

1] SEE THE SOLUTIONS OF THE EXERCISES 3] AND 4] OF THE SHEET CALLED "SOLUTIONS 03-18-2022".

2]

MATRIX A

$$A - \lambda I = \begin{pmatrix} 4-\lambda & -1 & 1 \\ 2 & -\lambda & 1 \\ -2 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (4-\lambda)(\lambda^2-1) - 2(\lambda-1) - 2(-1+\lambda) = \\ &= (4-\lambda)(\lambda^2-1) - 4(\lambda-1) = \\ &= (\lambda-1)[(4-\lambda)(\lambda+1) - 4] = (\lambda-1)(-\lambda^2+3\lambda) = \\ &= -\lambda(\lambda-1)(\lambda-3) \end{aligned}$$

$$\lambda_1 = 0$$

$$\lambda_2 = 1$$

$$\lambda_3 = 3$$

→ I KNOW THAT THE MATRIX IS  
DIAGONALIZABLE BECAUSE  
THE ALGEBRAIC MULTIPLICITY OF  
EACH EIGENVALUE IS ONE.

$$\lambda = 0$$

$$\begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{cases} 4x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_3 = 0 \\ -2x_1 + x_2 = 0 \end{cases}$$

$$x_2 = t$$

$$x_1 = +\frac{1}{2}t$$

$$x_3 = -t$$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$m_{v_1} = 1$$

$$\lambda = 1$$

$$\begin{pmatrix} 3 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix} \rightarrow \begin{cases} 3x_1 - x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \\ -2x_1 + x_2 - x_3 = 0 \end{cases}$$

$$\begin{aligned}x_3 &= t \\ x_1 &= 0 \\ x_2 &= t\end{aligned}$$

$$\rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$mg = 1$$

$$\boxed{\lambda = 3}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 1 \\ 2 & 1 & -3 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 - 3x_2 + x_3 = 0 \\ -2x_1 + x_2 - 3x_3 = 0 \end{cases}$$

$$x_1 = -2t$$

$$x_2 = -t$$

$$x_3 = t$$

$$\rightarrow v_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$mg = 1$$

So:

$$P = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

THE COLUMNS  
OF THE MATRIX  $P$   
ARE COMPOSED BY  
THE EIGENVECTORS.  
PAY ATTENTION: THE  
ORDER OF THE EIGENVECTORS  
HAS TO BE THE SAME OF  
THE EIGENVALUES IN  $D$ .

MATRIX B

$$B - \lambda I = \begin{pmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 1 & -\lambda \end{pmatrix}$$

$$\begin{aligned}\det(B - \lambda I) &= -\lambda [-\lambda(1-\lambda)] - 1(1-\lambda) = \\ &= \lambda^2(1-\lambda) - (1-\lambda) = (1-\lambda)(\lambda^2 - 1) = \\ &= -(1-\lambda)^2(1+\lambda)\end{aligned}$$



$$\lambda = 1 \quad m_a = 2$$

$$\lambda = -1 \quad m_a = -1$$

$$\boxed{\lambda = 1}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \rightarrow \begin{cases} -x_1 - x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \end{cases}$$

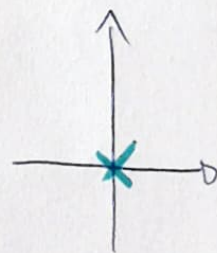
$$\begin{aligned} x_3 &= t \\ x_1 &= -t \\ x_2 &= 0 \end{aligned} \rightarrow v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad m_g = 1 \neq m_a = 2$$

$\Rightarrow$  THE MATRIX IS NOT DIAGONALIZABLE.

$\boxed{3}$  SEE THE EXERCISE  $\boxed{5}$  OF THE "SOLUTIONS 03-18-2022"

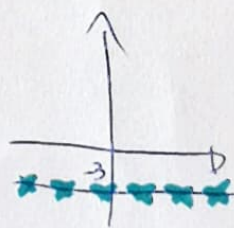
$\boxed{4}$  (a)  $D: x^2 + y^2 > 0 \leadsto (x, y) \neq (0, 0)$ .

$$\Rightarrow D = \mathbb{R}^2 \setminus \{(0, 0)\}$$

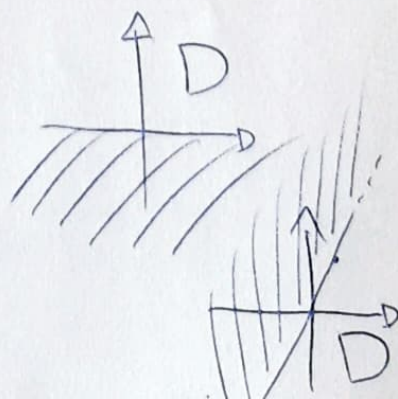


(b)  $D: y + 3 \neq 0 \leadsto y \neq -3$

$$D = \mathbb{R}^2 \setminus \{(x, y) \mid y = -3\}$$



(c)  $D: y \geq 0 \quad D = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$

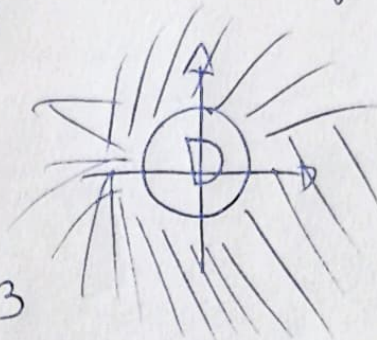


(d)  $D: 2x - y > 0 \leadsto D = \{(x, y) \in \mathbb{R}^2 \mid 2x - y > 0\}$



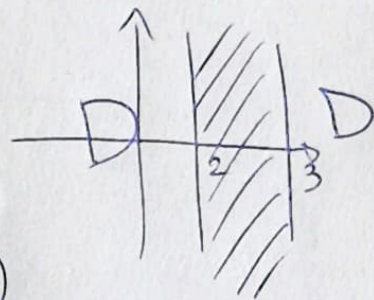
$$(e) D: 1-x^2-y^2 \geq 0 \leadsto -x^2-y^2 \geq -1 \leadsto x^2+y^2 \leq 1$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$$



$$(f) D: x^2-5x+6 > 0 \leadsto x < 2 \vee x > 3$$

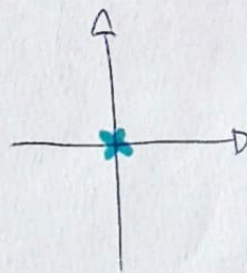
$$D = \{(x,y) \in \mathbb{R}^2 \mid x < 2 \vee x > 3\}$$



$$(g) D: \begin{cases} x^2+y^4 > 0 \\ x^2+y^2+3 > 0 \end{cases} \leadsto \begin{cases} (x,y) \neq (0,0) \\ \forall (x,y) \in \mathbb{R}^2 \end{cases}$$

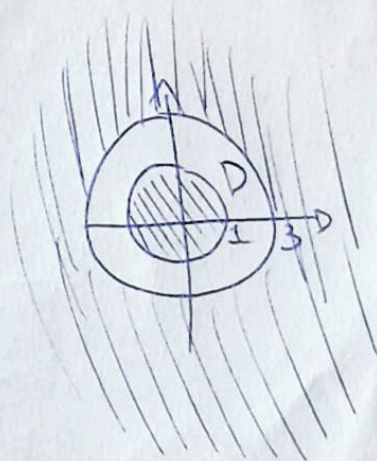
$$\rightarrow (x,y) \neq (0,0)$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$$



$$(h) D: \begin{cases} x^2+y^2-1 > 0 \\ 9-x^2-y^2 > 0 \end{cases} \leadsto \begin{cases} x^2+y^2 > 1 \\ x^2+y^2 < 9 \end{cases}$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 9\}$$

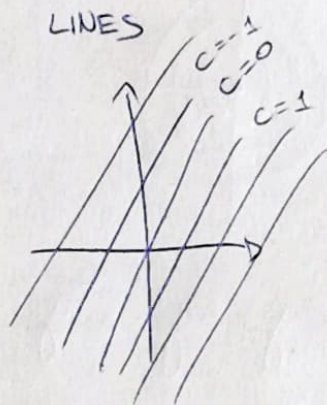




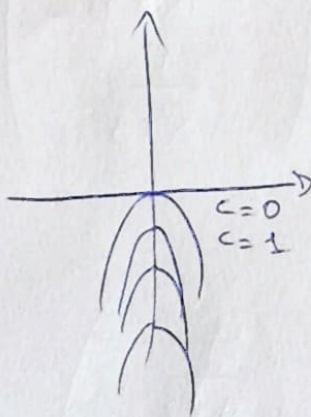
5

$$(a) f(x,y)=c \Leftrightarrow 3x-y=c \Leftrightarrow y-3x+c=0$$

$$\Leftrightarrow y=3x-c$$



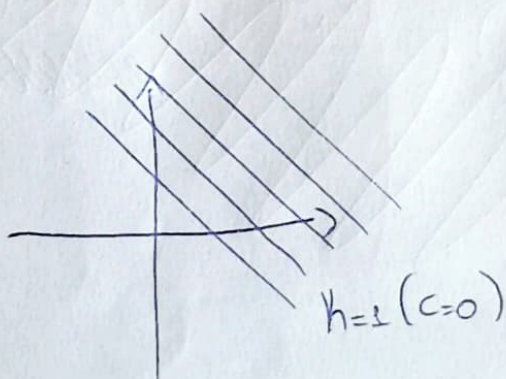
$$(b) f(x,y)=c \Leftrightarrow 2x^2+y=c \Leftrightarrow y=-2x^2+c \quad \text{PARABOLAS}$$



$$(c) f(x,y)=c \Leftrightarrow \log(x+y-3)=c \Leftrightarrow x+y-3=e^c$$

$$h=e^c > 0$$

$$y+x-3-h=0 \Leftrightarrow y=-x+3+h$$



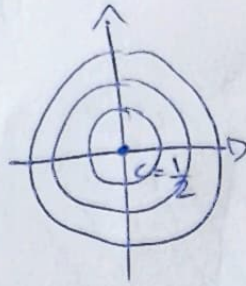
$$(d) f(x,y)=\frac{1}{2-x^2-y^2}=c \Leftrightarrow 2-x^2-y^2=\frac{1}{c} \Leftrightarrow$$

$$-x^2 - y^2 = \frac{1}{c} - 2 \Leftrightarrow x^2 + y^2 = 2 - \frac{1}{c} \quad \text{CIRCUMFERENCES}$$

$$\text{WITH } 2 - \frac{1}{c} \geq 0$$

$$2 \geq \frac{1}{c}$$

$$c \geq \frac{1}{2}$$



$$\boxed{6} \quad (a) \quad \nabla f(x, y) = \begin{pmatrix} 3x^2y + 5y^2 - 3\sqrt{y} \\ x^3 + 10xy - \frac{3x}{2\sqrt{y}} + 7 \end{pmatrix}$$

$$(b) \quad \nabla f(x, y) = \begin{pmatrix} \frac{-2x(y+1)}{(x^2+3)^2} \\ \frac{1}{x^2+3} \end{pmatrix}$$

$$(c) \quad \nabla f(x, y) = \begin{pmatrix} \frac{y}{y+2} \\ \frac{2x+1}{(y+2)^2} \end{pmatrix}$$



$$(d) \nabla f(x,y) = \begin{pmatrix} \sin(xy^3) [\sin(xy^3) + 2xy^3 \cos(xy^3)] \\ 3x^2 y^2 \sin(2xy^3) \end{pmatrix}$$

$$(e) \nabla f(x,y) = \begin{pmatrix} \frac{1}{x} + e^y \\ x e^y - \frac{1}{y} \end{pmatrix}$$

$$(f) \nabla f(x,y) = \begin{pmatrix} e^{y/x} (2x-y) \\ x e^{y/x} \end{pmatrix}$$

$$(g) \nabla f(x,y) = \begin{pmatrix} x^{\log(y^2)-1} [\cos(1-3x) \log(y^2) + 3x \sin(1-3x)] \\ \frac{2 \log(x) \cos(1-3x) x^{\log(y^2)}}{x} \end{pmatrix}$$

$$(h) \nabla f(x,y) = \begin{pmatrix} -\frac{(x^2-y^4-2) \sin(y)}{(x^2+y^4+2)^2} \\ x \frac{[(x^2+y^4+2) \cos(y) - 4y^3 \sin(y)]}{(x^2+y^4+2)^2} \end{pmatrix}$$

$$\boxed{7} \quad (a) \nabla f(x,y) = \begin{pmatrix} 6x + 6xy + y \\ 3x^2 + x \end{pmatrix}$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 6x + 6xy + y = 0 \\ 3x^2 + x = 0 \end{cases} \begin{cases} \nearrow \begin{cases} x=0 \\ y=0 \end{cases} \\ \searrow \begin{cases} x=-\frac{1}{3} \\ -2-2y+y=0 \end{cases} \end{cases}$$

$$\Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=-\frac{1}{3} \\ y=-2 \end{cases}$$

$$(b) \quad \nabla f(x,y) = \begin{pmatrix} 2xy + 3x \\ x^2 + 3x + 3y^2 \end{pmatrix}$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 2xy + 3x = 0 \\ x^2 + 3x + 3y^2 = 0 \end{cases} \rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$$

$$\rightarrow \begin{cases} y = -\frac{3}{2} \\ x^2 + 3x + \frac{27}{4} = 0 \quad \nexists x \end{cases}$$

$$\begin{cases} x=0 \\ y=0 \end{cases}$$

$$(c) \quad \nabla f(x,y) = \begin{pmatrix} 3x^2 + 6xy \\ 3x^2 + 2y \end{pmatrix}$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 3x^2 + 6xy = 0 \\ 3x^2 + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x(x+2y) = 0 \\ 3x^2 + 2y = 0 \end{cases}$$

$$\rightarrow \begin{cases} x=0 \\ 3x^2 + 2y = 0 \end{cases} \rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$$

$$\rightarrow \begin{cases} x = -2y \\ 12y^2 + 2y = 0 \end{cases} \rightarrow \begin{cases} y = -\frac{1}{6} \\ x = \frac{1}{3} \\ y=0 \\ x=0 \end{cases}$$



$$\boxed{x=0} \quad \boxed{x=\frac{1}{3}}$$

$$\boxed{y=0} \quad \boxed{y=-\frac{1}{6}}$$

$$(d) \nabla f(x,y) = \begin{pmatrix} 2xy^3 + y^2 \\ 3y^2x^2 + 2xy - 3 \end{pmatrix}$$

$$\nabla f = 0 \Leftrightarrow \begin{cases} 2xy^3 + y^2 = 0 \\ 3y^2x^2 + 2xy - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} y^2(2xy + 1) = 0 \\ 3y^2x^2 + 2xy - 3 = 0 \end{cases}$$

$$\swarrow \begin{cases} y=0 \\ 3y^2x^2 + 2xy - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ -3=0 \end{cases} \nexists (x,y)$$

$$\searrow \begin{cases} 2xy + 1 = 0 \\ 3y^2x^2 + 2xy - 3 = 0 \end{cases} \Leftrightarrow \begin{cases} 2xy = -1 \\ 3y^2x^2 = 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} xy = -\frac{1}{2} \\ \frac{3}{4} = 4 \end{cases} \nexists (x,y)$$

NO CRITICAL POINTS.

$$(e) \nabla f(x,y) = \begin{pmatrix} ye^{x-y} + xy e^{x-y} \\ xe^{x-y} + xy e^{x-y} \end{pmatrix}$$

$$\nabla f(x,y) = 0 \Leftrightarrow \begin{cases} ye^{x-y} + xy e^{x-y} = 0 \\ xe^{x-y} + xy e^{x-y} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} ye^{x-y}(x+1) = 0 \\ xe^{x-y}(1-y) = 0 \end{cases} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{cases} y=0 \\ xe^{x-y}(1-y) = 0 \\ x=-1 \\ xe^{x-y}(1-y) = 0 \end{cases}$$

$$\begin{cases} y=0 \\ xe^{x-y}=0 \end{cases} \rightarrow \boxed{\begin{cases} y=0 \\ x=0 \end{cases}}$$

$$\begin{cases} x=-1 \\ (y-1)e^{x-y}=0 \end{cases} \rightarrow \boxed{\begin{cases} x=-1 \\ y=1 \end{cases}}$$

$$(F) \nabla f(x,y) = \begin{pmatrix} \frac{1}{2\sqrt{x^3+2xy^2}} \cdot (3x^2+2y^2) \\ \frac{1}{2\sqrt{x^3+2xy^2}} (4xy) \end{pmatrix}$$

$$\nabla f(x,y) = 0 \Leftrightarrow \begin{cases} \frac{3x^2+2y^2}{2\sqrt{x^3+2xy^2}} = 0 \\ \frac{4xy}{2\sqrt{x^3+2xy^2}} = 0 \end{cases}$$



FROM THE FIRST EQUATION, THE ONLY WAY TO HAVE A NUMERATOR EQUAL TO ZERO IS  $(x, y) = (0, 0)$  BUT THIS IS NOT POSSIBLE!  
(LOOK AT THE DENOMINATOR):

$$g) \nabla f(x, y) = \begin{pmatrix} \frac{1}{2\sqrt{x^2+3x^2y^2+2y^4}} \cdot (2x+6xy^2) \\ \frac{1}{2\sqrt{x^2+3x^2y^2+2y^4}} (6x^2y+8y^3) \end{pmatrix}$$

$$\nabla f(x, y) = 0 \Leftrightarrow \begin{cases} \frac{2x+6xy^2}{2\sqrt{x^2+3x^2y^2+2y^4}} = 0 \\ \frac{6x^2y+8y^3}{2\sqrt{x^2+3x^2y^2+2y^4}} = 0 \end{cases}$$

$\rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$   
IS NOT ADMISSIBLE

NO CRITICAL POINTS

$$(h) \nabla f(x, y) = \begin{pmatrix} y \cos(x) \\ \sin(x) \end{pmatrix}$$

$$\nabla f(x, y) = 0 \Leftrightarrow \begin{cases} y \cos(x) = 0 \\ \sin(x) = 0 \end{cases}$$

$$\begin{cases} y=0 \\ \sin(x)=0 \end{cases} \rightarrow \begin{cases} \cos(x)=0 \\ \sin(x)=0 \end{cases} \nexists x$$

$$\begin{cases} y=0 \\ \sin(x)=0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} y=0 \\ x = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z} \end{cases}$$

INFINITE  
CRITICAL  
POINTS