MATHEMATICS

2nd Midterm Simulation

April 1, 2022, A.Y. 2021/2022

Any kind of electronic device (calculators, smartphones, smartwatches etc..) is FORBIDDEN. Put all your devices in the location indicated by the examiners. If an attendee violates this rule (even if the device is switched off or in offline mode), she/he will be immediately expelled from the exam session and her/his exam will be invalidated.

Examiners will mark only THIS A3 paper SHOWING THE UNIVERSITY LOGO on the upper left corner. Any other additional papers (drafts, notes, scribbles or anything else) will not be taken into consideration. Use a clear and clean handwriting. Unclear or ambiguous sentences may result in a negative impact on the final grade of the exam.

Remember to always double check the consistency of your results. EVEN IF CORRECT, inconsistent statements will result in a negative impact on the final grade of the exam.

MARKS: you get 3 points for each correct answer, 0 points for unanswered questions, -1 points for each incorrect answers. Open questions are not penalized.

Time for the test: 1 hour and 15 minutes

MATRICOLA .... *Lastname* .... *Name* ....

1. *Compute the following integral*

$$\int \frac{3x^2+2-4x}{3x}dx$$

☐ *A.* $\frac{x^2}{2}+\frac{2}{3}\ln|x|-\frac{4}{3}x+c$ ☐ *B.* $\frac{x^2}{2}+2\ln|x|-\frac{4}{3}x+c$

☐ *C.* $x+\frac{1}{x}-\frac{4}{3}+c$ ☐ *D.* $x^2+\ln|x|-\frac{4}{3}x+c$

Solution Assign the denominator $3x$ to each term of the numerator, simplify suitably and exploit linearity of the indefinite integral

$$\int \frac{3x^2}{3x}+\frac{2}{3x}-\frac{4x}{3x}dx=\int x\ dx+\frac{2}{3}\int\frac{1}{x}dx-\frac{4}{3}\int dx=\frac{x^2}{2}+\frac{2}{3}\ln|x|-\frac{4}{3}x+c$$

2. *Compute the following definite integral:*

$$\int_0^{\sqrt{\ln 3}} xe^{x^2}\,dx$$

☐ A. 1 ☐ B. 2

☐ C. 0 ☐ D. e

Solution By solving, through direct substitution the following indefinite integral

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$$

and applying the fundamental theorem of calculus, we get

$$\int_0^{\sqrt{\ln 3}} xe^{x^2}\,dx = \frac{1}{2}\left[e^{x^2}\right]_0^{\sqrt{\ln 3}} = \frac{1}{2}(e^{(\sqrt{\ln 3})^2} - 1) = 1$$

3. *Compute the domain and the range of the following function of two variables*

$$z = f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$

☐ A. $D_f = \{(x,y) \in \mathbb{R}^2 \ ; \ y < -x-1\} \quad R_f = (0,+\infty)$

☐ B. $D_f = \{(x,y) \in \mathbb{R}^2 \ ; \ y \geq -x-1 \ \text{ and } x \neq 1\} \quad R_f = (-\infty,+\infty)$

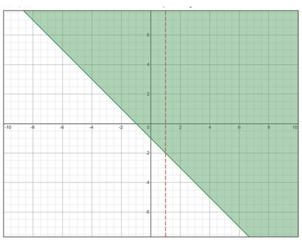
☐ C. $D_f = \{(x,y) \in \mathbb{R}^2 \ ; \ x \geq y-1 \ ; \ x \neq 1\} \quad R_f = (-\infty,+\infty)$

☐ D. $D_f = \{(x,y) \in \mathbb{R}^2 \ ; \ y \geq -x-1 \ \text{ and } x \neq 1\} \quad R_f = [0,+\infty)$

Solution The domain of the function is the collection of points $(x,y) \in \mathbb{R}$ such that

$$\begin{cases} x+y+1 \geq 0 \\ x \neq 1 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} y \geq -x-10 \\ x \neq 1 \end{cases}$$

hence $D_f = \{(x,y) \in \mathbb{R}^2 \ ; \ y \geq -x-1 \ \text{ and } x \neq 1\}$, as shown in the figure



Regarding the Range, surely the function attains both positive and negative values, as the denominator $x-1$ covers all real values except for 0. Moreover if we analyze what happens in the direction $(x,0)$ for $x \to 1^-$ and for $x \to 1^+$, we get that

$$\lim_{(x,0)\to(1^-,0)} \frac{\sqrt{x+0+1}}{x-1} = -\infty \qquad \text{and} \qquad \lim_{(x,0)\to(1^+,0)} \frac{\sqrt{x+0+1}}{x-1} = +\infty,$$

and the function is equal to zero for all the points (x,y) that lay on the line $x+y+1=0$. Hence $R_f = (-\infty,+\infty)$.

4. *Discuss if the following system is consistent as k changes and specify the number of solutions.*

$$\begin{cases} x+y+kz &= 2 \\ x+y+3z &= 2 \\ 2x+ky-z &= 1 \end{cases}$$

- ☐ A. *The system is consistent with a unique solution if $k \neq 2$ and $k \neq 3$, if $k=3$ the system is consistent with ∞^1 solutions, if $k=2$ the system is inconsistent*
- ☐ B. *The system is consistent with a unique solution if $k \neq 2$ and $k \neq 3$, if $k=2$ the system is consistent with ∞^1 solutions, if $k=3$ the system is inconsistent*
- ☐ C. *The system is consistent with a unique solution if $k \neq -2$ and $k \neq -3$, if $k=-3$ and $k=-2$ the system is consistent*
- ☐ D. *The system is consistent with a unique solution if $k \neq 2$ and $k \neq 3$, if $k=3$ and $k=2$ the system is consistent with ∞^1 solutions.*

Solution Le t us consider the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & k \\ 1 & 1 & 3 \\ 2 & k & -1 \end{pmatrix},$$

whose determinant is

$$det(A) = -1+6+k^2-2k-3k+1 = k^2-5k+6 = (k-2)(k-3)$$

Hence if $k \neq 2$ and $k \neq 3$ the system is consistent and admits one and only one solution.
If $k=2$ the coefficient matrix becomes

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 2 & -1 \end{pmatrix},$$

and $rk(A)=2$ as the highlighted (in red) minor of order 2 is different from zero. Let us analyze the rank of the augmented matrix

$$A|b = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & -1 & 1 \end{pmatrix},$$

and by bordering the highlighted (in red) submatrix of order 2, we just have to check the value of the following determinant

$$det \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix} = 3 + 8 - 2 - 12 + 2 - 2 = -3 \neq 0$$

hence $rk(A|b) = 3 \neq 2 = rk(A)$, and the system is inconsistent.
If $k = 3$ the coefficient matrix becomes

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 2 & 3 & -1 \end{pmatrix},$$

and $rk(A) = 2$ as the highlighted in red minor of order 2 is different from zero. Let us analyze the rank of the augmented matrix

$$A|b = \begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & -1 & 1 \end{pmatrix},$$

and by bordering the highlighted (in red) submatrix of order 2, we just have to check the value of the following determinant

$$det \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 3 & -1 & 1 \end{pmatrix} = 0$$

hence $rk(A|b) = 2 = rk(A)$, and the system is consistent, with ∞^1 solutions

5. *Evaluate the following limit*

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x (1 - \cos^2 t) dt$$

Then,

☐ *A.* 1	☐ *B.* 3
☐ *C.* 0	☐ *D.* $\frac{1}{3}$

Solution The limit is of the form $\frac{0}{0}$ and we can solve it by applying the de l'Hopital Theorem and by exploiting the Fundamental Theorem of Calculus (when differentiating the integral function at the numerator)

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x (1 - \cos^2 t) dt = \lim_{x \to 0} \frac{\int_0^x (1 - \cos^2 t)}{x^3} dt = \lim_{x \to 0} \frac{1 - \cos^2 x}{3x^2} = \lim_{x \to 0} \frac{1}{3} \frac{\sin^2 x}{x^2} = \frac{1}{3}$$

6. *Identify the cartesian equation of the plane passing through the point* $A(1,2,0)$ *and orthogonal to the vector* $\mathbf{u} = (1,1,1)$

- ☐ *A.* $x + y + z = 3$
- ☐ *B.* $x - y - z + 3 = 0$
- ☐ *C.* $2x + 2y - z = 3$
- ☐ *D.* $x - y + z = 2$

Solution Let $P(x,y,z)$ *be a generic point of the plane. Hence the vector* $PA = (x-1, y-2, z)$ *lies on the plane. All vectors on the plane have to be orthogonal to the vector* $\mathbf{u} = (1,1,1)$*; hence*

$$PA \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad (x-1, y-2, z) \cdot (1,1,1) = 0 \quad \Longrightarrow \quad x-1+y-2+z = 0 \quad \Longrightarrow \quad x+y+z = 3$$

7. *For which values of* k *are the following three vectors dependent?*

$$\mathbf{v} = (1,1,1) \quad \mathbf{u} = (3,2,k) \quad \mathbf{w} = (0,k,k)$$

- ☐ *A.* $k = 1, \ \ k = 0$
- ☐ *B.* $k = -1, \ \ k = 2$
- ☐ *C.* $k = 0, \ \ k = 2$
- ☐ *D.* $k = 1, \ \ k = 2$

Solution Display the vectors as columns of a 3×3 *square matrix* A*, and the vectors will be dependent if the determinant of this matrix is null*

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & k \\ 1 & k & k \end{pmatrix},$$

and

$$det(A) = 2k + 3k - k^2 - 3k = 2k - k^2 = k(2-k),$$

hence if $k = 0$ *or if* $k = 2$ *the vectors are linearly dependent.*

8. *Compute all eigenvalues of the matrix*

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$$

and indicate their algebraic multiplicity (AM).

- ☐ A. $\lambda_1 = 1$ *with* $AM(\lambda_1) = 1$, $\lambda_2 = 2$ *with* $AM(\lambda_2) = 2$
- ☐ B. $\lambda_1 = -1$ *with* $AM(\lambda_1) = 1$, $\lambda_2 = 1$ *with* $AM(\lambda_2) = 1$, $\lambda_3 = 2$ *with* $AM(\lambda_3) = 1$,
- ☐ C. $\lambda_1 = 1$ *with* $AM(\lambda_1) = 1$, $\lambda_2 = -1$ *with* $AM(\lambda_2) = 2$
- ☐ D. $\lambda_1 = -2$ *with* $AM(\lambda_1) = 1$, $\lambda_2 = -1$ *with* $AM(\lambda_2) = 1$, $\lambda_3 = 2$ *with* $AM(\lambda_3) = 1$,

Solution The characteristic equation is given by

$$det(A - \lambda I_3) = det \begin{pmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & -2 \\ 2 & 0 & 1-\lambda \end{pmatrix} = (-1-\lambda)(-1-\lambda)(1-\lambda) = (-1-\lambda)^2(1-\lambda) = 0$$

hence $\lambda_1 = 1$ *with* $AM(\lambda_1) = 1$ *and* $\lambda_2 = -1$ *with* $AM(\lambda_2) = 2$

9. *(Open question: please answer in the space below. Indicate all necessary steps to solve the exercise) Describe the integration by parts method*

Solution The Product Rule says that if f *and* g *are differentiable functions of* x*, then* $(fg)' = f'g + fg'$*. For simplicity, we have written* f *for* $f(x)$ *and* g *for* $g(x)$*. Suppose we integrate both sides with respect to* x*. This gives*

$$\int (fg)'dx = \int (f'g + fg')dx$$

By the fundamental Theorem of calculus, the left side integrates to fg*. The right side can be broken into two integrals*

$$fg = \int f'gdx + \int fg'dx,$$

and solving for the second integral

$$\int fg'dx = fg - \int f'gdx.$$

10. *(Open question: please answer in the space below) Give definition of similar matrices and prove that similar matrices have same eigenvalues*

Solution A $n \times n$ matrix B is called similar to matrix A if there exists an invertible matrix P such that $B = P^{-1}AP$.

If $n \times n$-matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.
Proof. If $B = P^{-1}AP$, then $B - \lambda I_n = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I_n)P$. Using the multiplicative property of determinant, we have $det(B - \lambda I_n) = det(P^{-1}(A - \lambda I_n)P) = detP^{-1}det(A - \lambda I_n)detP = det(A - \lambda I_n)$. Hence, matrices A and B have the same eigenvalues.

11. *(Open question: please answer in the space below. Indicate all necessary steps to solve the exercise) Compute all stationary points of the function below and determine their nature:*

$$f(x, y) = y^2[x^2 + y^2 - 2(x + y) + 2]$$

Solution Let us look for critical points by applying the first order conditions

$$\begin{cases} f_x(x, y) &= y^2(2x - 2) = 2y^2(x - 1) = 0 \\ f_y(x, y) &= 2y(x^2 + y^2 - 2(x + y) + 2) + y^2(2y - 2) = 2y[(x - 1)^2 + (y - 1)^2 + y(y - 1)] = 0 \end{cases}$$

The first equation gives two alternatives $y = 0$ or $x = 1$.
If $y = 0$, also the second equation is verified $\forall x \in \mathbb{R}$. Hence $(x, 0)$ $\forall x \in \mathbb{R}$ are all critical points.
If $x = 1$

$$\begin{cases} x = 1 \\ (y - 1)^2 + y(y - 1) = 0 \end{cases} \implies \begin{cases} x = 1 \\ (y - 1)(2y - 1) = 0 \end{cases} \implies \begin{cases} x = 1 \\ y = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = \frac{1}{2} \end{cases}$$

Let us pass to the second order conditions

$$\begin{cases} f_{xx}(x, y) = 2y^2 \\ f_{xy}(x, y) = f_{yx}(x, y) = 4y(x - 1) \\ f_{yy}(x, y) = 2[(x - 1)^2 + (y - 1)^2 + y(y - 1)] + 2y[2(y - 1) + 2y - 1] \end{cases}$$

$$detH(1, 1) = det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 > 0$$

and being $f_{xx}(1, 1) = 2 > 0$, then $(1, 1)$ is a local minimum, and $f(1, 1) = 0$

$$detH\left(1, \frac{1}{2}\right) = det \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{2} < 0$$

hence $\left(1, \frac{1}{2}\right)$ is a saddle point, and $f\left(1, \frac{1}{2}\right) = \frac{1}{16}$.

$$detH(x,0) = det \begin{pmatrix} 0 & 0 \\ 0 & 2((x-1)^2+1) \end{pmatrix} = 0 \quad \forall x \in \mathbb{R}$$

hence the method is inconclusive for the points $(x,0)$. *But notice that the function can be written as* $f(x,y) = y^2[(x-1)^2+(y-1)^2] \geq 0$ *and* $f(x,0) = 0 \;\; \forall x \in \mathbb{R}$, *therefore the points* $(x,0)$ *and also* $(1,1)$ *are points of global minimum.*