Quantitative Methods – I

Practice 9

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THEME #1

Sampling

Populations and Samples

- A **Population** is the set of all items or individuals of interest

Examples: All likely voters in the next election

All jet engines produced this year

All tax receipts over this year

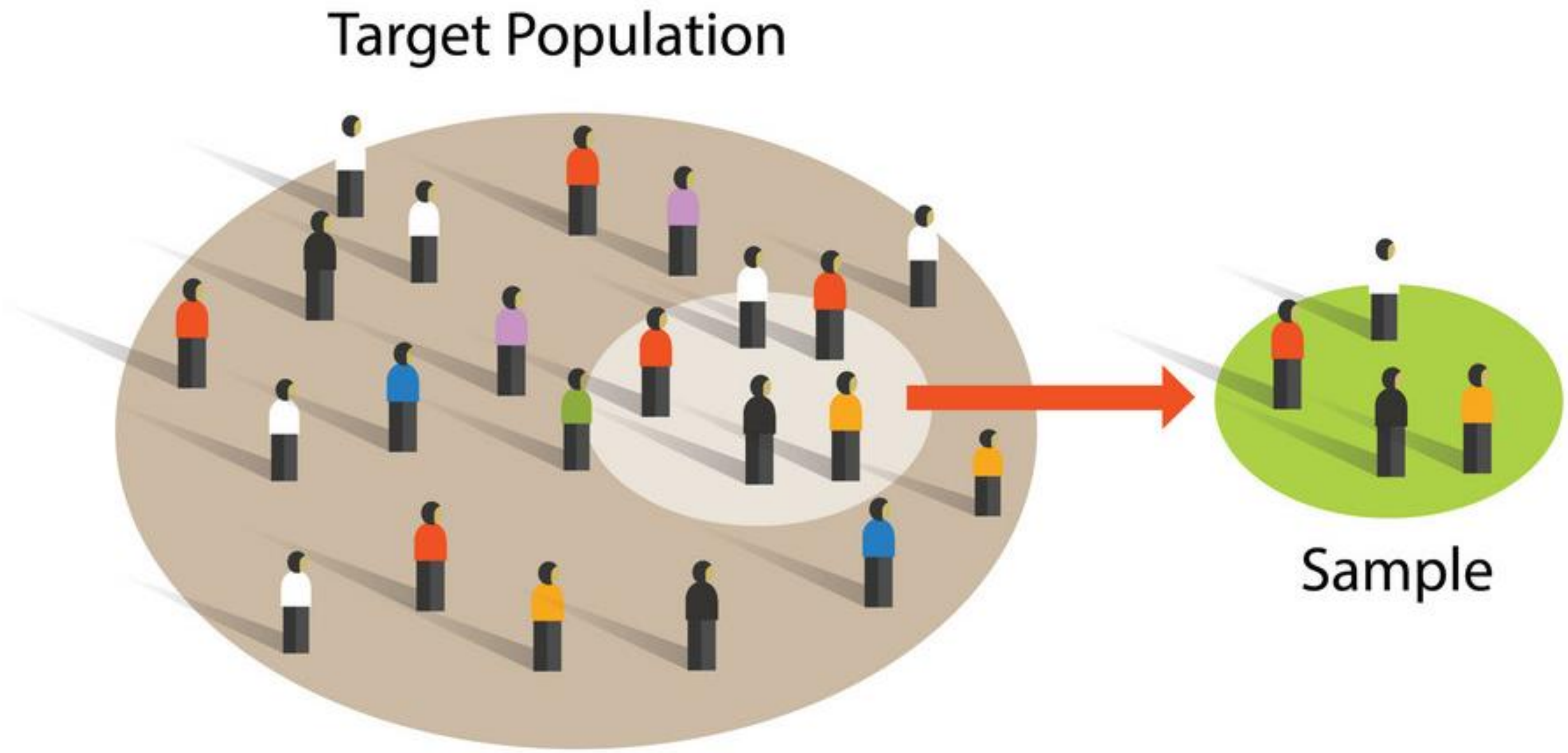
- A **Sample** is a subset of the population

Examples: 1,000 voters selected at random for interview

50 engines selected for testing

30 random receipts selected for audit

Population vs. Sample

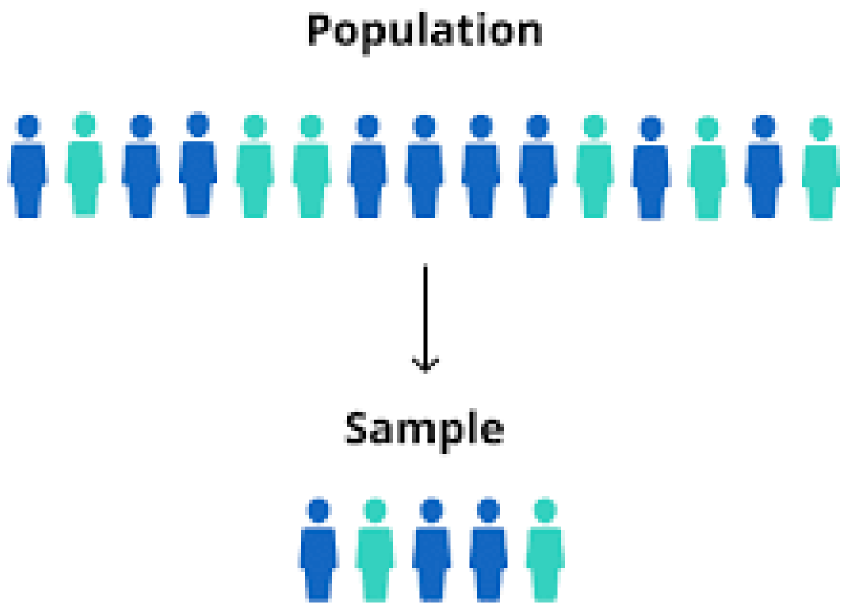


Census vs. (sample) Survey

In a **census**, data about all individual units (e.g. people or households) are collected in the **population**.

In a **survey**, data are only collected for a sub-part of the population; this part is called a **sample**.
These data are then used to estimate the characteristics of the whole population.

For example, the proportion of people below the age of 18 or the proportion of women and men in the selected sample of households has to reflect the reality in the total population.

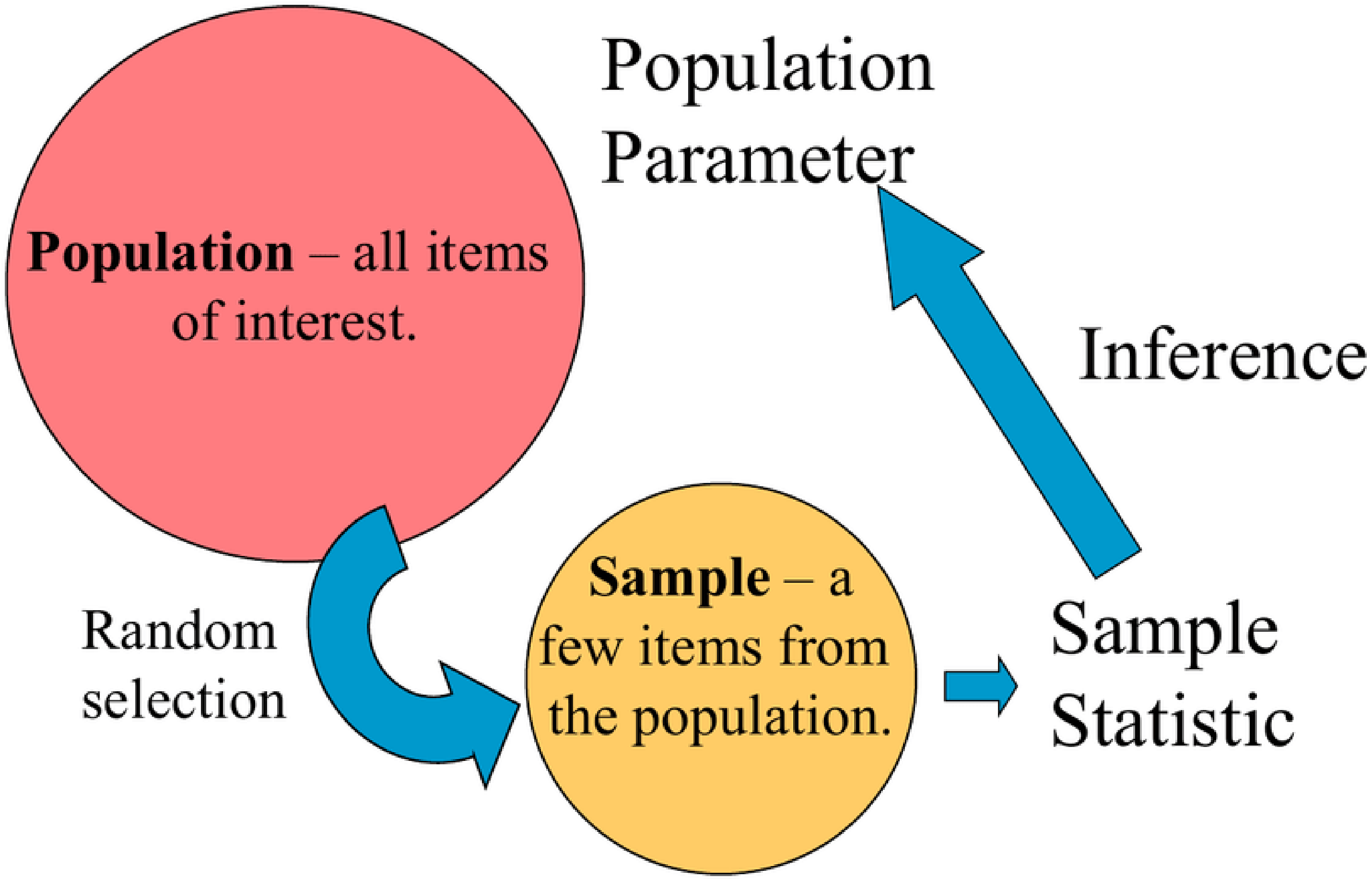


Why Sample?

- Less time consuming than a census
- Less costly to administer than a census
- It is possible to obtain statistical results of a sufficiently high precision based on samples.

Inferential Statistics

Making statements about a population by examining sample results



Inferential Statistics

Drawing conclusions and/or making decisions concerning a population based on sample results.

Main tools:

Estimation: the process of finding a value of a statistic derived from a sample to estimate the value of a corresponding population parameter.

- e.g., Estimate the population mean weight using the sample mean weight

Hypothesis Testing: the process to use sample evidence to test a claim or hypothesis.

- e.g., Use sample evidence to test the claim that the population mean weight is 60 kilos

Estimation

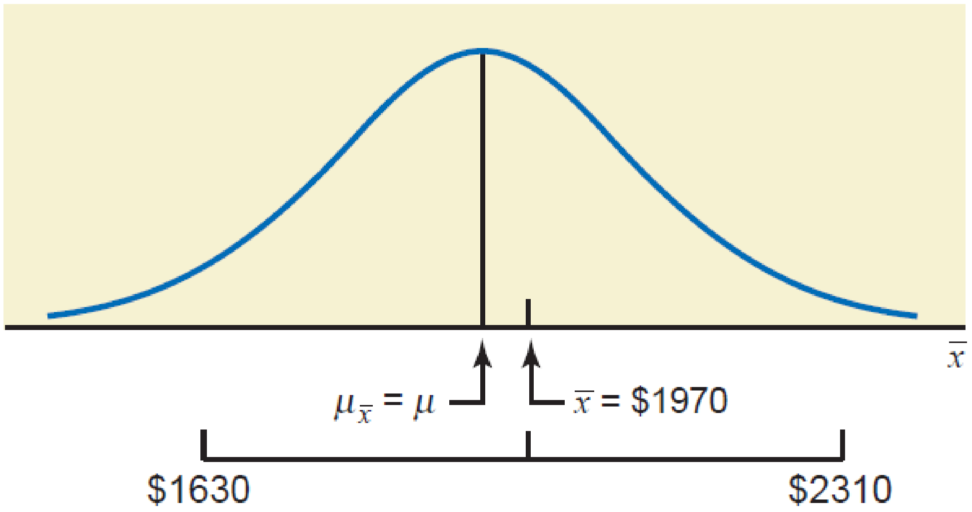
Point Estimate

The value assigned to a population parameter based on the value of a sample statistic is a point estimate. The sample statistic used to estimate a population parameter is called an estimator.

Ex. $\bar{x}$ is a point estimation of μ

Interval Estimate

Instead of assigning a single value to a population parameter, an interval is constructed around the point estimate, and then a probabilistic statement that this interval contains the corresponding population parameter is made.



THEME #2

Estimation of a Population mean

Expected Value of Sample Mean

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population

- That is, each observation is the realisation of a random variable X_i
- If the sample is random all X_i are independent and follow the same distribution.

The sample mean value of these observations is defined as

$$\boxed{\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

Standard Error of the Sample Mean

Different samples of the same size from the same population will yield different sample means

A measure of the variability in the mean from sample to sample is given by the Standard Error of the (sample) Mean:

$$\boxed{\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}}$$

Note that the standard error of the mean decreases as the sample size increases

Estimation of a Population Mean: σ Known

Three Possible Cases

Case I. If the following three conditions are fulfilled:

1. The population standard deviation σ is known
2. The sample size is small (i.e., $n < 30$)
3. The population from which the sample is selected is approximately normally distributed.

Case II. If the following two conditions are fulfilled:

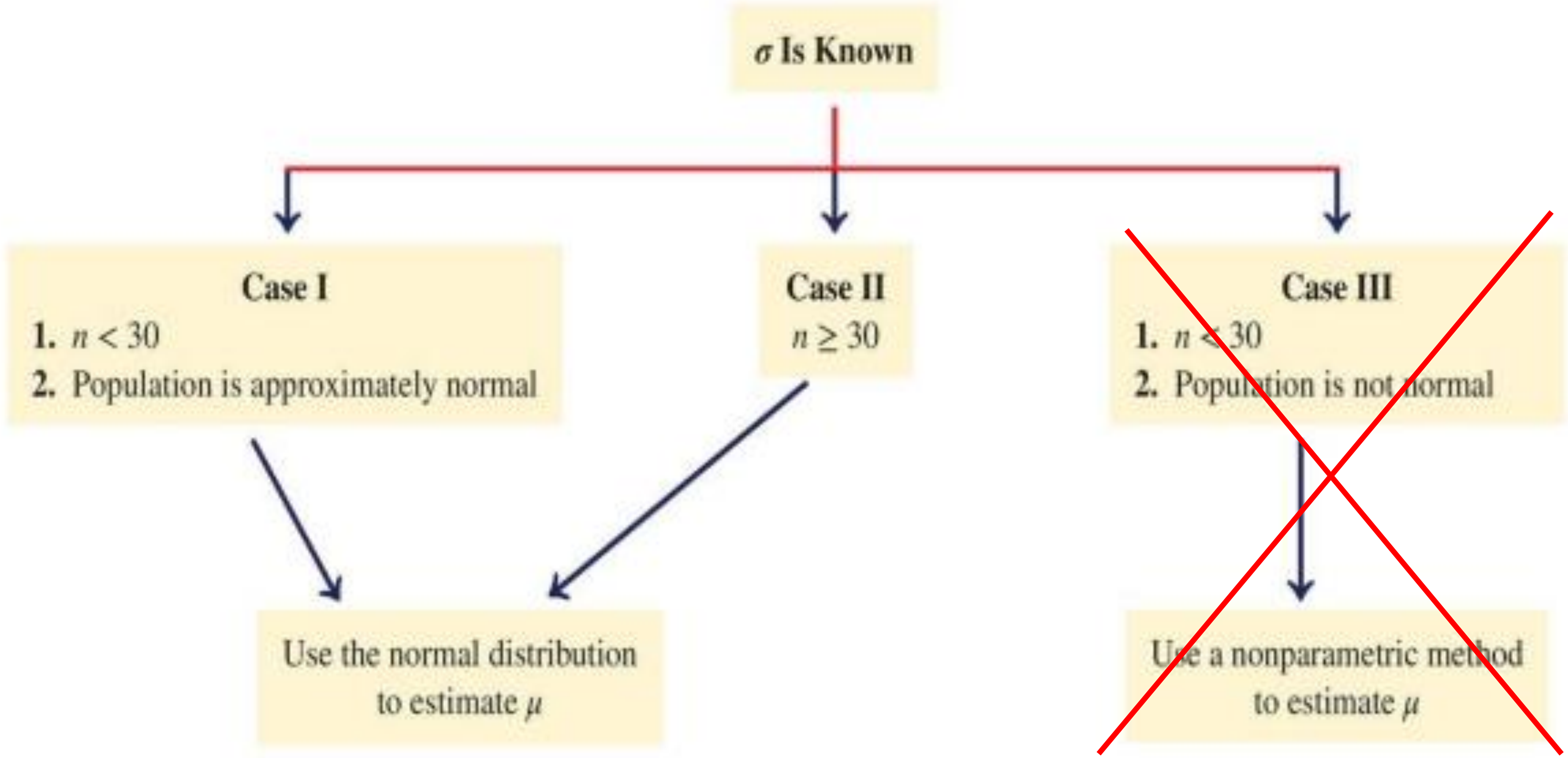
1. The population standard deviation σ is known
2. The sample size is large (i.e., $n \geq 30$)

Case III. If the following three conditions are fulfilled:

1. The population standard deviation σ is known
2. The sample size is small (i.e., $n < 30$)
3. The population from which the sample is selected is not normally distributed (or its distribution is unknown)

Estimation of a Population Mean: σ Known

Three Possible Cases



If the Population is Normal

- If a population is normal with mean μ and standard deviation σ, the sampling distribution of $\bar{X}$ is also normally distributed with

$$\boxed{\mu_{\bar{X}} = \mu}$$

and

$$\boxed{\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}}$$

X is a r.v. of population normally distributed:

$$X \sim N(\mu; \sigma^2)$$

$\bar{x}$ is a sample r.v. extract from a population normally distributed:

$$E(\bar{x}) = \mu_{\bar{x}} = \mu \qquad E(\sigma) = \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$\bar{x} \sim N\left(\mu_{\bar{x}}; \sigma_{\bar{x}}^2\right) = N(\mu; \frac{\sigma^2}{n})$$

Exercise a.1

The exam scores of all examinees is normally distributed with a **mean** of **26** and a **standard deviation** of **4**. Let $\bar{x}$ be the average score of a random sample of a set of examinees. Describe the shape of its sampling distribution when the sample size is:

a. n = 16
b. n = 100
c. n = 40,000

Solution
X is the average score of all the examinees and $\bar{x}$ is the average score of a random sample of a set of examinees.

X is normally distributed with mean μ = 26 and σ = 4

$$X \sim N(\mu; \sigma^2) = N(26; 16)$$

Also $\bar{x}$ is normally distributed with $\mu_{\bar{x}} = \mu = 26$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

$$\bar{x} \sim N\left(\mu_{\bar{x}}; \sigma_{\bar{x}}^2\right) = N(\mu; \frac{\sigma^2}{n})$$

Exercise a.1

The exam scores of all examinees is normally distributed with a mean of 26 and a **standard deviation** of **4**. Let $\bar{x}$ be the average score of a random sample of a set of examinees. Describe the shape of its sampling distribution when the sample size is:

a. $n = 16$
b. $n = 100$
c. $n = 40{,}000$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$\bar{x} \sim N\left(\mu_{\bar{x}}; \sigma_{\bar{x}}^2\right) = N\left(\mu; \frac{\sigma^2}{n}\right)$$

Solution

So,

a. $\mu_{\bar{x}} = \mu = 26$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{16}} = 4 / 4 = 1$

$$\bar{x} \sim N(26; 1)$$

b. $\mu_{\bar{x}} = \mu = 26$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{100}} = 4 / 10 = 0.4$

$$\bar{x} \sim N(26; 0.4^2)$$

c. $\mu_{\bar{x}} = \mu = 26$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{40{,}000}} = 4 / 200 = 0.02$

$$\bar{x} \sim N(26; 0.02^2)$$

Z-value for Sampling Distribution of the Mean

Z-value for the sampling distribution of $\bar{X}$:

$$Z = \frac{(\bar{X} - \mu_{\bar{X}})}{\sigma_{\bar{X}}} = \frac{(\bar{X} - \mu)}{\frac{\sigma}{\sqrt{n}}} = \frac{(\bar{X} - \mu)}{\sigma / \sqrt{n}}.$$

where: $\bar{X}$ = **sample mean**

μ = population mean

σ = population standard deviation

n = sample size

Exercise a.2

The weight of the packages of a certain brand of cookies is normally distributed with $\boldsymbol{\mu}$ = **130** and $\boldsymbol{\sigma}$ = **3.3** grams. Find the probability that the average weight of a random sample of **20** packages is between 131 and 132 grams.

Solution

$$Z = \frac{(\overline{X} - \mu_{\overline{X}})}{\sigma_{\overline{X}}} = \frac{(\overline{X} - \mu)}{\sigma / \sqrt{n}}.$$

If $X \sim N(\mu = 130;\ \sigma^2 = 3.3^2 = 10.89)$

$\bar{x} \sim N(\mu, \sigma^2/n) = N(\mu = 130,\ \sigma_{\bar{x}}^2 = 10.89/20 = 0.5445)$ and $\sigma_{\bar{x}} = 0.74$

$P(131 < \bar{x} < 132)$, than standardize, $P(131\text{-}130/0.74 < z < 132\text{-}130/0.74) = P(1.35 < z < 2.70) =$

$= P(z < 2.7) - P(z<1.35) = 0.9965 - 0.9115 = 0.085 = 8.5\%$

Exercise a.3

The daily food expenditure of the Italian population has a normal distribution with **average** € **25**.00, and **variance** € **4**.00. A sample of **10** units is extracted with re-entry. Find the probability that $\bar{x}$ is greater than 27.

Solution

X is the daily food expenditure, the text indicates that $X \sim N(25, 4)$.

$\bar{x} \sim N(\mu, \sigma^2/n)$, or $\bar{x} \sim N(25, 4^2/10)$.

$P(\bar{x} > 27) = P(z > \frac{27-25}{2/\sqrt{10}}) = P(z > 3{,}16) = 1 - P(Z \leq 3.16) = 1 - 0.9992 = 0.0008 = 0.08\%$

Exercise a.4

Scores on a common final exam in a large enrollment, multiple-section freshman course are normally distributed with **mean 72.7** and **standard deviation** 13.1. Find the probability that the score X on a randomly selected exam paper is between 70 and 80. Find the probability that the mean score $\bar{x}$ of n=**38** randomly selected exam papers is between 70 and 80.

Solution

P(70<X<80)= P(70 < $\frac{x-\mu}{\sigma}$ < 80) = P(-2.7/13.1 < z < 7.3/ 13.1) = P(-0.21<z<0.56) = 0.7123 – 0.4168 = 0.2955 = =29.55%

P(70<$\bar{x}$<80) = P(70 < $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ < 80) = P($\frac{-2.7}{13.1/6.16}$ < z < $\frac{7.3}{13.1/6.16}$) = P(-2.7/2.13<z<7.3/2.13) = P(-1.27<z<3.43) = =0.9997 – 0.1020 = 0.8977 = 89.77%

Exercise a.5

John eats at the same fast food restaurant every day. Suppose the time X between the moment John enters the restaurant and the moment he is served his food is normally distributed with **mean 4.2** minutes and **standard deviation 1.3** minutes.

a. Find the probability that when he enters the restaurant today it will be less than 5 minutes until he is served.
b. Find the probability that average time until he is served in **four** randomly selected visits to the restaurant will be more than 5 minutes.

Solution

a. P(X<5)= P($\frac{x-\mu}{\sigma}$ < 5) = P(z < 0.8/1.3) = P(z< 0.62) = 0.7324 = 73.24%

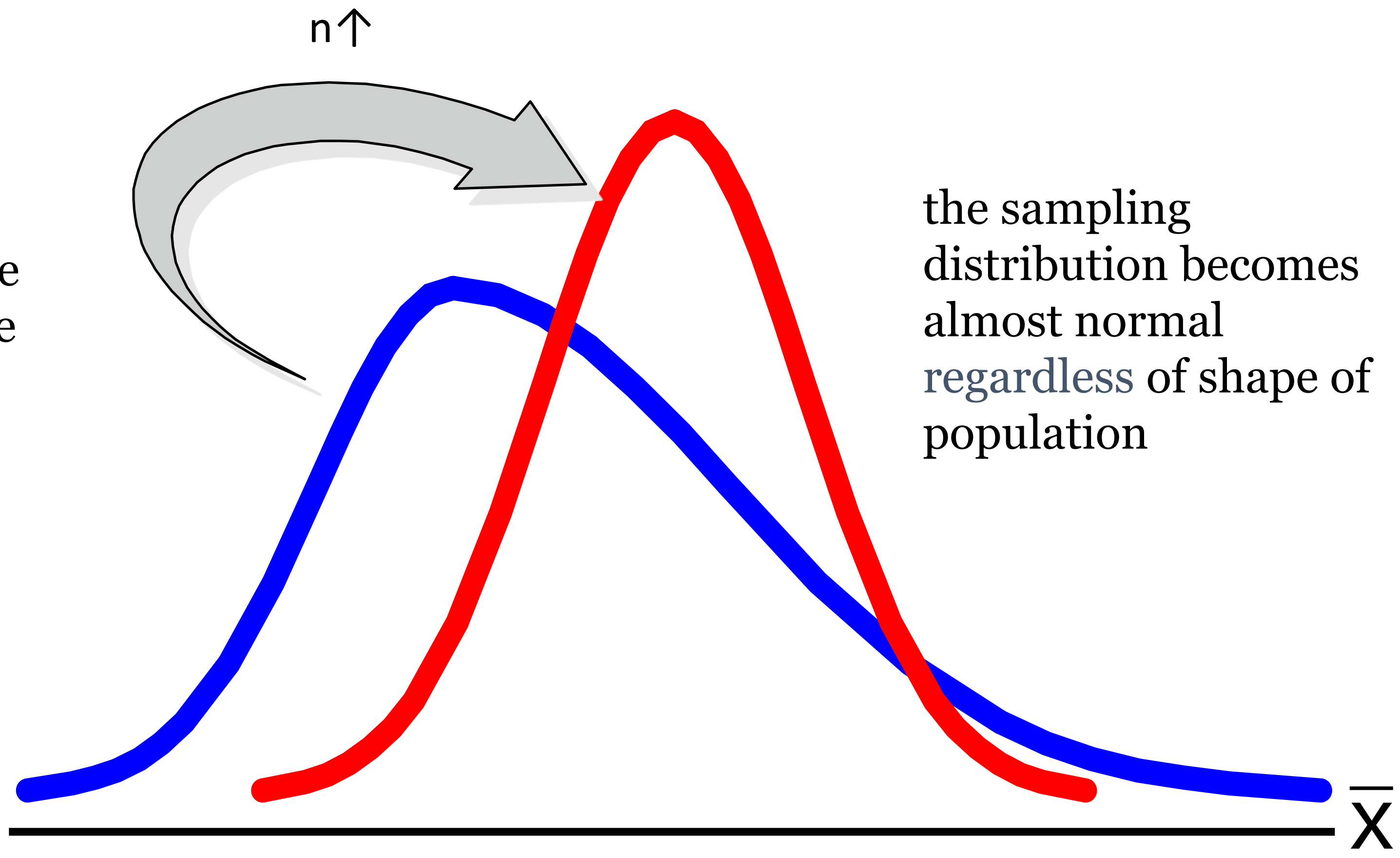
b. P($\bar{x}$ > 5) = P($\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ > 5) = P(z > 0.8/ 1.3/2) = 1 - P(z<1.23) = 1 - 0.8907 = 0.1093 = 10.93%

And if the Population is **not** Normal?

We can apply the Central Limit Theorem:

- Even if the population is not normal,
- ...sample means from the population will be approximately normal as long as the sample size is large enough.

As the sample size gets large enough...



the sampling distribution becomes almost normal regardless of shape of population

How Large is Large Enough?

- For most distributions, $n > 25$ will give a sampling distribution that is nearly normal
- For normal population distributions, the sampling distribution of the mean is always normally distributed

Ex.3. Suppose a population has mean $\mu = 8$ and standard deviation $\sigma = 3$. Suppose a random sample of size $n = 36$ is selected.

What is the probability that the sample mean is between 7.8 and 8.2?

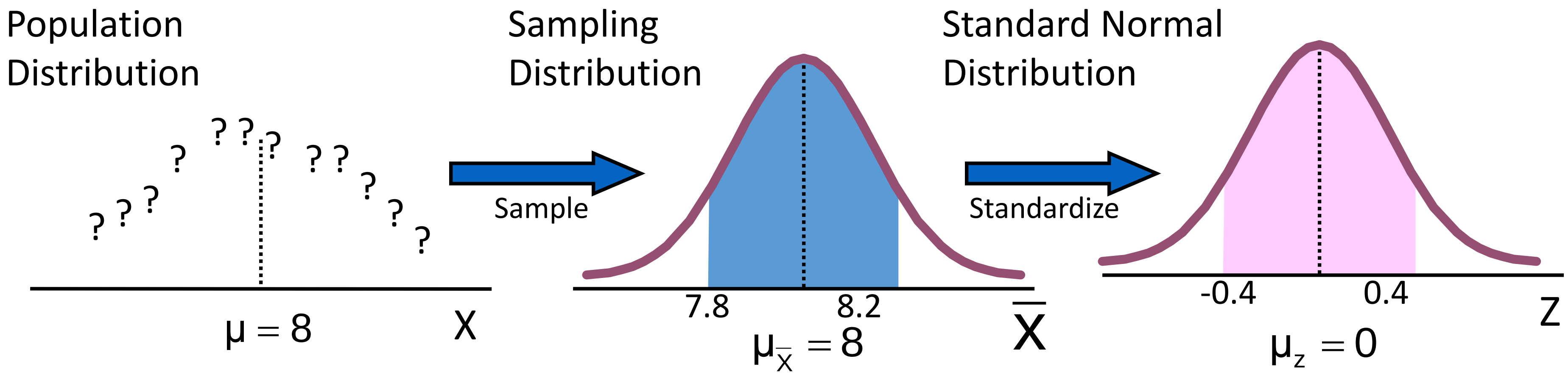
Solution

Even if the population is not normally distributed, the central limit theorem can be used (n > 25)

So the sampling distribution of $\bar{X}$ is approximately normal,

with mean $\mu_{\bar{x}} = 8$ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{36}} = 0.5$

$$P(7.8 < \mu_{\bar{X}} < 8.2) \approx P\left(\frac{7.8-8}{3/\sqrt{36}} < \frac{\mu_{\bar{X}} - \mu}{\sigma/\sqrt{n}} < \frac{8.2-8}{3/\sqrt{36}}\right) = P(-0.4 < Z < 0.4) = 0.3108$$



Population Proportions, p

p = the proportion of the population having some characteristic

Sample proportion ($\hat{p}$) provides an estimate of p:

$$\hat{p} = \frac{X}{n} = \frac{\text{number of items in the sample having the characteristic of interest}}{\text{sample size}}$$

- $0 \leq \hat{p} \leq 1$
- $\hat{p}$ has a binomial distribution, but can be approximated by a normal distribution when $np(1 - p) > 9$

Z-Value for Proportions

Standardize $\hat{p}$ to a Z value with the formula:

$$Z = \frac{\hat{p} - p}{\sigma_{\hat{p}}} = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

Ex.4. If the true proportion of voters who support a metro line in Dublin is **p = .70**, what is the probability that a sample of size **200** yields a sample proportion between 0.70 and 0.75?

Solution

If p = 0.7 and n = 200, what is $P(0.70 \le \hat{p} \le 0.75)$?

$$Z = \frac{\hat{p}-p}{\sigma_{\hat{p}}} = \frac{\hat{p}-p}{\sqrt{\dfrac{p(1-p)}{n}}}$$

Find $\sigma_{\hat{p}}$:

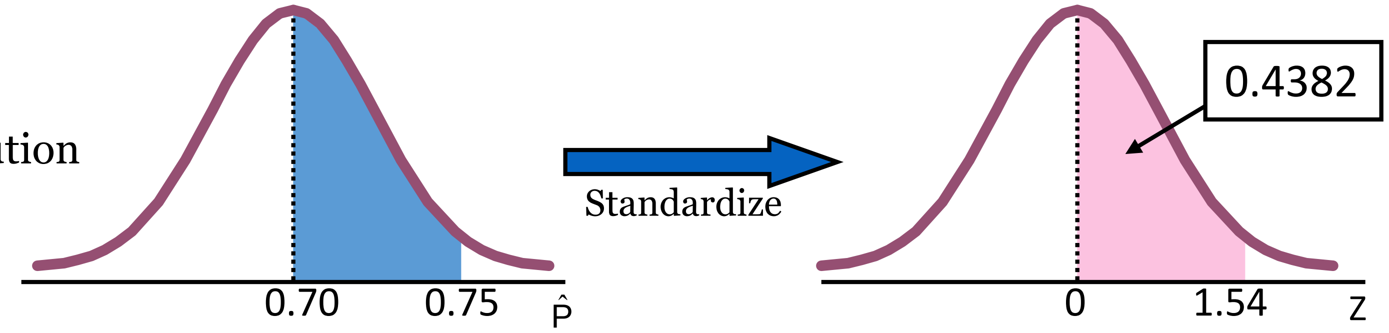
$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{.7(1-.7)}{200}} = .0324$$

Convert to standard normal: $P(.70 \le \hat{p} \le .75) \approx P\left(\frac{.70-.70}{.0323} \le Z \le \frac{.75-.70}{.0324}\right) = P(0 \le Z \le 1.54)$

Use standard normal table:

$$P(0 \le Z \le 1.54) = 0.4382$$

Sampling Distribution



Standardized Normal Distribution

Estimation of a Population Mean: <u>σ Not Known</u>

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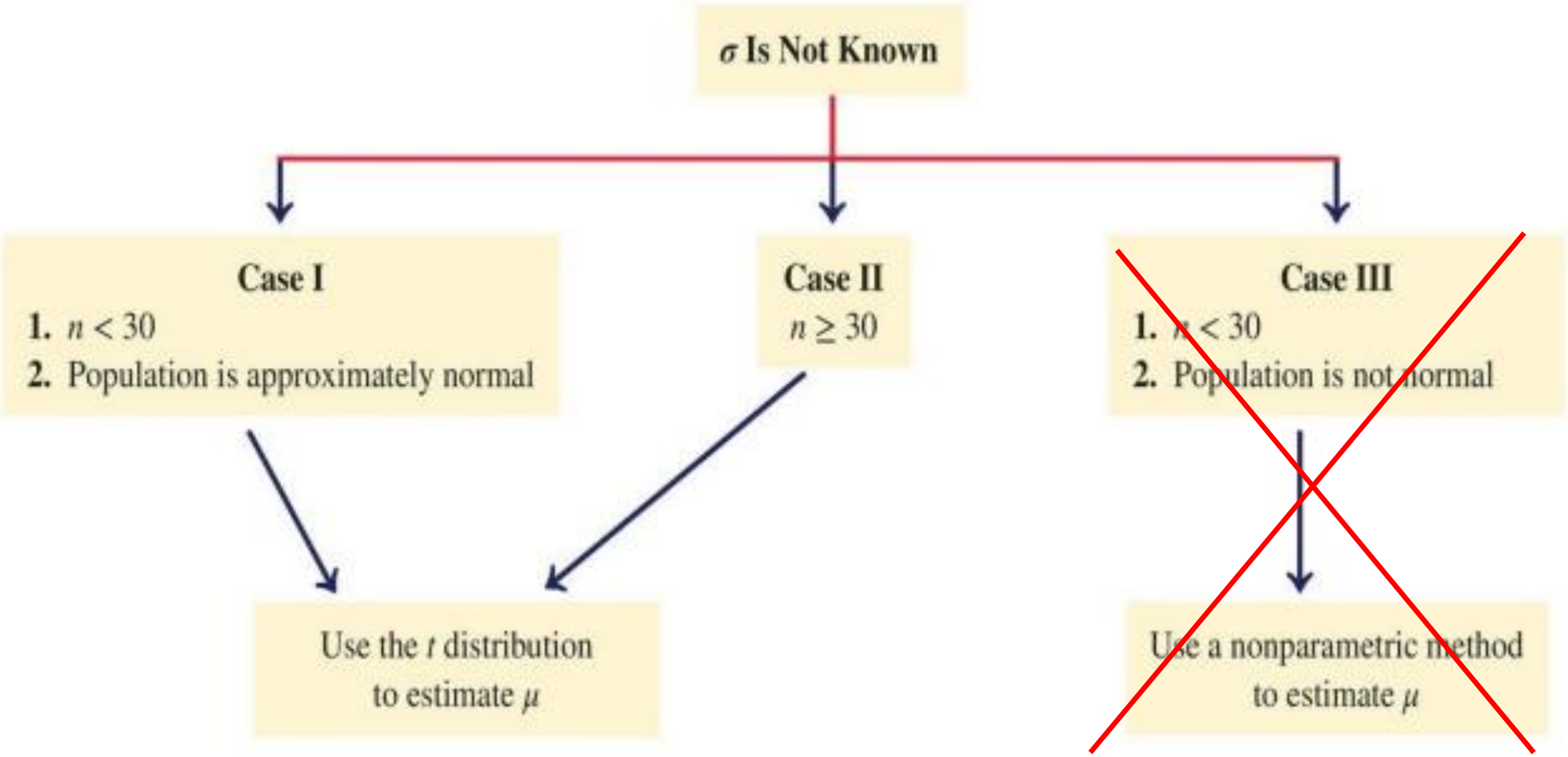
Estimation of a Population Mean: σ Not Known

Three Possible Cases



THEME #3

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The T distribution

The t Distribution

The t distribution is a specific type of bell-shaped distribution with a lower height and a greater spread than the standard normal distribution.

As the sample size becomes larger, the t distribution approaches the standard normal distribution.

For $n > 60$, X converges to $Z \sim N(0,1)$

The t distribution has only one parameter, called the degrees of freedom (df).

$$df = n - 1$$

The t distribution is symmetric.

The mean is equal to 0 and its standard deviation is:
$\sqrt{df\,/(df\ -\ 2)}$.

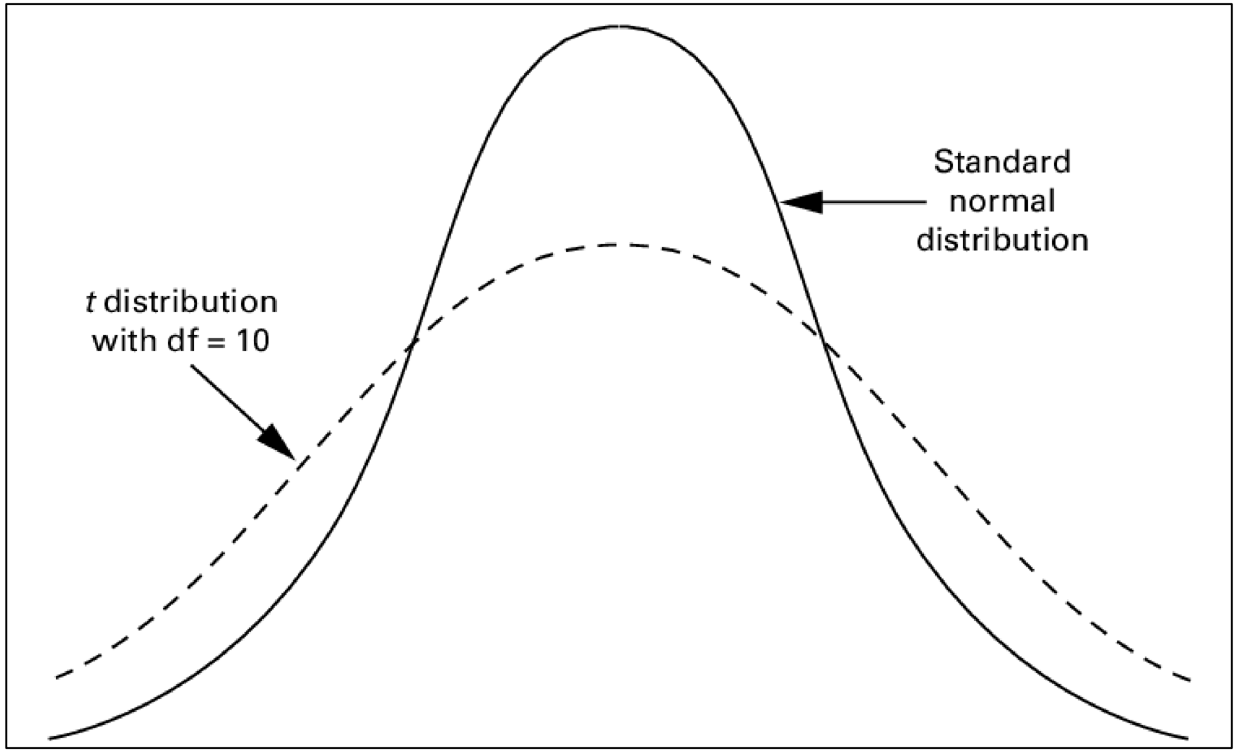
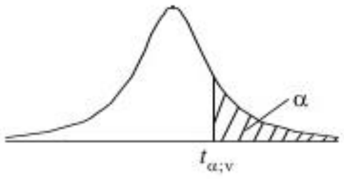


Table of the Student's t-distribution

The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom



ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Ex. 5 Find the value of **t** for **16 degrees of freedom** and an area of **0.05** in the right tail of a t distribution curve.

Solution

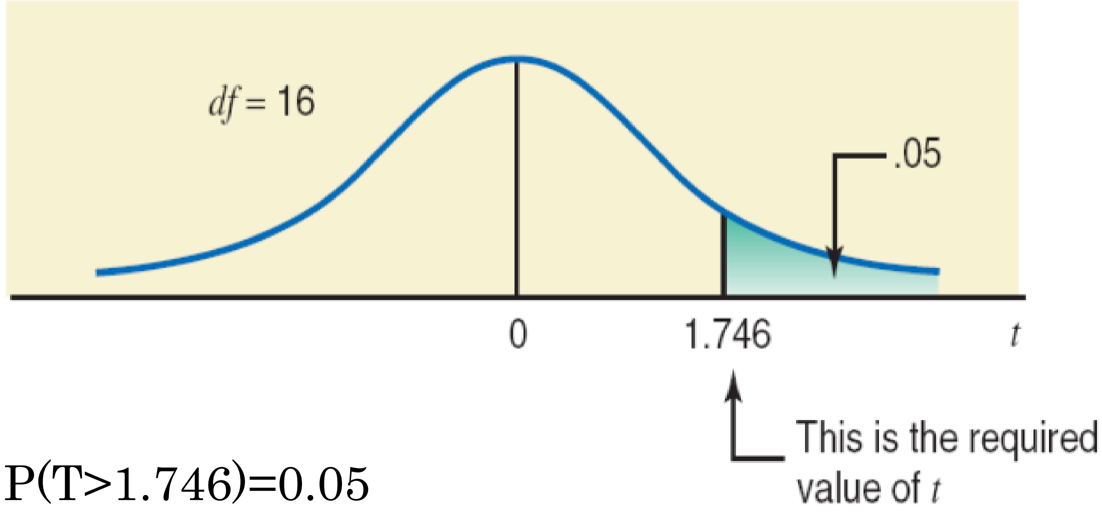
Find in the table of the t distribution for 16 degrees of freedom the t value for an area of 0.05.

The t value that gave to us an area **from t to +∞** equal to 0.05 is 1.746.

Area in the right tail

df	Area in the Right Tail Under the *t* Distribution Curve				
	.10	.05	.025	. . .	.001
1	3.078	6.314	12.706	. . .	318.309
2	1.886	2.920	4.303	. . .	22.327
3	1.638	2.353	3.182	. . .	10.215
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
df → 16	1.337	1.746	2.120	. . .	3.686
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
.	. . .	. . .	. . .	. . .	. . .
75	1.293	1.665	1.992	. . .	3.202
∞	1.282	1.645	1.960	. . .	3.090

The required value of *t* for 16 *df* and .05 area in the right tail

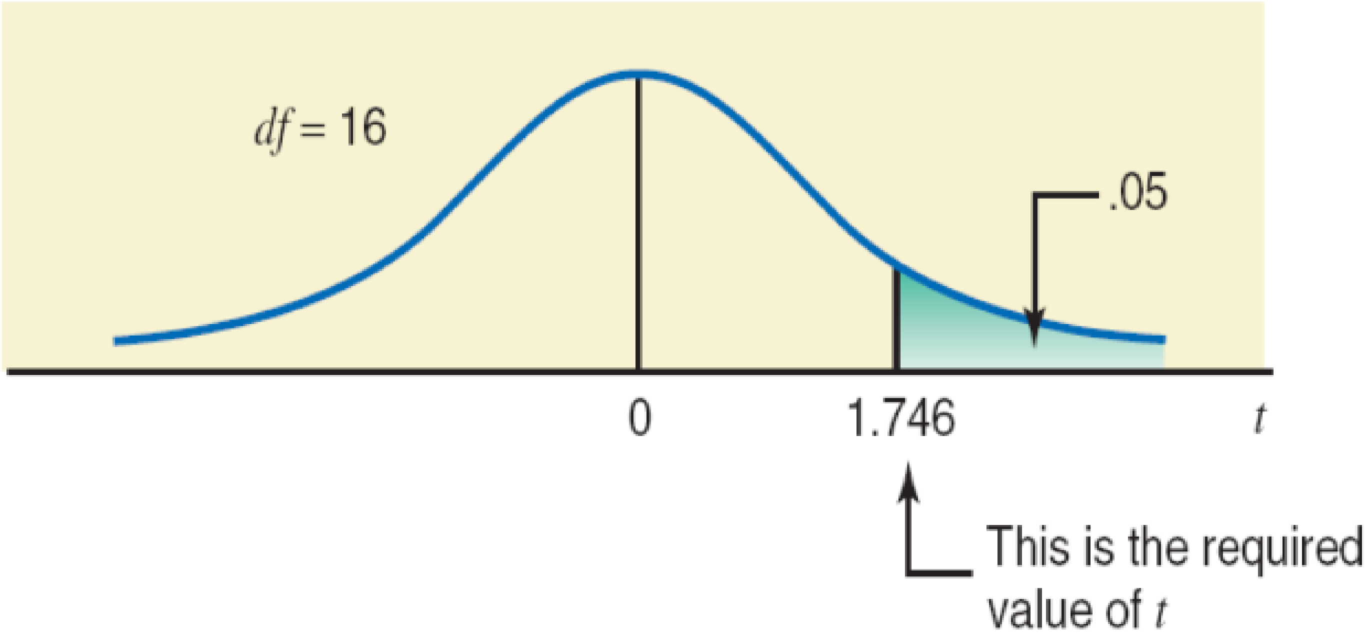


P(T>1.746)=0.05

Ex. 5 Find the value of t for 16 degrees of freedom and an area of 0.05 in the right tail of a t distribution curve.

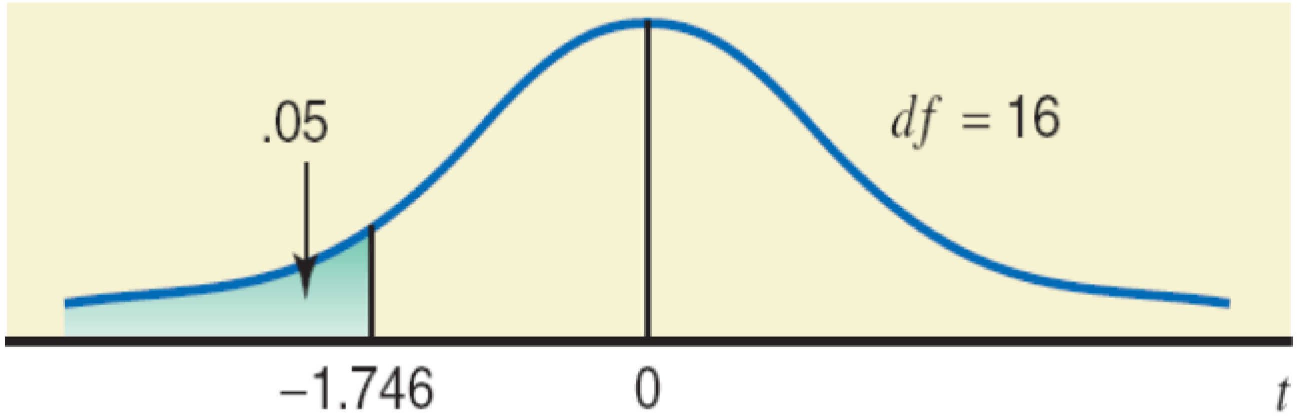
Solution

$P(T>1.746)=0.05$



The t distribution is simmetric

Hence the area **from** $[-\infty;\ \mathbf{-1.746}]$ is also 0.05



Exercise 6

a. Find the value of **t** for **24** df and **0.05** area in the right tail (*one-sided*);

b. Find the value of **t** for **24** df and **0.05** in the ***two-sided*** area ;

c. Find the **probability** that a r.v. t for **24** df assumes a value greater than **2.5**;

d. Find the **probability** that a r.v. t for **24** df assumes a value greater than **1.5**;

Solution

a. We have to find the 95% percentile of the t di Student with 24 df : $t_{24,0.05}$.

   From the table the t value is: 1.711

b. We have to find the 2.5% e 97.5% percentile that border the two-sided area that leaves in the tail a probability of 5%.

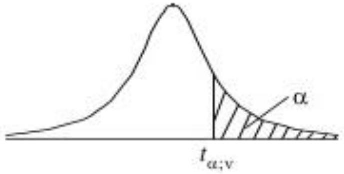
   From the table $t_{24,0.025} = 2.064$ and $t_{24,0.975} = -2.064$

c. If $X \sim t_{24}$ then, $P(X > 2.5)=0.01$

d. If $X \sim t_{24}$ then, $0.05 \leq P(X > 1.5) \leq 0.1$

Table of the Student's *t*-distribution

The table gives the values of $t_{\alpha;\nu}$ where $\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom



Ex 6 d

Ex 6 c

Ex 5

Ex 6 a

Ex 6 b

ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
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5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

THEME #4

The Chi square and F distributions

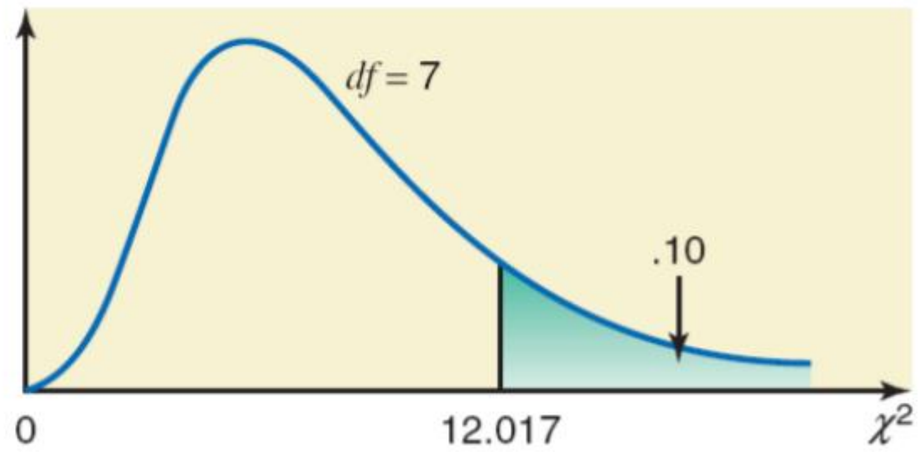
Chi square distribution, $\chi 2$

Let $Z_1, Z_2, Z_3, \ldots, Z_n$, be n i.i.d. random variables $Z_i \sim N(0,1)$

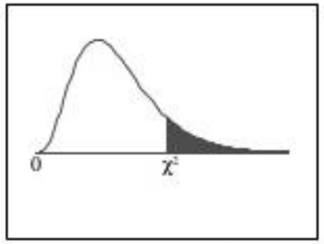
$$X = \sum_{i=1}^{n} Z^2{}_i \sim \chi_n^2$$

where **n**=degrees of freedom (only parameter)

$E(X) = n$ and $V(X) = 2n$



Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

F distribution, $F_{n,m}$

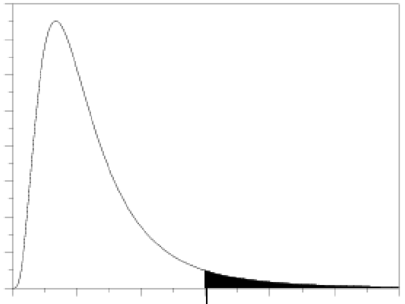
Let X~χ_m^2 and Y~χ_n^2,

$$F = \frac{X/m}{Y/n} \sim F_{m,n}$$

where m = numerator's degree of freedom, n = denominator's degree of freedom (2 parameters)

The F distribution is the ratio of 2 Chi square distributions.

The total area under F distribution curve is 1



Critical F

F - Distribution ($\alpha = 0.05$ in the Right Tail)

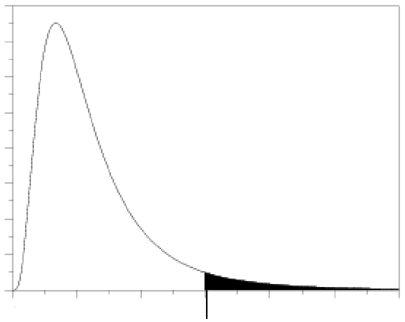
df_2 \ df_1 (Numerator Degrees of Freedom)	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8867	8.8452	8.8123
4	7.7086	9.9443	6.5914	6.3882	6.2561	6.1631	6.0942	6.0410	6.9988
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8759	4.8183	4.7725
6	5.9874	5.1433	4.7571	4.5337	4.3874	4.2839	4.2067	4.1468	4.0990
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7870	3.7257	3.6767
8	5.3177	4.4590	4.0662	3.8379	3.6875	3.5806	3.5005	3.4381	3.3881
9	5.1174	4.2565	3.8625	3.6331	3.4817	3.3738	3.2927	3.2296	3.1789
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.1355	3.0717	3.0204
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	3.0123	2.9480	2.8962
12	4.7472	3.8853	3.4903	3.2592	3.1059	2.9961	2.9134	2.8486	2.7964
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.8321	2.7669	2.7144
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.7642	2.6987	2.6458
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.7066	2.6408	2.5876
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.6572	2.5911	2.5377
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.6143	2.5480	2.4943
18	4.4139	3.5546	3.1599	2.9277	2.7729	2.6613	2.5767	2.5102	2.4563
19	4.3807	3.5219	3.1274	2.8951	2.7401	2.6283	2.5435	2.4768	2.4227
20	4.3512	3.4928	3.0984	2.8661	2.7109	2.5990	2.5140	2.4471	2.3928
21	4.3248	3.4668	3.0725	2.8401	2.6848	2.5727	2.4876	2.4205	2.3660
22	4.3009	3.4434	3.0491	2.8167	2.6613	2.5491	2.4638	2.3965	2.3419
23	4.2793	3.4221	3.0280	2.7955	2.6400	2.5277	2.4422	2.3748	2.3201
24	4.2597	3.4028	3.0088	2.7763	2.6207	2.5082	2.4226	2.3551	2.3002
25	4.2417	3.3852	2.9912	2.7587	2.6030	2.4904	2.4047	2.3371	2.2821
26	4.2252	3.3690	2.9752	2.7426	2.5868	2.4741	2.3883	2.3205	2.2655
27	4.2100	3.3541	2.9604	2.7278	2.5719	2.4591	2.3732	2.3053	2.2501
28	4.1960	3.3404	2.9467	2.7141	2.5581	2.4453	2.3593	2.2913	2.2360
29	4.1830	3.3277	2.9340	2.7014	2.5454	2.4324	2.3463	2.2783	2.2229
30	4.1709	3.3158	2.9223	2.6896	2.5336	2.4205	2.3343	2.2662	2.2107
40	4.0847	3.2317	2.8387	2.6060	2.4495	2.3359	2.2490	2.1802	2.1240
60	4.0012	3.1504	2.7581	2.5252	2.3683	2.2541	2.1665	2.0970	2.0401
120	3.9201	3.0718	2.6802	2.4472	2.2899	2.1750	2.0868	2.0164	1.9588
∞	3.8415	2.9957	2.6049	2.3719	2.2141	2.0986	2.0096	1.9384	1.8799

Denominator Degrees of Freedom

F - Distribution ($\alpha = 0.05$ in the Right Tail)

df_2 \ df_1 (Numerator Degrees of Freedom)	10	12	15	20	24	30	40	60	120	∞
1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	19.396	19.413	19.429	19.446	19.454	19.462	19.471	19.479	19.487	19.496
3	8.7855	8.7446	8.7029	8.6602	8.6385	8.6166	8.5944	8.5720	8.5494	8.5264
4	5.9644	5.9117	5.8578	5.8025	5.7744	5.7459	5.7170	5.6877	5.6581	5.6281
5	4.7351	4.6777	4.6188	4.5581	4.5272	4.4957	4.4638	4.4314	4.3985	4.3650
6	4.0600	3.9999	3.9381	3.8742	3.8415	3.8082	3.7743	3.7398	3.7047	3.6689
7	3.6365	3.5747	3.5107	3.4445	3.4105	3.3758	3.3404	3.3043	3.2674	3.2298
8	3.3472	3.2839	3.2184	3.1503	3.1152	3.0794	3.0428	3.0053	2.9669	2.9276
9	3.1373	3.0729	3.0061	2.9365	2.9005	2.8637	2.8259	2.7872	2.7475	2.7067
10	2.9782	2.9130	2.8450	2.7740	2.7372	2.6996	2.6609	2.6211	2.5801	2.5379
11	2.8536	2.7876	2.7186	2.6464	2.6090	2.5705	2.5309	2.4901	2.4480	2.4045
12	2.7534	2.6866	2.6169	2.5436	2.5055	2.4663	2.4259	2.3842	2.3410	2.2962
13	2.6710	2.6037	2.5331	2.4589	2.4202	2.3803	2.3392	2.2966	2.2524	2.2064
14	2.6022	2.5342	2.4630	2.3879	2.3487	2.3082	2.2664	2.2229	2.1778	2.1307
15	2.5437	2.4753	2.4034	2.3275	2.2878	2.2468	2.2043	2.1601	2.1141	2.0658
16	2.4935	2.4247	2.3522	2.2756	2.2354	2.1938	2.1507	2.1058	2.0589	2.0096
17	2.4499	2.3807	2.3077	2.2304	2.1898	2.1477	2.1040	2.0584	2.0107	1.9604
18	2.4117	2.3421	2.2686	2.1906	2.1497	2.1071	2.0629	2.0166	1.9681	1.9168
19	2.3779	2.3080	2.2341	2.1555	2.1141	2.0712	2.0264	1.9795	1.9302	1.8780
20	2.3479	2.2776	2.2033	2.1242	2.0825	2.0391	1.9938	1.9464	1.8963	1.8432
21	2.3210	2.2504	2.1757	2.0960	2.0540	2.0102	1.9645	1.9165	1.8657	1.8117
22	2.2967	2.2258	2.1508	2.0707	2.0283	1.9842	1.9380	1.8894	1.8380	1.7831
23	2.2747	2.2036	2.1282	2.0476	2.0050	1.9605	1.9139	1.8648	1.8128	1.7570
24	2.2547	2.1834	2.1077	2.0267	1.9838	1.9390	1.8920	1.8424	1.7896	1.7330
25	2.2365	2.1649	2.0889	2.0075	1.9643	1.9192	1.8718	1.8217	1.7684	1.7110
26	2.2197	2.1479	2.0716	1.9898	1.9464	1.9010	1.8533	1.8027	1.7488	1.6906
27	2.2043	2.1323	2.0558	1.9736	1.9299	1.8842	1.8361	1.7851	1.7306	1.6717
28	2.1900	2.1179	2.0411	1.9586	1.9147	1.8687	1.8203	1.7689	1.7138	1.6541
29	2.1768	2.1045	2.0275	1.9446	1.9005	1.8543	1.8055	1.7537	1.6981	1.6376
30	2.1646	2.0921	2.0148	1.9317	1.8874	1.8409	1.7918	1.7396	1.6835	1.6223
40	2.0772	2.0035	1.9245	1.8389	1.7929	1.7444	1.6928	1.6373	1.5766	1.5089
60	1.9926	1.9174	1.8364	1.7480	1.7001	1.6491	1.5943	1.5343	1.4673	1.3893
120	1.9105	1.8337	1.7505	1.6587	1.6084	1.5543	1.4952	1.4290	1.3519	1.2539
∞	1.8307	1.7522	1.6664	1.5705	1.5173	1.4591	1.3940	1.3180	1.2214	1.0000

Denominator Degrees of Freedom



Critical F

F - Distribution (α = 0.01 in the Right Tail)

Numerator Degrees of Freedom (df_1); Denominator Degrees of Freedom (df_2)

df_2 \ df_1	1	2	3	4	5	6	7	8	9
1	4052.2	4999.5	5403.4	5624.6	5763.6	5859.0	5928.4	5981.1	6022.5
2	98.503	99.000	99.166	99.249	99.299	99.333	99.356	99.374	99.388
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158
6	13.745	10.925	9.7795	9.1483	8.7459	8.4661	8.2600	8.1017	7.9761
7	12.246	9.5466	8.4513	7.8466	7.4604	7.1914	6.9928	6.8400	6.7188
8	11.259	8.6491	7.5910	7.0061	6.6318	6.3707	6.1776	6.0289	5.9106
9	10.561	8.0215	6.9919	6.4221	6.0569	5.8018	5.6129	5.4671	5.3511
10	10.044	7.5594	6.5523	5.9943	5.6363	5.3858	5.2001	5.0567	4.9424
11	9.6460	7.2057	6.2167	5.6683	5.3160	5.0692	4.8861	4.7445	4.6315
12	9.3302	6.9266	5.9525	5.4120	5.0643	4.8206	4.6395	4.4994	4.3875
13	9.0738	6.7010	5.7394	5.2053	4.8616	4.6204	4.4410	4.3021	4.1911
14	8.8616	6.5149	5.5639	5.0354	4.6950	4.4558	4.2779	4.1399	4.0297
15	8.6831	6.3589	5.4170	4.8932	4.5556	4.3183	4.1415	4.0045	3.8948
16	8.5310	6.2262	5.2922	4.7726	4.4374	4.2016	4.0259	3.8896	3.7804
17	8.3997	6.1121	5.1850	4.6690	4.3359	4.1015	3.9267	3.7910	3.6822
18	8.2854	6.0129	5.0919	4.5790	4.2479	4.0146	3.8406	3.7054	3.5971
19	8.1849	5.9259	5.0103	4.5003	4.1708	3.9386	3.7653	3.6305	3.5225
20	8.0960	5.8489	4.9382	4.4307	4.1027	3.8714	3.6987	3.5644	3.4567
21	8.0166	5.7804	4.8740	4.3688	4.0421	3.8117	3.6396	3.5056	3.3981
22	7.9454	5.7190	4.8166	4.3134	3.9880	3.7583	3.5867	3.4530	3.3458
23	7.8811	5.6637	4.7649	4.2636	3.9392	3.7102	3.5390	3.4057	3.2986
24	7.8229	5.6136	4.7181	4.2184	3.8951	3.6667	3.4959	3.3629	3.2560
25	7.7698	5.5680	4.6755	4.1774	3.8550	3.6272	3.4568	3.3239	3.2172
26	7.7213	5.5263	4.6366	4.1400	3.8183	3.5911	3.4210	3.2884	3.1818
27	7.6767	5.4881	4.6009	4.1056	3.7848	3.5580	3.3882	3.2558	3.1494
28	7.6356	5.4529	4.5681	4.0740	3.7539	3.5276	3.3581	3.2259	3.1195
29	7.5977	5.4204	4.5378	4.0449	3.7254	3.4995	3.3303	3.1982	3.0920
30	7.5625	5.3903	4.5097	4.0179	3.6990	3.4735	3.3045	3.1726	3.0665
40	7.3141	5.1785	4.3126	3.8283	3.5138	3.2910	3.1238	2.9930	2.8876
60	7.0771	4.9774	4.1259	3.6490	3.3389	3.1187	2.9530	2.8233	2.7185
120	6.8509	4.7865	3.9491	3.4795	3.1735	2.9559	2.7918	2.6629	2.5586
∞	6.6349	4.6052	3.7816	3.3192	3.0173	2.8020	2.6393	2.5113	2.4073

F - Distribution (α = 0.01 in the Right Tail)

Numerator Degrees of Freedom (df_1); Denominator Degrees of Freedom (df_2)

df_2 \ df_1	10	12	15	20	24	30	40	60	120	∞
1	6055.8	6106.3	6157.3	6208.7	6234.6	6260.6	6286.8	6313.0	6339.4	6365.9
2	99.399	99.416	99.433	99.449	99.458	99.466	99.474	99.482	99.491	99.499
3	27.229	27.052	26.872	26.690	26.598	26.505	26.411	26.316	26.221	26.125
4	14.546	14.374	14.198	14.020	13.929	13.838	13.745	13.652	13.558	13.463
5	10.051	9.8883	9.7222	9.5526	9.4665	9.3793	9.2912	9.2020	9.1118	9.0204
6	7.8741	7.7183	7.5590	7.3958	7.3127	7.2285	7.1432	7.0567	6.9690	6.8800
7	6.6201	6.4691	6.3143	6.1554	6.0743	5.9920	5.9084	5.8236	5.7373	5.6495
8	5.8143	5.6667	5.5151	5.3591	5.2793	5.1981	5.1156	5.0316	4.9461	4.8588
9	5.2565	5.1114	4.9621	4.8080	4.7290	4.6486	4.5666	4.4831	4.3978	4.3105
10	4.8491	4.7059	4.5581	4.4054	4.3269	4.2469	4.1653	4.0819	3.9965	3.9090
11	4.5393	4.3974	4.2509	4.0990	4.0209	3.9411	3.8596	3.7761	3.6904	3.6024
12	4.2961	4.1553	4.0096	3.8584	3.7805	3.7008	3.6192	3.5355	3.4494	3.3608
13	4.1003	4.9603	3.8154	3.6646	3.5868	3.5070	3.4253	3.3413	3.2548	3.1654
14	3.9394	3.8001	3.6557	3.5052	3.4274	3.3476	3.2656	3.1813	3.0942	3.0040
15	3.8049	3.6662	3.5222	3.3719	3.2940	3.2141	3.1319	3.0471	2.9595	2.8684
16	3.6909	3.5527	3.4089	3.2587	3.1808	3.1007	3.0182	2.9330	2.8447	2.7528
17	3.5931	3.4552	3.3117	3.1615	3.0835	3.0032	2.9205	2.8348	2.7459	2.6530
18	3.5082	3.3706	3.2273	3.0771	2.9990	2.9185	2.8354	2.7493	2.6597	2.5660
19	3.4338	3.2965	3.1533	3.0031	2.9249	2.8442	2.7608	2.6742	2.5839	2.4893
20	3.3682	3.2311	3.0880	2.9377	2.8594	2.7785	2.6947	2.6077	2.5168	2.4212
21	3.3098	3.1730	3.0300	2.8796	2.8010	2.7200	2.6359	2.5484	2.4568	2.3603
22	3.2576	3.1209	2.9779	2.8274	2.7488	2.6675	2.5831	2.4951	2.4029	2.3055
23	3.2106	3.0740	2.9311	2.7805	2.7017	2.6202	2.5355	2.4471	2.3542	2.2558
24	3.1681	3.0316	2.8887	2.7380	2.6591	2.5773	2.4923	2.4035	2.3100	2.2107
25	3.1294	2.9931	2.8502	2.6993	2.6203	2.5383	2.4530	2.3637	2.2696	2.1694
26	3.0941	2.9578	2.8150	2.6640	2.5848	2.5026	2.4170	2.3273	2.2325	2.1315
27	3.0618	2.9256	2.7827	2.6316	2.5522	2.4699	2.3840	2.2938	2.1985	2.0965
28	3.0320	2.8959	2.7530	2.6017	2.5223	2.4397	2.3535	2.2629	2.1670	2.0642
29	3.0045	2.8685	2.7256	2.5742	2.4946	2.4118	2.3253	2.2344	2.1379	2.0342
30	2.9791	2.8431	2.7002	2.5487	2.4689	2.3860	2.2992	2.2079	2.1108	2.0062
40	2.8005	2.6648	2.5216	2.3689	2.2880	2.2034	2.1142	2.0194	1.9172	1.8047
60	2.6318	2.4961	2.3523	2.1978	2.1154	2.0285	1.9360	1.8363	1.7263	1.6006
120	2.4721	2.3363	2.1915	2.0346	1.9500	1.8600	1.7628	1.6557	1.5330	1.3805
∞	2.3209	2.1847	2.0385	1.8783	1.7908	1.6964	1.5923	1.4730	1.3246	1.0000