDP^{-1}PD^{n-1}P^{-1}$$
$$=PDD^{n-1}P^{-1}=PD^nP^{-1}$$

DEFINITION: A square matrix $A\in M_n$ is said to be DIAGONALIZABLE if A is similar to a diagonal matrix, i.e. $\exists P\in M_n$, invertible matrix, such that $A=PAP^{-1}$, where D is diagonal

But when is a matrix $A\in M_n$ diagonalizable and how do we compute D and P? The following theorem answers to this question

THEOREM: A square matrix $A \in M_n$ is diagonalizable if and only if A has n independent eigenvectors. Infact $A = PAP^{-1}$, where D is diagonal, if and only if the columns of P are n linearly independent eigenvectors of A. In this case the diagonal entries of D are eigenvalues of A, corresponding to eigenvectors columns of P

THEOREM: If $v_1, v_2, \dots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_r$ for a square matrix $A \in M_n$, then $v_1, v_2, \dots, v_r$ are linearly independent

COROLLARY: If a square matrix $A \in M_n$ has n distinct eigenvalues, then A is diagonalizable

DIAGONALIZATION THEOREM A square matrix $A \in M_n$ is diagonalizable if and only if

$$m_A(\lambda) = m_G(\lambda)$$

for all eigenvalues λ of A.

Putting together all of these results we get that

i) A is diagonalizable if and only if $\forall \lambda$ eigenvalues of A $m_A(\lambda) = m_G(\lambda)$ and the sum of the algebraic multiplicites of all eigenvalues is equal to n

ii) if A is diagonalizable we can compute n independent eigenvectors $v_1, v_2, \dots, v_n$ that form a base for $\mathbb{R}^n$, and moreover D will be given by displaying $v_1, \dots, v_n$ as columns of a square matrix

$$P = (v_1 | v_2 | \dots | v_n)$$

and D will be the diagonal matrix where $a_{11} = \lambda_1$ is the eigenvalue that corresponds to v_1, $a_{22} = \lambda_2$ is the eigenvalue that corresponds to v_2 and so on

iii) as $v_1, v_2, \dots, v_n$ are independent eigenvectors, hence

$$P = (v_1 | v_2 | \dots | v_n)$$

is invertible

iv) The eigenvalue λ appears in the diagonal matrix so many times as the number that represents the algebraic multiplicity

v) Doesn't matter the order in which you display the eigenvectors in P. Does matter that, once you have chosen the disposition of the eigenvectors in P, the eigenvalues in the diagonal entries of D have to respect the disposition chosen for P.

EXAMPLES:

1. Going back to example

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \qquad \text{we had}$$

λ	$\lambda_1 = 3$	$\lambda_2 = -2$	$\lambda_3 = 2$
E_λ	$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$	$\text{span}\left\{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}\right\}$
$m_A(\lambda)$	1	1	1
$m_G(\lambda)$	1	1	1

hence as $m_A(\lambda_i) = m_G(\lambda_i)$ for all $\lambda_i, i = 1,2,3$ and $m_A(\lambda_1) + m_A(\lambda_2) + m_A(\lambda_3) = 3 \Rightarrow A$ is diagonalizable and a possible choice for P and D is

$$P = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Generator of E_3 — Generator of E_{-2} — Generator of E_2

To check if the similarity is respected verify that as

$$A = PDP^{-1} \Longleftrightarrow \boxed{AP = PD}$$

$$AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -2 & 2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -2 & 2 \end{pmatrix}$$

verified $AP = PD$

EXAMPLE:

in the Homework example

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

you will find that there are two eigenvalues

$\lambda_1 = 2$ such that $m_A(2) = 1 \rightarrow m_G(2) = 1$

$\lambda_2 = -1$ such that $m_A(-1) = 2$ but $m_G(-1) = 1$

hence A is not diagonalizable

EXAMPLE: Diagonalize, if possible

$$A = \begin{pmatrix} 2 & -6 & -6 \\ 0 & 2 & 0 \\ 0 & -3 & -1 \end{pmatrix}$$

Let us compute eigenvalues:

$$\det(A-\lambda I_3) = \det\begin{pmatrix} 2-\lambda & -6 & -6 \\ 0 & 2-\lambda & 0 \\ 0 & -3 & -1-\lambda \end{pmatrix}$$

$$= (2-\lambda)\cdot(2-\lambda)(-1-\lambda) = -(\lambda+1)(2-\lambda)^2$$

Hence by solving

$$\det(A-\lambda I_3) = -(\lambda+1)(2-\lambda)^2 = 0 \quad \text{we get}$$

$$\lambda_1 = 2 \quad \text{with } m_A(2)=2$$

$$\lambda_2 = -1 \quad \text{with } m_A(-1) = 1$$

Let us compute eigenvectors:

if $\lambda_1 = 2 \Longrightarrow A-2I_3 = \begin{pmatrix} 0 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & -3 & \boxed{-3} \end{pmatrix}$

it is quite clear that $rk(A-2I_3) = 1$ hence the system admits $\infty^{3-1} = \infty^2$ solutions and

$m_G(2) = 2 = m_A(2)$

By considering the order 1 highlighted minor we get as solutions

$$\begin{cases} x_1 = s \in \mathbb{R} \\ x_2 = t \in \mathbb{R} \\ -3x_3 = 3t \end{cases} \qquad \begin{cases} x_1 = s \in \mathbb{R} \\ x_2 = t \in \mathbb{R} \\ x_3 = -t \end{cases}$$

hence $v(s,t) = \begin{pmatrix} x_1(s,t) \\ x_2(s,t) \\ x_3(s,t) \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

and $E_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$ hence

two independent eigenvectors to display here

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

if $\lambda_2 = -1 \Rightarrow A-(-1)I_3 = \begin{pmatrix} 3 & -6 & -6 \\ 0 & 3 & 0 \\ 0 & -3 & 0 \end{pmatrix}$

hence $rk(A+I_3) = 2$ and the system admits ∞^{3-2} solutions given by

$$\begin{cases} 3x_1 - 6x_2 = 6t \\ 3x_2 = 0 \\ x_3 = t \in \mathbb{R} \end{cases} \qquad \begin{cases} 3x_1 = 6t \\ x_2 = 0 \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 2t \\ x_2 = 0 \\ x_3 = t \end{cases}$$

$$v(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$E_{-1} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ v \in \mathbb{R}^3 ;\ v(t) = t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

hence $v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ could be the third independent vector to display in $P \Rightarrow$:

$$P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

2 is the eigenvalue corresponding to v_1 and to v_2

-1 is the eigenvalue corresponding to v_3

the disposition of v_i in P has been respected

$\lambda_1 = 2$ appears 2 times as $m_A(2)$

$\lambda_2 = -1$ appears 1 time as $m_A(-1)$

CHECK:

$$AP = \begin{pmatrix} 2 & -6 & -6 \\ 0 & 2 & 0 \\ 0 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & -2 & -1 \end{pmatrix} \Rightarrow AP = PD \text{ verified}$$

HOMEWORK:

1. Diagonalize if possible the following matrices

i) $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & 8 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

ii) $A=\begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$

iii) $A=\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

2. Discuss the diagonalizability of the following matrices, as k changes

i) $A=\begin{pmatrix} 0 & k \\ 2 & k-2 \end{pmatrix}$

ii) $A=\begin{pmatrix} 1 & 0 & 0 \\ k-1 & -4 & -3 \\ 2-k & 10 & 7 \end{pmatrix}$

iii) $A=\begin{pmatrix} -2 & 0 & -2 \\ -1 & k & -1 \\ 2 & 0 & 2 \end{pmatrix}$