Quantitative Methods – I

Practice 5

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THEME #1

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Sample space & Event

The basic concepts of probability

Experiment: a measurement process that produces quantifiable results (e.g. throwing two dice, dealing cards, at poker, measuring heights of people, recording proton-proton collisions)

Outcome: a single result from a measurement (e.g. the numbers shown on the two dice)

Sample space (S or Ω or ξ): the set of **all possible** outcomes from an experiment (e.g. the set of all possible five-card hands)

The number of all possible outcomes may be

(a) **finite** (e.g. all possible outcomes from throwing a single die; all possible 5-card poker hands)
(b) **countably infinite** (e.g. number of proton-proton events to be made before a Higgs boson event is observed)
or (c) **constitute a continuum** (e.g. heights of people)

In case (a), the sample space is said to be **finite**
in cases (a) and (b), the sample space is said to be **discrete**
in case (c), the sample space is said to be **continuous**

In this practice we consider discrete, mainly finite, sample spaces

An **event** is any subset of a sample set (including the empty set, and the whole set)

Two events that have no outcome in common are called **mutually exclusive** events.

In discussing discrete sample spaces, it is useful to use **Venn diagrams** and basic set-theory.

Therefore we will refer to the **union** ($A \cup B$)**, intersection,** ($A \cap B$) and **complement** ($\bar{A}$ or A^c) of events A and B.

THEME #2

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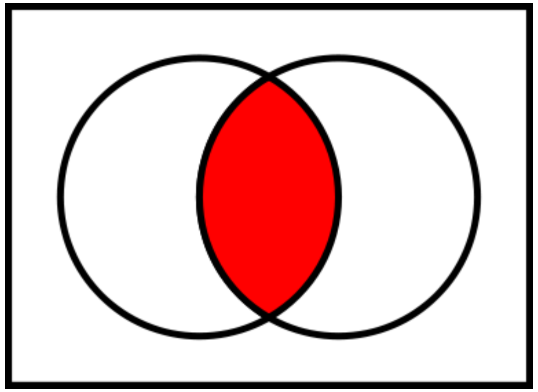
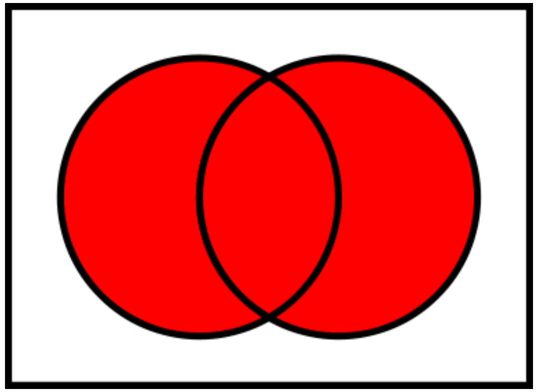
Venn Diagrams

Intersection $\cap$

Union $\cup$

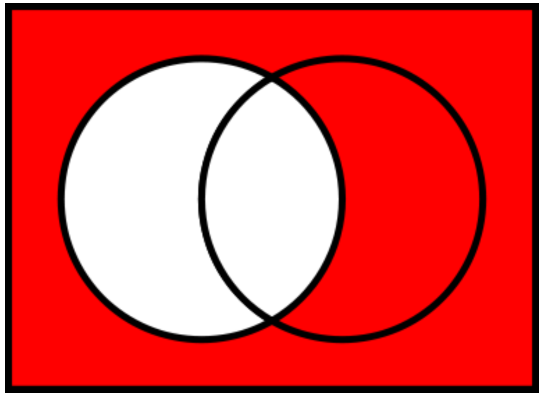


$A \cap B$

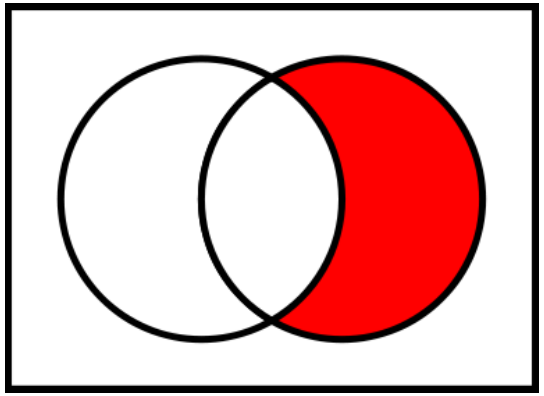


$A \cup B$

Complement $\bar{A}$ or A^C



$A^c = U \setminus A$

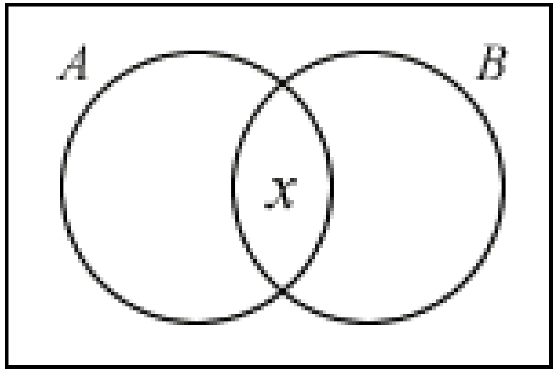


$A^c \cap B = B \setminus A$

Intersect

$A \cap B = x$

$A \cup B = n(A) + n(B) - x$

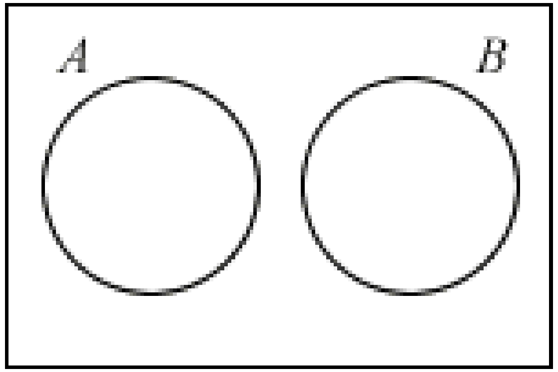


Because x is in A e B

Disjoint

$A \cap B = \varnothing$

$A \cup B = n(A) + n(B)$

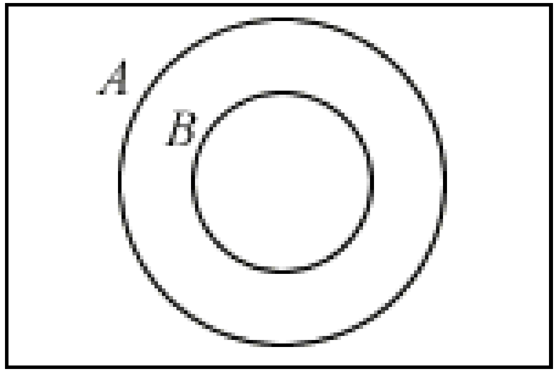


Subset

$B \subseteq A$

$A \cap B = B$

$A \cup B = A$



Properties of the operations between events		
	Union	Intersection
Idempotency	A ∪ A = A	A ∩ A = A
Neutral event	A ∪ Ø = A	A ∩ Ω = A
Commutativity	A ∪ B = B ∪ A	A ∩ B = B ∩ A
Associativity	(A ∪ B) ∪ C = A ∪ (B ∪ C)	(A ∩ B) ∩ C = A ∩ (B ∩ C)

Exercise 0: S = (3, 4, 2, 8, 9, 10, 27, 23, 14)

A = (2, 4, 8) B = (3, 4, 8, 27)

Calculate

$\bar{A}$ = (3, 9, 10, 27, 23, 14)

$\bar{B}$ = (2, 9, 10, 23, 14)

$A \cup B$ = (2, 3, 4, 8, 27)

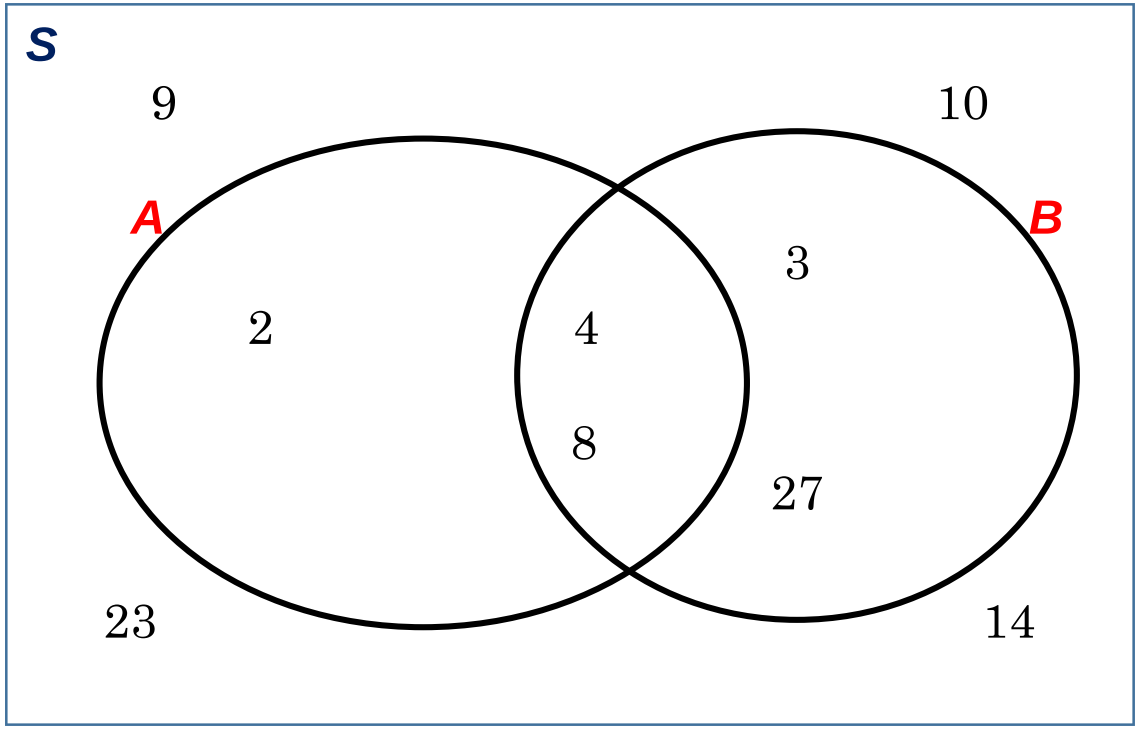
$A \cap B$ = (4, 8)

$A - B$ = (2)

$\overline{A \cup B}$ = (9, 10, 23, 14)

$\overline{A \cap B}$ = (3, 9, 10, 27, 23, 14, 2)

Draw the Venn diagram



Exercise 1. $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

A = multiples of 3
B = multiples of 5

(a) Draw the Venn diagram
(b) Find $A \cap B$

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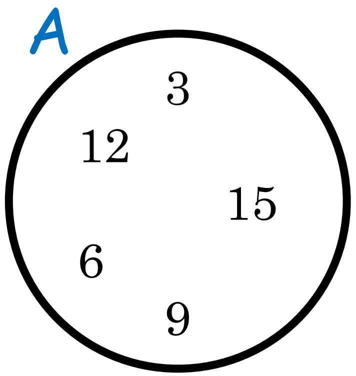

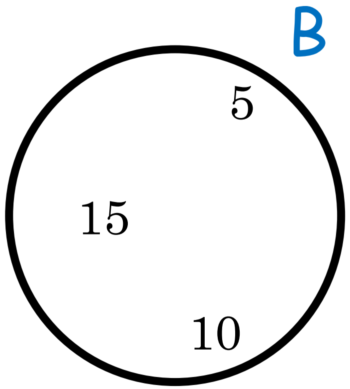

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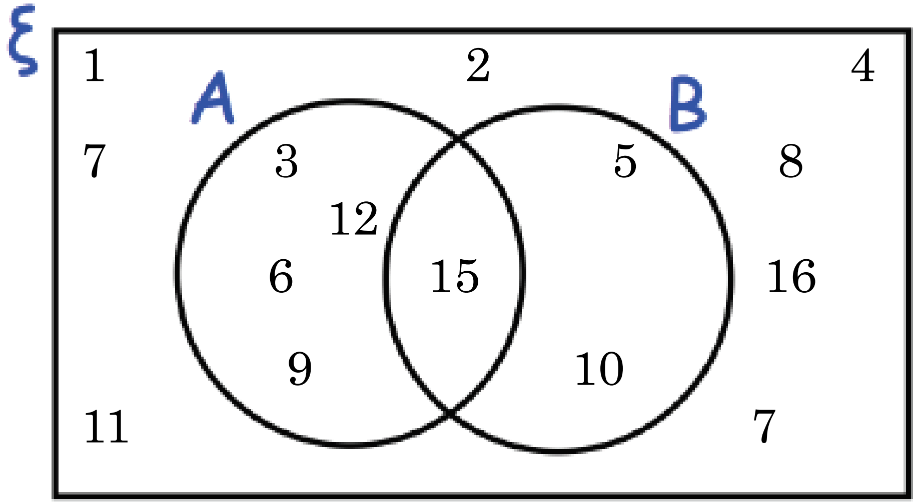

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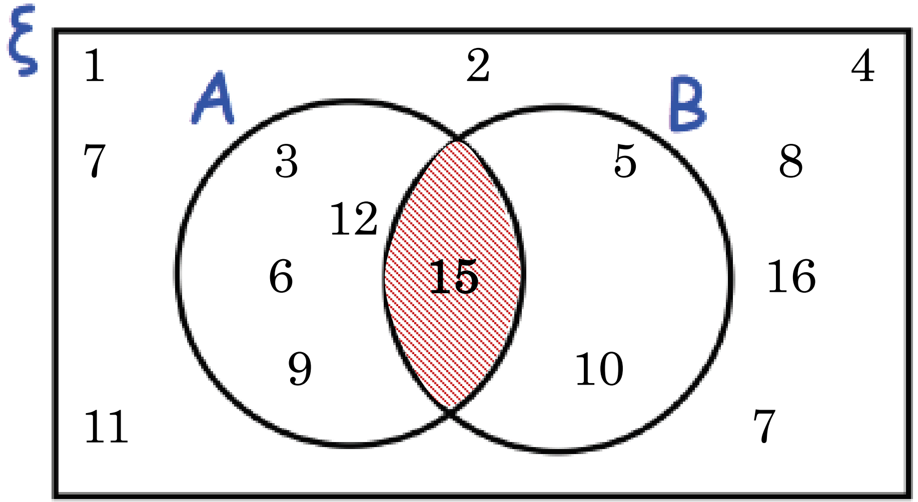
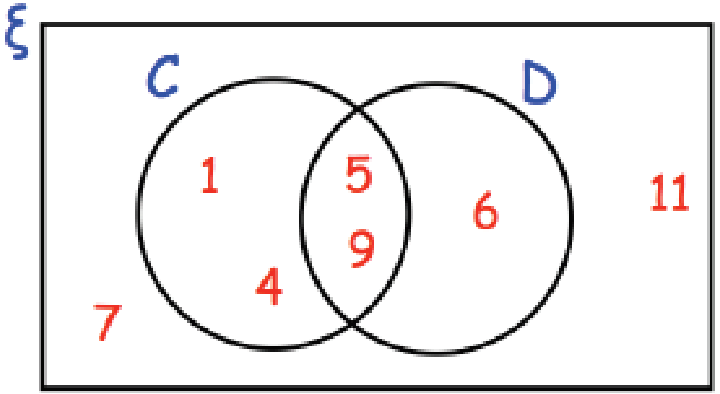


Exercise 2. Here is a Venn diagram



Write down the numbers that are in set

(a) D

$(6, 5, 9)$

(b) $C \cup D$

$C \cup D = (1, 4, 5, 9) \cup (5, 9, 6)$

$C + D - C \cap D = (1,4,5,9) + (5,9,6) - (5,9)$

$C \cup D = (1, 4, 5, 9, 6)$

(c) $\bar{C}$

$\bar{C} = \xi - C = (1, 4, 5, 9, 6, 7, 11) \cap (1, 4, 5, 9)$
$= (6, 7, 11)$

Exercise 3. There are 80 students in year 11.

9 students study French and German.
35 students only study French
2 students do not study French or German.

(a) Complete the Venn diagram

$n(\xi) = 80$

$n(G \cap F) = 9$

$n(F) - n(G \cap F) = 35 = n(\bar{G} \cap F)$

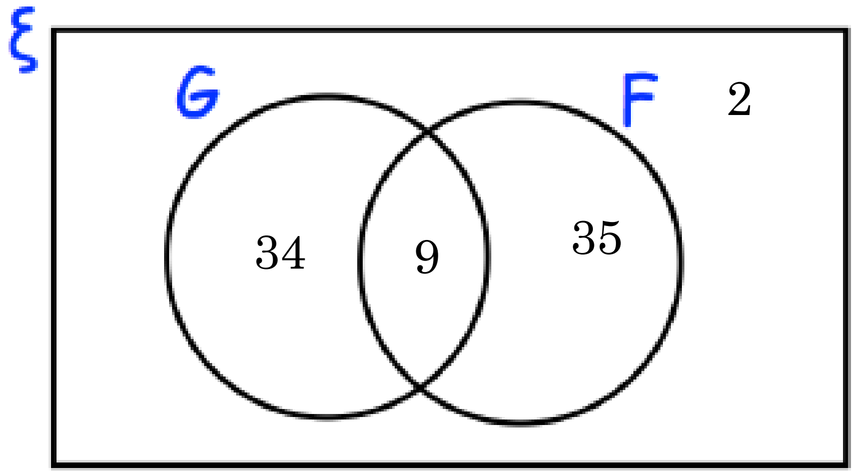
$n(F) = 35 + 9 = 44$

$n(\overline{G \cup F}) = 2$

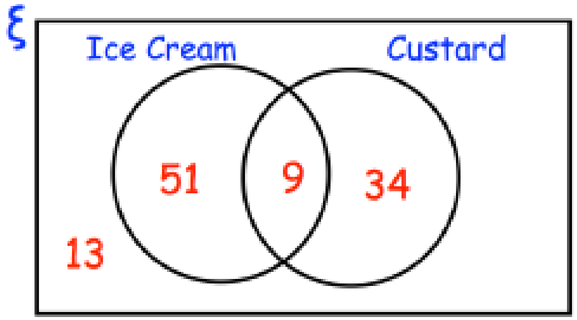
$n(G \cup F) = 80 - 2 = 78$

$n(G) = 78 - 35 = n(G \cup F) - n(\bar{G} \cap F) = 43$

$n(G) - n(G \cap F) = 43 - 9 = n(\bar{F} \cap G) = 34$



Exercise 4. At a wedding, the guests may have ice cream or custard with their dessert.
The Venn diagram shows information about the choices the guests made.



(a) How many guests had custard?

$35 + 9 = 44$

Event «Custard»

(b) How many guests had ice cream and custard?

9

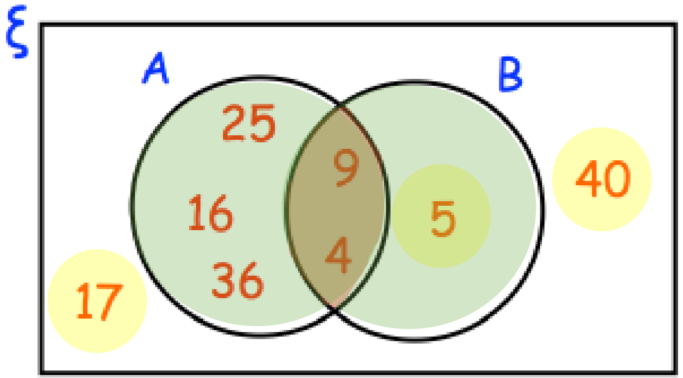
Intersection of Events «Custard» and «Ice Cream»

(c) How many guests went to the wedding?

$51 + 9 + 34 + 13 = 107$

Sample Size (S): «Desserts»

Exercise 5. Here is a Venn diagram.



Write down the numbers that are in set

(a) $A \cap B$ (9, 4)

(b) $A \cup B$ (25, 16, 36, 9, 4, 5)

(c) A^c or $\bar{A}$ (5, 17 , 40)

THEME #3

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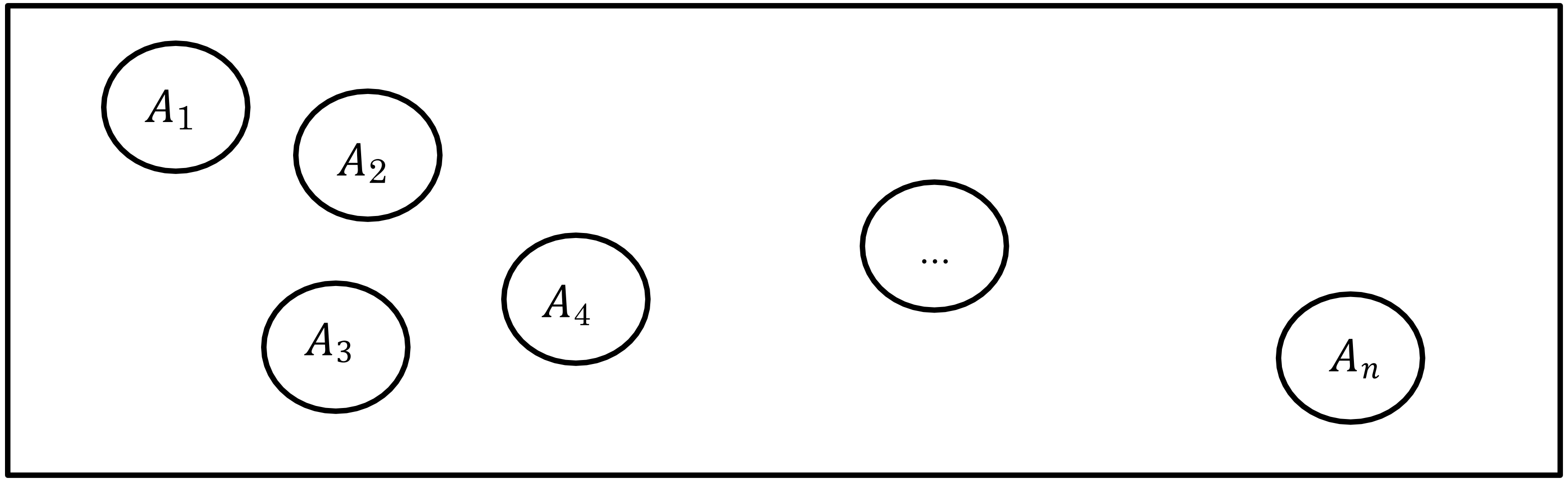
Counting principles

If the events are mutually distinct…

A_1, A_2, … A_k are events mutually distinct (no outcomes in common) with n_1 outcomes in event A_1, n_2 outcomes in event A_2, … n_k outcomes in event A_k:

The total number of outcomes in $\mathbf{A_1 \cup A_2 \cup \ldots \cup A_k}$ is $\mathbf{n_1 + n_2 + \ldots + n_k}$



This is called the **addition principle**

Fundamental Counting Principle

The **fundamental counting principle** states that if there are **p** ways to do one thing, and **q** ways to do another thing, then there are **p×q** ways to do both things.

Example 1:
Suppose you have 3 shirts (call them A, B, and C), and 4 pairs of pants (call them w, x, y, and z).
Then you have:

$$3 \times 4 = 12 \text{ possible outfits:}$$

Aw, Ax, Ay, Az,
Bw, Bx, By, Bz,
Cw, Cx, Cy, Cz

The **multiplication principle**

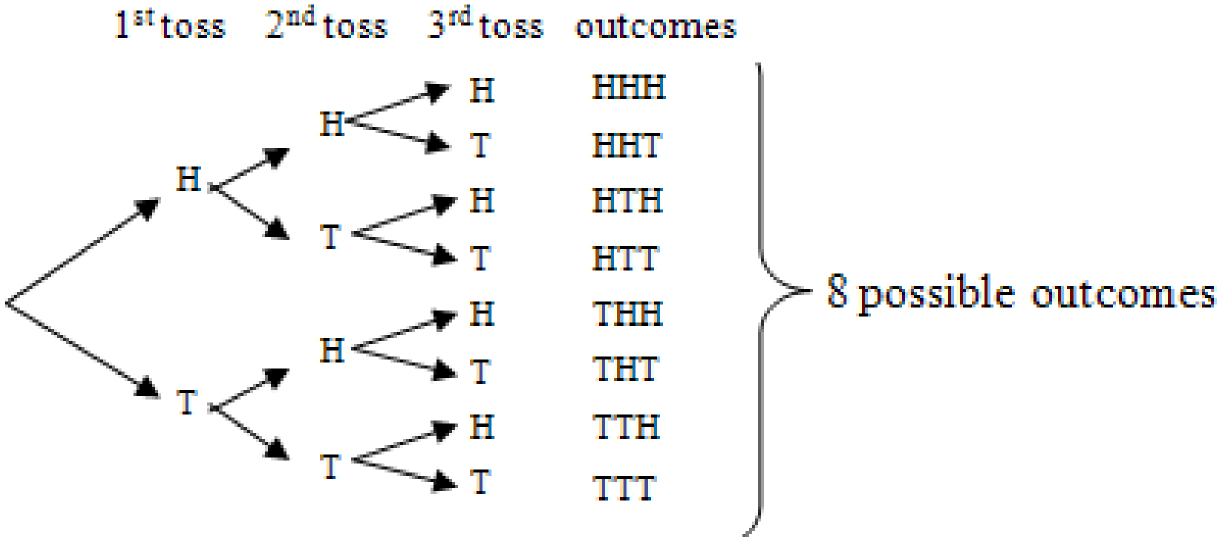
Example 2:
Suppose you roll a 6 sided die and draw a card from a deck of 52 cards. There are 6 possible outcomes on the die, and 52 possible outcomes from the deck of cards.
So, there are a total of:

$$6 \times 52 = 312$$

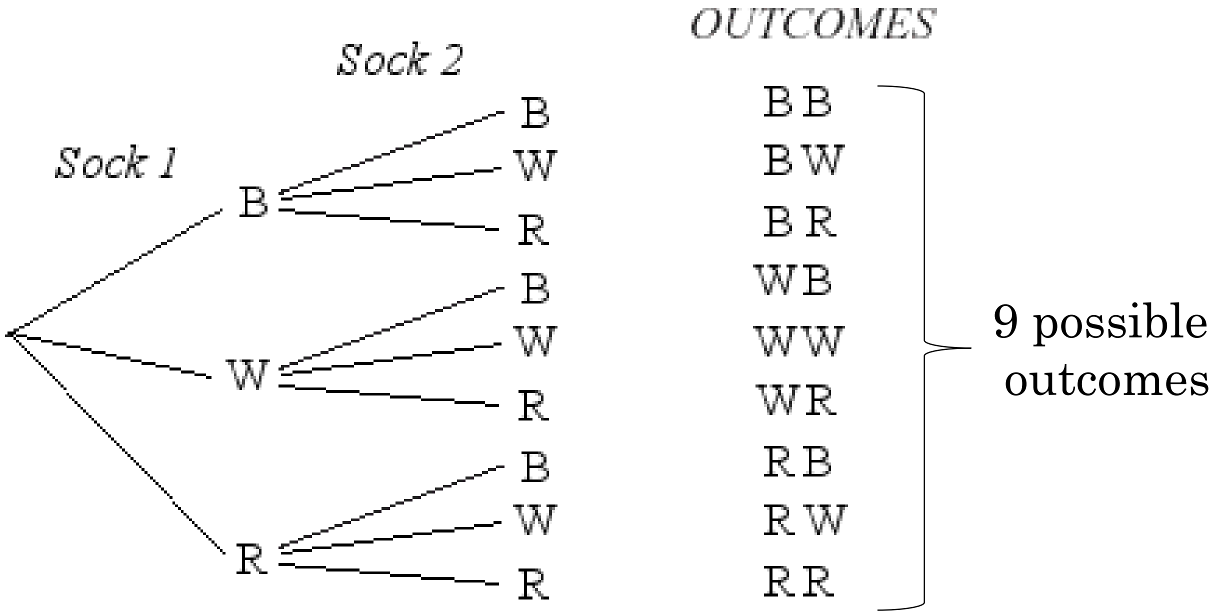
possible outcomes of the experiment.

The counting principle can be extended to situations where you have more than 2 choices. For instance, if there are **p** ways to do one thing, **q** ways to a second thing, and **r** ways to do a third thing, then there are **p×q×r** ways to do all three things.

Ex.1 - How many outcomes are there when flipping 3 coins?



$2 \times 2 \times 2 = 8$ WAYS



3 x 3 = 9 WAYS

Ex.2 - I have 6 socks of 3 colors (red, white, red). One at a time, I grab a sock from the drawer without a replacement. How many outcomes are there?

Counting rules

We use this rule to count the number of possible and favourable outcomes of an experiment.

Factorials: n! is represents the product of all the integers from n to 1.

$$n! = n\,(n-1)\,(n-2)\,(n-3)\,\ldots\,3 \cdot 2 \cdot 1$$

Dispositions with replacement

Total outcomes of an experiment = $n_1 \times n_2 \times n_3 \times \ldots n_k$ (multiplication principle)

$$\text{If } n_1 = n_2 = n_3 = \ldots = n_k \rightarrow \text{Total outcomes (k steps)} = n^k$$

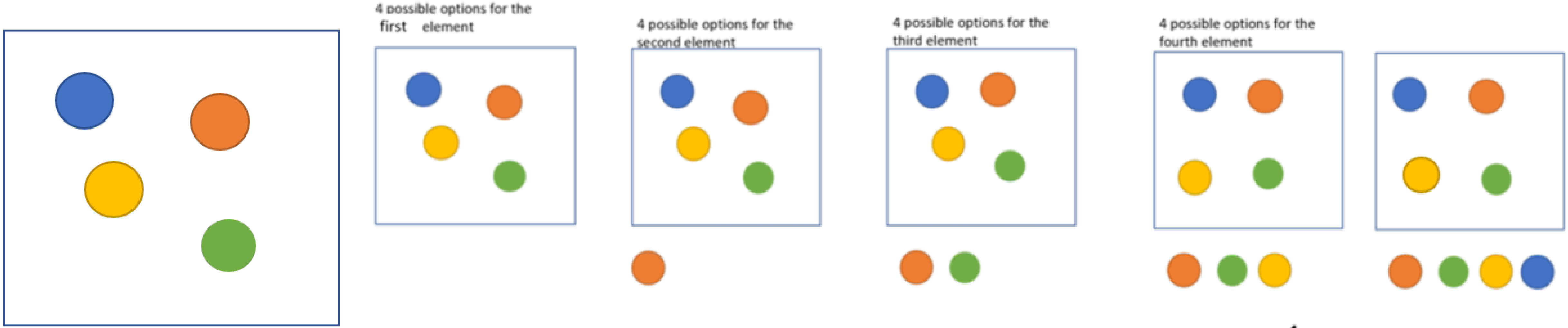
In case of selection of few elements from a group:

Combination: number of combination of n things, taken k at a time

$$_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

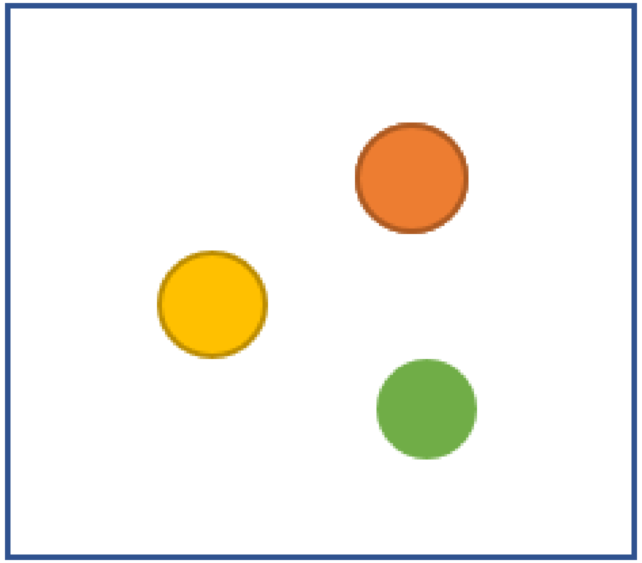
Combinations are dispositions where the order does not matter.

Dispositions with Repetition

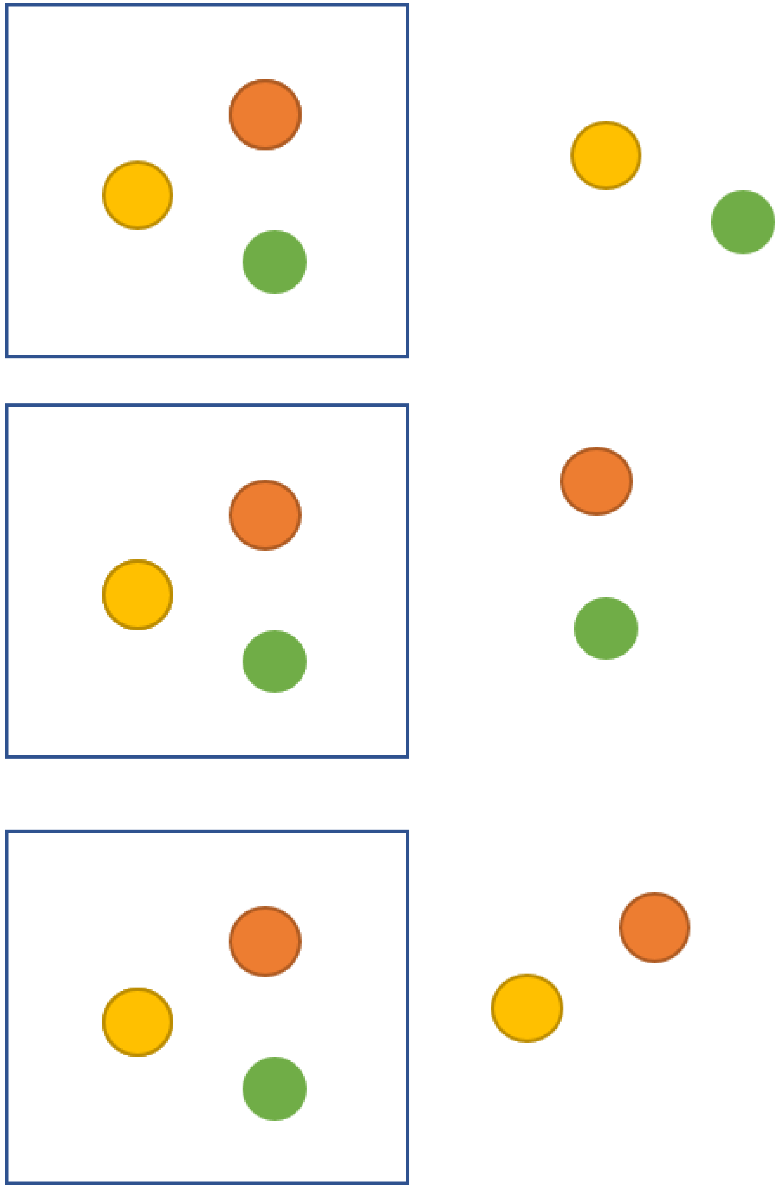


$$4 \quad * \quad 4 \quad * \quad 4 \quad * \quad 4 = 4^4$$

Combinations



$${}_{n}C_{k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$



$n=3$

$k=2$

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Question 1 A man just bought 4 suits, 8 shirts, and 12 ties. All of these suits, shirts, and ties coordinate with each other. If he is to randomly select one suit, one shirt, and one tie to wear on a certain day, how many different outcomes (selections) are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 4 \times 8 \times 12 = 384$

Question 2 A student is to select three classes for next semester. If this student decides to randomly select one course from each of eight economics classes, six mathematics classes, and five computer classes, how many different outcomes are possible?

Solution

Multiplication principle: Total outcomes = $n_1 \times n_2 \times n_3 = 8 \times 6 \times 5 = 240$

Question 3 An environmental agency will randomly select 4 houses from a block containing 25 houses for a radon check.

How many total selections are possible?

Solution

$${}_{25}C_4 = \frac{n!}{k!(n-k)!} = 25!/(4!\times21!) = (25\times24\times23\times22\times\cancel{21\times\ldots\times1})/(4\times3\times2\times\cancel{21\times20\times\ldots\times1}) = (25\times24\times23\times22)/(4\times3\times2) = 12.650$$

Combination

Question 4 You just got a free ticket for a boat ride, and you can bring along 2 friends! Unfortunately, you have 5 friends who want to come along.

How many different groups of friends could you take with you?

Solution

$${}_5C_2 = \frac{n!}{k!(n-k)!} = 5!/(2!\times3!)=(5\times4\times3\times2\times1)/(2\times1\times3\times2\times1)=5\times2=10$$

Question 5 Emily is packing her bags for her vacation. She has 6 shirts, but only 3 fit in her bag.

How many different groups of 3 shirts can she take?

Solution

$${}_6C_3 = 6!/(3!\times3!)=(6\times5\times4\times3\times2\times1)/(3\times2\times1\times3\times2\times1)=5\times4=20$$

Question 6 Suppose we have an office of 5 women and 6 men and need to select a 4 person committee. How many ways can we select

a) 2 men and 2 women?

Men (n=5) Woman (m=6)

k=2 Men: ${}_5C_2 = \frac{n!}{k!(n-k)!}$ = 5!/(2!×3!) = (5×4×3×2×1)/(2×1×3×2×1)=5×2=10

k=2 Woman: ${}_6C_2 = \frac{m!}{k!(m-k)!}$ = 6!/(2!×4!) = (6×5×4×3×2×1)/(2×1×4×3×2×1)=3×5=15

2 Men and 2 Woman: ${}_5C_2 \times {}_6C_2$ = 10 × 15 = 150

b) 3 men and 1 woman?

k=3 Men: ${}_5C_3 = \frac{n!}{k!(n-k)!}$ = 5!/(3!×2!) = (5×4×3×2×1)/(3×2×1×2×1)=5×2=10

k=1 Woman: ${}_6C_1 = \frac{m!}{k!(m-k)!}$ = 6!/(1!×5!) !)=(6×5×4×3×2×1)/(×1×5×4×3×2×1)=6

3 Men and 1 Woman: ${}_5C_3 \times {}_6C_1$ = 10 × 6 = 60

c) All women?

Men (n=5) Woman (m=6)

k=0 Men: ${}_5C_0 = \frac{n!}{k!(n-k)!}$ = 5!/(0!×5!)=1

k=4 Woman: ${}_6C_4 = \frac{m!}{k!(m-k)!}$ = 6!/(4!×2!) = (6×5×4×3×2×1)/(4×3×2×1×2×1)=3×5=15

All Woman: ${}_5C_0 \times {}_6C_4$ = 1 × 15 = 15