Bachelor in in Business Administration and Economics
Quantitative Methods
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Additional exercises on Bayes Theorem

Exercise 1.
A firm producing electronic devices distributes its products only after they pass the quality control. Historically the firm has observed that:
⇒ The probability that a device is defective is 0.1;
⇒ A defective device does not pass the quality control with probability 0.9.
⇒ A working device pass the quality control with probability 0.8.

Compute:
a) The probability that a randomly selected device passes the quality control
b) The probability that the device is working given that it passed the quality control

Solution
1) Spot the two events involved in the problem:
D = defective device
C = device passing the quality control

2) Assign the correct probabilities:
$P(D) = 0.1 \Rightarrow P(\bar{D}) = 1 - P(D) = 1 - 0.1 = 0.9$
$P(\bar{C} \mid D) = 0.9 \Rightarrow P(C \mid D) = 1 - 0.9 = 0.1$
$P(C \mid \bar{D}) = 0.8 \Rightarrow P(\bar{C} \mid \bar{D}) = 1 - 0.8 = 0.2$

3) Solve
a)
$$P(C) = P(C \mid D) * P(D) + P(C \mid \bar{D}) * P(\bar{D}) = 0.1 * 0.1 + 0.8 * 0.9 = 0.01 + 0.72 = 0.73$$
b)
$$P(\bar{D} \mid C) = \frac{P(C \mid \bar{D}) \times P(\bar{D})}{P(C)} = \frac{0.8 * 0.9}{0.73} = \frac{0.72}{0.73} = 0.9863$$

Excercise 2.
In a bar there are two slot machines. If you play with machine A you win with probability 1/2, while if you play with machine B you win with probability 1/4.
Suppose you do not know which one is machine A and B and you decide to play choosing one of them **randomly**. You win: what is the probability that you ended up playing with slot machine A?

Solution

1) Locate the events
A = you choose slot machine A (the complementary event, slot machine B, can either be denoted with $\bar{A}$ or with B)
W = you win

2) Attach probabilities

$$P(A) = \frac{1}{2} \Rightarrow P(\overline{A}) \text{ or } P(B) = \frac{1}{2}$$

$$P(W \mid A) = \frac{1}{2} \Rightarrow P(\overline{W} \mid A) = \frac{1}{2}$$

$$P(W \mid B) = \frac{1}{4} \Rightarrow P(\overline{W} \mid B) = \frac{3}{4}$$

3) Understand what is the required probability and solve

You are asked to find the probability of having chosen A given that you won, i.e.

$$P(A|W) = \frac{P(W \mid A) * P(A)}{P(W \mid A) * P(A) + P(W \mid B) * P(B)}$$

$$= \frac{\frac{1}{2} * \frac{1}{2}}{\frac{1}{2} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2}} = \frac{2}{3}$$

Exercise 3.

Consider the same situation of the exercise above. How would the solution change if you did not choose the slot machine probability of playing with slot machine B is 4/5?

Solution

1) Locate the events: same as above
2) Attach probabilities

$$P(B) = \frac{4}{5} \Rightarrow P(A) = \frac{1}{5}$$

$$P(W \mid A) = \frac{1}{2} \Rightarrow P(\overline{W} \mid A) = \frac{1}{2}$$

$$P(W \mid B) = \frac{1}{4} \Rightarrow P(\overline{W} \mid B) = \frac{3}{4}$$

3) Solve

$$P(A|W) = \frac{P(W \mid A) * P(A)}{P(W \mid A) * P(A) + P(W \mid B) * P(B)}$$

$$= \frac{\frac{1}{2} * \frac{1}{5}}{\frac{1}{2} * \frac{1}{5} + \frac{1}{4} * \frac{4}{5}} = \frac{1}{3}$$

Question: does it make sense that now this probability is lower than before?

Excercise 4.

80% of Tor Vergata students commute using the car, while the rest use the public transport service. Those commuting by car arrive on time for attending the lessons 60% of the times, while those commuting with public services, which rely on fast tracks, arrive on time with probability 90%.

A student arrives late in our class: what is the probability that he/she used the car?

Solution

1) Locate the events

C = student commutes with the car

L= student arrives late (i.e. does not arrive on time)

2) Attach probabilities

$P(C)=0.8 \Rightarrow P(\overline{C})=1-0.8=0.2$

$P(L \mid C)=1-0.6=0.4$

$P(L \mid \overline{C})=0.1 \Rightarrow P(\overline{L} \mid \overline{C})=0.9$

3)

$$P(C \mid L)=\frac{P(L \mid C) * P(C)}{P(L \mid C) * P(C)+P(L \mid \overline{C}) * P(\overline{C})}$$

$$=\frac{0.4 * 0.8}{0.4 * 0.8+0.1 * 0.2}=\frac{0.32}{0.34}=0.9412$$

Exercise 5.

10% of the students of a Statistic course attend classes exclusively in person, 40% attend all classes online, while the rest attend some classes online and some in person. Of those attending fully in person, 10% get a final mark above 28/30. Such evaluation is obtained by 5% of the students attending the course in blended modality, and by the 2% of the students attending the course online only. A student takes the Statistics exam and gets a 30 cum Laude. How is he/she more likely to have attended the course?

a) Fully online
b) Fully in presence
c) In blended modality

Solution

1) Locate the events

O = attending online only

P = attending in person only

B = blended modality

H = gets high mark (above 28/30)

2) Attach probabilities

$P(O)=0.4, P(P)=0.1 \Rightarrow P(B)=1-0.1-0.4=0.5$

$P(H \mid O)=0.02$

$P(H \mid P)=0.1$

$P(H \mid B)=0.05$

3)

$$P(O \mid H)=\frac{P(H \mid O) * P(O)}{P(H \mid O) * P(O)+P(H \mid P) * P(P)+P(H \mid B) * P(B)}=$$

$$=\frac{0.02 * 0.4}{0.02 * 0.4+0.1 * 0.1+0.05 * 0.5}=\frac{0.02 * 0.4}{0.043}=0.186$$

$$P(P \mid H)=\frac{P(H \mid P) * P(P)}{P(H)}=\frac{0.1 * 0.1}{0.043}=0.233 \rightarrow P(B \mid H)=1-0.186-0.233=0.581$$

The student getting a high mark has most likely attended the course in blended modality.

Exercise 6.
Tonight your friend Albert will watch Masterchef Australia with probability 10%, while the other friends Betty and Clark will watch it with probability 50% and 70%, respectively. During the evening, you get a message from one of them revealing the winner of tonight's episode. Whom is the message most likely?

Solution
Denoting with E = "friend watching the episode" and with A, B and C the events denoting the three friends. Then,

$P(E\,|\,A)=0.1$

$P(E\,|\,B)=0.5$

$P(E\,|\,C)=0.7$

You receive a message with the winner, hence the sending person has watched the episode. So, in order to answer you need to find which among P(A | E), P(B | E) and P(C | E) is the highest.

$$P(A|E)=\frac{P(A)*P(E\,|\,A)}{P(A)*P(E\,|\,A)+P(B)*P(E\,|\,B)+P(C)*P(E\,|\,C)}$$

➔
$$=\frac{\frac{1}{3}*0.1}{\frac{1}{3}*0.1+\frac{1}{3}*0.5+\frac{1}{3}*0.7}=\frac{\frac{0.1}{3}}{\frac{1.3}{3}}=\frac{0.1}{1.3}=0.077$$

$$P(B|E)=\frac{P(B)*P(E\,|\,B)}{P(A)*P(E\,|\,A)+P(B)*P(E\,|\,B)+P(C)*P(E\,|\,C)}$$

➔
$$=\frac{\frac{1}{3}*0.5}{\frac{1}{3}*0.1+\frac{1}{3}*0.5+\frac{1}{3}*0.7}=\frac{\frac{0.5}{3}}{\frac{1.3}{3}}=\frac{0.5}{1.3}=0.3846$$

➔
$$P(C\,|\,E)=1-P(A|E)-P(B|E)$$
$$=1-0.077-0.3846=0.5384$$

The message is most likely from your friend Clark.

Exercise 7.
70% of Tor Vergata students are satisfied with their Bachelor degree course. On average, 85% of the students satisfied with their bachelor degree graduate with a very high mark, while this happens only for 25% of those who are not satisfied with their bachelor degree.

a. Compute the probability that a randomly selected student will graduate with a good mark
b. A student graduated with a high mark: what is the probability that he/she is satisfied with the Bachelor degree course.

Solution

a) 0.67
b) 0.888

Exercise 8.
Every diagnosis test has a degree of specificity, i.e. the ability to correctly identify healthy patients (or, to have low chance of false positives), and a sensitivity, i.e. the ability to correctly identify the patients carrying the illness (i.e., to have low chance of false negatives). The covid-19 rapid tests have a specificity of 95% and a sensibility of 86%. Knowing that the prevalence of covid-19 infection is currently equal to 1.1%, compute the probability:

a) Of getting a positive covid-19 rapid test
b) Of actually carrying covid-19 infection given a positive covid-19 rapid test

Solution

1) Let's denote with T="rapid test is positive" and with P="person is actually positive".

2) Assign the correct probabilities

- P(P)=0.01 ➔ current prevalence of infection
- $P(\overline{T} \mid \overline{P}) = 0.95$ ➔ Specificity of rapid tests (with $P(T \mid \overline{P}) = 0.05$ chances of false positive)
- $P(T \mid P) = 0.86$ ➔ Sensibility of rapid tests (with $P(\overline{T} \mid P) = 0.14$ chances of false negatives)

3) Solve

a. Probability of getting a positive rapid test

$$P(T) = P(T \mid P) * P(P) + P(T \mid \overline{P}) * P(\overline{P})$$
$$= 0.86 * 0.01 + 0.05 * 0.99 = 0.0581$$

b. Probability of begin actually infected given a positive rapid test

$$P(P \mid T) = \frac{P(T \mid P) * P(P)}{P(T)} = \frac{0.86 * 0.01}{0.0581} = 0.148$$

Exercise 9.
Consider two boxes, containing 100 pencils each. Box 1 has 30 red pencils and 70 white, while Box 2 has 35 red and 65 black. You can extract a pencil from Box 1 only if tossing a coin three times you get three same outcomes (i.e. all three Heads or three Tails). Otherwise, you extract from Box 2.
Compute the following probabilities

1. You get a red pencil
2. You extracted from Box 2 given that you get a red pencil

Solution
First, consider the sample space of tossing the coin three times: it has 8 elements and only 2 have three identical outcomes, i.e. (TTT) and (HHH).
Hence, the extraction will be done from Box 1 with probability 2/8=1/4 and from Box 2 with probability 6/8=3/4.

1)
Let R = "you get a red pencil", U1="extraction from Box1" and U2 ="extraction from Box 2". Then, you are asked P(R), i.e.

$$= P(U_1) * P(R \mid U_1) + P(U_2) * P(R \mid U_2)$$
$$= \frac{1}{4} * \frac{30}{100} + \frac{3}{4} * \frac{35}{100} = \frac{30}{400} + \frac{105}{400}$$
$$= \frac{135}{400} = 0.3375$$

2) we are asked , i.e.

$$P(U_2 \mid R) = \frac{P(U_2)P(R \mid U_2)}{P(U_1)P(R \mid U_1) + P(U_2)P(R \mid U_2)}$$
$$= \frac{\frac{3}{4} * \frac{35}{100}}{\frac{1}{4} * \frac{30}{100} + \frac{3}{4} * \frac{35}{100}} = \frac{\frac{105}{400}}{\frac{135}{400}} = \frac{105}{400} * \frac{400}{135} = \frac{105}{135} = 0.7778$$

Exercise 10.

During the exam, a student is asked to answer to a multiple choice question with 4 alternatives. We know that the student knows the right answer with probability 40% and that if he/she does not know the answer will try one of the possible alternatives randomly. The students got the right answer: what is the probability that he/she actually know the answer?

Solution

S = "student knows the answer"
C = "student answers correctly"

$P(S) = 0.4$ => $P(\bar{S}) = 0.6$

$P(C \mid \bar{S}) = \frac{1}{4}$ ➔ if he/she does not know, answers randomly

$P(C \mid S) = 1$ ➔ if he/she knows…answers correctly for sure!

Then P(S | C) = 0.72