Ex3 pag7 of Linear Spaces part 2

r: $OA(1,-1,2)$ $OB(-2,0,1)$

s: $C(1,3,-3)$ $\overline{OD}=(2,-2,3)$

$$r(t)=OA+t(OB-OA)$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\left[\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right]$$

$$r: \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \ t\in\mathbb{R}$$

$$s: \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + u\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \ u\in\mathbb{R}$$

are $\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ dependent or independent?

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \\ -1 & 3 \end{pmatrix} \rightarrow \begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix} = 6-2=4\neq 0$$

$\Rightarrow$ r and s aren't parallel

$$\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 6 \\ 1 & -2 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} \quad \begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix} \rightarrow rkA=1$$

$\det = 6-6=0$ $\quad\quad$ $\det = -6+6=0$

$$r\begin{cases} x(t)=1-3t \\ y(t)=-1+t \\ z(t)=2-t \end{cases} \qquad s\begin{cases} x(u)=1+2u \\ y(u)=3-2u \\ z(u)=-3+3u \end{cases}$$

$$\begin{cases} \cancel{1}-3t=\cancel{1}+2u \\ -1+t=3-2u \\ 2-t=-3+3u \end{cases} \quad \begin{cases} t=-\frac{2}{3}u \\ -1-\frac{2}{3}u=3-2u \\ \text{—} \end{cases}$$

$$\begin{cases} t=-\frac{2}{3}u \\ 2u-\frac{2}{3}u=4 \\ \text{—} \end{cases} \quad \begin{cases} t=-\frac{2}{3}u \\ \frac{\cancel{4}}{3}u=\cancel{4} \\ \text{—} \end{cases} \quad \begin{cases} t=-\frac{2}{3}u \\ u=3 \\ \text{—} \end{cases} \quad \begin{cases} t=-\frac{2}{\cancel{3}}\cdot\cancel{3}=-2 \\ u=3 \\ 2-(-2)=-3+3\cdot 3 \\ 4 \quad =6 \quad \text{NO!!} \end{cases}$$

$\longrightarrow$ Hence they are not intersecting

$\Longrightarrow$ hence they are SKEW

EX 6

r: $A(2,3,1)$ $B(0,0,1)$

s: $C(0,0,0)$ $D(4,6,0)$

$$r(t)=\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}+t\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}-\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right]$$

$$=\begin{pmatrix} 2-2t \\ 3-3t \\ 1 \end{pmatrix} \qquad \begin{cases} x(t)=2-2t \\ y(t)=3-3t \\ z(t)=1 \end{cases}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}+t\begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$$

$$s(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} , \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix} = (-2) \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}$$

write the equation of the plane that passes through $A(2,3,1)$ $B(0,0,1)$
$D(-2,-3,0)$ $C(0,0,0)$

$$p(t) = p_0 + su + tv$$

$$\begin{array}{cccc} \downarrow & & \downarrow & \downarrow \\ p_0 = C & & CA & CB \end{array}$$

u
v

$$P(s,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} x(s,t) = 2s \\ y(s,t) = 3s \\ z(s,t) = s + t \end{cases}$$

EX3 $\begin{cases} -x+y-z=-1 \\ x-y+kz=k \\ -2x+ky-2z=-3 \end{cases}$

$$A=\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & k \\ -2 & k & -2 \end{pmatrix}$$

if $A \in M_n$
check if $\det A \neq 0$
$\Rightarrow$ CRAMER'S THM

for which $\exists$ only one solution whose components are $x_i = \dfrac{\det A_i}{\det A}$ — from A by substituting to the i-th column the column of the given terms

$\det A = -2-2k-k-(-2-k^2-2)$
$= -\cancel{2}-2k-k+\cancel{2}+k^2+2$
$= k^2-3k+2 = (k-2)(k-1)$

if $k \neq 2$, $k \neq 1$ $\det A \neq 0$

$$x = \frac{\begin{vmatrix} -1 & 1 & -1 \\ k & -1 & k \\ -3 & k & -2 \end{vmatrix}}{\underbrace{(k-2)(k-1)}_{\det A}} = \frac{-2-3k-k^2-(-3-k^2-2k)}{(k-2)(k-1)}$$

$$= \frac{-2-3k-\cancel{k^2}+3+\cancel{k^2}+2k}{(k-2)(k-1)} = \frac{1-k}{(k-2)(k-1)}$$

$$= -\frac{\cancel{(k-1)}}{(k-2)\cancel{(k-1)}} = -\frac{1}{k-2} = \frac{1}{2-k}$$

$$y=\frac{\begin{vmatrix}-1 & -1 & -1\\ 1 & k & k\\ -2 & -3 & -2\end{vmatrix}}{(k-2)(k-1)}=\frac{2k+2k+3-(2k+3k+2)}{(k-2)(k-1)}$$

$$=\frac{2k+\cancel{2k}+3-\cancel{2k}-3k-2}{(k-2)(k-1)}=\frac{1-k}{(k-2)(k-1)}$$

$$=-\frac{\cancel{(k-1)}}{(k-2)\cancel{(k-1)}}=-\frac{1}{k-2}=\frac{1}{2-k}$$

$$z=\frac{\begin{vmatrix}-1 & 1 & -1\\ 1 & -1 & k\\ -2 & k & -3\end{vmatrix}}{(k-2)(k-1)}=\frac{-3-2k-k-(-2-k^2-3)}{(k-2)(k-1)}$$

$$=\frac{\cancel{-3}-2k-k+2+k^2\cancel{+3}}{(k-2)(k-1)}$$

$$=\frac{k^2-3k+2}{(k-2)(k-1)}=\frac{\cancel{(k-2)}\cancel{(k-1)}}{\cancel{(k-2)}\cancel{(k-1)}}=1$$

If $k=2$

$$A=\begin{pmatrix}-1 & 1 & -1\\ 1 & -1 & 2\\ -2 & 2 & -2\end{pmatrix} \qquad \begin{vmatrix}-1 & 2\\ 2 & -2\end{vmatrix}=2-4=-2\neq 0$$

$\mathrm{rk}\,A=2$ (R-C: The system is consistent iff $\mathrm{rk}(A)=\mathrm{rk}(A|b)$ and admits in this case $\infty^{n-\mathrm{rk}(A)}$ solutions)

$$(A|b)=\left(\begin{array}{ccc|c}-1 & 1 & -1 & -1\\ 1 & -1 & 2 & 2\\ -2 & 2 & -2 & -3\end{array}\right)$$

$$2 \leq rk(A|b) \leq 3$$

$$\begin{vmatrix} 1 & -1 & -1 \\ -1 & 2 & 2 \\ 2 & -2 & -3 \end{vmatrix} = -6-4-2-(-4-4-3)$$

$$= -1 \neq 0$$

$$\Rightarrow rk(A|b) = 3 \neq 2 = rk(A)$$

hence by R-C the system doesn't admit solutions

if $k=1$

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & 1 & -2 \end{pmatrix}$$

$$(A|b) = \begin{pmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -2 & 1 & -2 & -3 \end{pmatrix}$$

$$\begin{vmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -2 & 1 & -3 \end{vmatrix} = 0 \Rightarrow rk(A|b) = 2 = rk(A)$$

$\Rightarrow$ the system is CONSISTENT

$\infty^{3-2} = \infty^1$ solutions

$$\begin{cases} x-y=1-t \\ -2x+y=-3+2t \\ z=t \quad t\in\mathbb{R} \end{cases} \qquad \tilde{A}=\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$\det\tilde{A}=1-2=-1$$

$$x=\frac{\begin{vmatrix} 1-t & -1 \\ -3+2t & 1 \end{vmatrix}}{-1}=\frac{1-t+2t-3}{-1}$$

$$=\frac{t-2}{-1}=2-t$$

$$y=\frac{\begin{vmatrix} 1 & 1-t \\ -2 & -3+2t \end{vmatrix}}{-1}=\frac{-3+2t+2(1-t)}{-1}$$

$$=\frac{-3+\cancel{2t}+2-\cancel{2t}}{-1}=\frac{-1}{-1}=1$$

$$z=t$$

$$\begin{cases} x=2-t \\ y=1 \\ z=t \end{cases}, \quad t\in\mathbb{R}$$

Ex. $A = \begin{pmatrix} 1 & 0 & 0 \\ k-1 & -4 & -3 \\ 2-k & 10 & 7 \end{pmatrix}$

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ k-1 & -4-\lambda & -3 \\ 2-k & 10 & 7-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left[(-4-\lambda)\cdot(7-\lambda) + 30\right]$$

$$= (1-\lambda)\left[-28 + 4\lambda - 7\lambda + \lambda^2 + 30\right]$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (1-\lambda)(\lambda-1)(\lambda-2) =$$

$$= -(\lambda-1)^2(\lambda-2)^1$$

$\lambda_1 = 1 \Rightarrow m_a(1) = 2$

$\lambda_2 = 2 \Rightarrow m_a(2) = 1$

If $\lambda = 1$

If $\lambda = 1$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ k-1 & -5 & \boxed{-3} \\ 2-k & 10 & 6 \end{pmatrix}}_{A-1\cdot I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

E_1 is a LINEAR SUBSPACE

$\dim E_1 = m_a(1)$

$1 \leq rk(A - 1 \cdot I_3) \leq 2$

$$\begin{vmatrix} -5 & -3 \\ 10 & 6 \end{vmatrix} = -30 + 30 = 0$$

$$\begin{vmatrix} k-1 & -3 \\ 2-k & 6 \end{vmatrix} = 6(k-1) + 3(2-k)$$
$$= \cancel{6}k \cancel{-6} + \cancel{6} - 3k$$
$$= 3k \neq 0$$

the $rk(A - I_3) = 2$ if $k \neq 0$

$\Rightarrow$ the system $(A - 1I_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

admits $\infty^{3-rk(A-I_3)} = \infty^{3-2} = \infty^1$

solutions

$m_G(1) = 1 \neq m_A(1)$ if $k \neq 0$

if $k \neq 0$ A is not DIAGONALIZABLE

If $k = 0$ and $\lambda = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -5 & \boxed{-3} \\ 2 & 10 & 6 \end{pmatrix} \quad 2^{nd} \cdot (-2) = 3^{rd}$$

$\rightarrow rk(A - 1I_3) = 1$

the system admits $\infty^{3-1} = \infty^2$ sol

$m_G(1) = 2 = m_A(1)$

$$\begin{cases} x = t \in \mathbb{R} \\ y = s \in \mathbb{R} \\ -3z = t + 5s \end{cases} \qquad \begin{cases} x = t \\ y = s \\ z = -\frac{1}{3}t - \frac{5}{3}s \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ -\frac{5}{3} \end{pmatrix}$$

if $t = 3$, $s = 3$

$$\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$$

if $\lambda_2 = 2$

$$\underbrace{\begin{pmatrix} -1 & 0 & 0 \\ -1 & -6 & -3 \\ 2 & 10 & 5 \end{pmatrix}}_{A-2I_3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -1 & 0 \\ -1 & -6 \end{vmatrix} = 6 \qquad \underset{\text{Sol.}}{\infty^{3-2}} = \infty^1$$

$$\begin{cases} -x = 0 \\ -x - 6y = 3t \\ z = t \in \mathbb{R} \end{cases} \qquad \begin{cases} x = 0 \\ -6y = 3t \\ z = t \end{cases}$$

$$\begin{cases} x = 0 \\ y = -\frac{1}{2}t \\ z = t \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}$$

$t = 2$

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & -1 \\ -1 & -5 & 2 \end{pmatrix}$$

$$A = PDP^{-1} \longleftrightarrow$$

$$AP = PD \rightarrow \text{CHECK}$$