Solution of Exercises requested on Meeting of March 31 2022

8) Consider the following planes

$\pi: x-y+z=0$ and $\pi': 8x+y-z=0$

i) establish their reciprocal position

ii) find the cartesian equation of the plane passing through $P=(1,1,1)$ and orthogonal to π and π'

(Hint: ii) extrapulate normal directions to π and π' from their point-normal equation)

Solution: i) to find reciprocal position I see if the following system is consistent

$$\begin{cases} x-y+z=0 \\ 8x+y-z=0 \end{cases}$$ this is a Homogeneous system, hence it is necessarily consistent

and being the coefficient matrix A:

$$A=\begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 8 & 1 \end{matrix}} & \begin{matrix} 1 \\ -1 \end{matrix} \end{pmatrix} \Rightarrow rk(A)=2$$, hence the system admits $\infty^{3-rkA}=\infty^1$ solutions, meaning that

$$\begin{vmatrix} 1 & -1 \\ 8 & 1 \end{vmatrix} = 1+8=9\neq 0$$

the set of solutions is a line, whose parametric equations are:

$$\begin{cases} x-y=-t \\ 8x+y=t \\ z=t\in\mathbb{R} \end{cases} \Longleftrightarrow \begin{cases} x=\dfrac{\begin{vmatrix} -t & -1 \\ t & 1 \end{vmatrix}}{9}=\dfrac{-t+t}{9}=0 \\ y=\dfrac{\begin{vmatrix} 1 & -t \\ 8 & t \end{vmatrix}}{9}=\dfrac{t+8t}{9}=\dfrac{\cancel{9}t}{\cancel{9}}=t \\ z=t \end{cases}$$

$$\rightarrow \begin{cases} x(t)=0 \\ y(t)=t \\ z(t)=t \end{cases}$$ $\Rightarrow$ the two planes intersect in the line given by

$$l(t)=\left\{(x,y,z)\in\mathbb{R}^3 ; \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}=t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, t\in\mathbb{R}\right\}$$

$$=\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

ii) as suggested from the hint, the equations of the planes π and π' :

$\pi: x-y+z=0$ and $\pi': 8x+y-z=0$

give the normal directions to each plane:

$n_\pi = (1,-1,1)$ and $n_{\pi'} = (8,1,-1)$

these two directions are independent, and hence they generate a plane. The one that passes through the point $P=(1,1,1)$ is given by

$$p(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 8 \\ 1 \\ -1 \end{pmatrix} \Longleftrightarrow$$

$$\begin{cases} x(t) = 1+s+8t \\ y(t) = 1-s+t \\ z(t) = 1+s-t \end{cases}$$

we can solve this system for s and t:

- solve the 1^{st} equation for s:

$$s = x-1-8t$$

- plug s in the 2^{nd} equation:

$$y = 1-(x-1-8t)+t \Longleftrightarrow y = 1-x+1+8t+t$$

$$\Longleftrightarrow y = 2-x+9t$$

- solve the last equation for t

$$9t = y+x-2 \Rightarrow t = \frac{y+x-2}{9}$$

- substitute $t = \frac{y+x-2}{9}$ into $s = x-1-8t$

$$s = x-1-8 \cdot \frac{y+x-2}{9} = \frac{9x-9-8y-8x+16}{9}$$

$$= \frac{x-8y+7}{9}$$

- Recalling that $t = \frac{y+x-2}{9}$ and plugging s and t written in terms of x, y and z in the 3^{rd} equation

$z = 1 + s - t$, we get

$$z = 1 + \frac{x - 8y + 7}{9} - \frac{y + x - 2}{9} \Longrightarrow$$

$$\cancel{9}.\ \frac{9z}{\cancel{9}} = \frac{9 + x - 8y + 7 - y - x + 2}{\cancel{9}} . \cancel{9}$$

$$9y + 9z - 18 = 0 \longleftrightarrow y + z - 2 = 0 \leftarrow \text{SOLUTION}$$

From Extra exercises in File of Eigenvalues and eigenvectors Ex 1 c&d)
Compute all Eigenvalues and Eigenvectors of the following matrices and for each Eigenvalue specify the algebraic and the geometric multiplicity

c) $$\begin{pmatrix} 3 & 0 & 0 \\ 9 & 3 & 10 \\ -5 & 0 & -12 \end{pmatrix} \longrightarrow A - \lambda I_3 = \begin{pmatrix} 3-\lambda & 0 & 0 \\ 9 & 3-\lambda & 10 \\ -5 & 0 & -12-\lambda \end{pmatrix}$$

$$\Rightarrow \det(A - \lambda I_3) = (3-\lambda)^2(-12-\lambda)$$

hence $\lambda_1 = 3$ and $m_A(\lambda_1) = 2$

$\lambda_2 = -12$ and $m_A(\lambda_2) = 1$

Let us compute m_G for each eigenvalue

if $\lambda_1 = 3 \Longrightarrow A - 3I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 9 & 0 & 10 \\ -5 & 0 & -15 \end{pmatrix}$

as the highlighted submatrix $\begin{pmatrix} 9 & 10 \\ -5 & -15 \end{pmatrix}$

has $\det\begin{pmatrix} 9 & 10 \\ -5 & -15 \end{pmatrix} = -135 + 50 = -85 \Longrightarrow$

$rk(A - 3I_3) = 2 \longrightarrow m_G(\lambda_1) = m_G(3) = n - rk(A - 3I_3)$

$\Longrightarrow m_G(\lambda_1) = 3 - 2 = 1$

$\Longrightarrow m_A(\lambda_1) = 2$ and $m_G(\lambda_1) = 1$

if $\lambda_2 = -12$ as $m_A(\lambda_2) = 1 \Rightarrow$ necessarily $m_G(\lambda_2) = 1$

d)
$$A=\begin{pmatrix}-4 & 0 & 7\\ 0 & -1 & -6\\ 0 & 0 & -1\end{pmatrix} \Rightarrow A-\lambda I_3=\begin{pmatrix}-4-\lambda & 0 & 7\\ 0 & -1-\lambda & -6\\ 0 & 0 & -1-\lambda\end{pmatrix}$$

hence $\det(A-\lambda I_3)=(-4-\lambda)(-1-\lambda)^2$ and

$\lambda_1=-4 \quad m_A(\lambda_1)=1$

$\lambda_2=-1 \quad m_A(\lambda_2)=2$

Surely $m_G(\lambda_1)=1$ as $m_A(\lambda_1)=1$

Instead if $\lambda_2=-1 \Rightarrow A-(-1)I_3=\begin{pmatrix}-4+1 & 0 & 7\\ 0 & 0 & -6\\ 0 & 0 & 0\end{pmatrix}$

$=\begin{pmatrix}-3 & 0 & 7\\ 0 & 0 & -6\\ 0 & 0 & 0\end{pmatrix}$ as the highlighted submatrix of order 2 $\begin{pmatrix}-3 & 7\\ 0 & -6\end{pmatrix}$

has $\det\begin{pmatrix}-3 & 7\\ 0 & -6\end{pmatrix}=18>0 \Rightarrow rk(A+I_3)=2$

and hence $m_G(\lambda_2)=3-2=1\neq m_A(\lambda_2)=2$