TOR VERGATA

UNIVERSITÀ DEGLI STUDI DI ROMA

Quantitative Methods III

Matrices

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Matrix

- A matrix is an array of numbers that are arranged in rows and columns.
- A matrix is "square" if it has the same number of rows as columns.
- We will consider only 2x2 and 3x3 square matrices

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{bmatrix}$$

Matrix Multiplication

Matrix Multiplication Let A be an $m \times n$ matrix and B an $n \times p$ matrix; then the product AB is an $m \times p$ matrix. The ij term of AB is the dot product of the ith row vector of A with the jth column vector of B, so that

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \end{bmatrix}$ «Matrix 2x3»

$B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ «Matrix 3x3»

Matrix Multiplication

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \end{bmatrix}$$ «Matrix 2x3»

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Matrix multiplication: the rows of the first matrix are multiplied by the columns of the second one.

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 & 5 \\ -1 & 4 & -2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 3+3\cdot(-1)+0\cdot 1 & 1\cdot(-2)+3\cdot 4+0\cdot 0 & 1\cdot 5+3\cdot(-2)+0\cdot 3 \\ 2\cdot 3+1\cdot(-1)+(-3)\cdot 1 & 2\cdot(-2)+1\cdot 4+(-3)\cdot 0 & 2\cdot 5+1\cdot(-2)+(-3)\cdot 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 10 & -1 \\ 2 & 0 & -1 \end{pmatrix}$$

***AB* is a «Matrix 2x3»**

- **The rows of A (2x3)**
- **The columns of B (3x3)**

https://matrixcalc.org/

Transpose of a Matrix

The *transpose* of a matrix is obtained by interchanging the rows and columns of a matrix.

For example, the transpose of the matrix

$$A = \begin{bmatrix} \boxed{1 \quad 2 \quad -3} \\ \boxed{0 \quad 1 \quad 4} \\ \boxed{-1 \quad 2 \quad 1} \end{bmatrix} \qquad \text{is} \qquad A' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

Determinants

- Every square matrix has a determinant.
- The determinant of a matrix is a number.
- We will consider the determinants only of 2x2 and 3x3 matrices.
- It is used to help us calculate the inverse of a matrix and it is used when finding the area of a triangle

$$\begin{vmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{vmatrix}$$

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

Finding Determinants of Matrices 2x2

$$\begin{vmatrix} 3 & 2 \\ -5 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 2 \\ -5 & 4 \end{vmatrix} = (3 * 4) - (-5 * 2)$$

$$= 12 - (-10)$$

$$= 22$$

Finding Determinants of Matrices 3x3

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & -2 & 5 \\ -1 & 4 & 2 \end{vmatrix} \begin{matrix} 2 & 0 \\ 1 & -2 \\ -1 & 4 \end{matrix}$$

$= [(2)(-2)(2) + (0)(5)(-1) + (3)(1)(4)]$

$- [(3)(-2)(-1) + (2)(5)(4) + (0)(1)(2)]$

$= [-8 + 0 + 12] - [6 + 40 + 0]$

$= 4 - 6 - 40 = -42$

Using matrix equations

Identity matrix (I): Square matrix with 1's on the diagonal and zeros everywhere else

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 x 2 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 x 3 identity matrix

The identity matrix is to matrix multiplication as **1** is to regular multiplication!!!!

Multiply:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$$

So, the identity matrix (I) multiplied by any matrix lets the "any" matrix keep its identity!

Mathematically, I · A = A and A · I = A !!

Using matrix equations

Inverse Matrix: 2 x 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In words:

- Take the original matrix.
- Switch *a* and *d*.
- Change the signs of *b* and *c*.
- Multiply the new matrix by 1 over the determinant of the original matrix.

Using matrix equations

Example: Find the inverse of A.

$$A = \begin{bmatrix} 2 & 4 \\ -4 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(2)(-10)-(-4)(4)}\begin{bmatrix} -10 & -4 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4}\begin{bmatrix} -10 & -4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

Find the inverse matrix

$$\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

Matrix A

Inverse = $\frac{1}{\det}\begin{bmatrix} \text{Matrix} \\ \text{Reloaded} \end{bmatrix}$

Det A = 8(2) – (-5)(-3) = 16 – 15 = 1

$$= \frac{1}{1}\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

What happens when you multiply a matrix by its inverse?

1st: What happens when you multiply a **number** by its inverse? $7 \bullet \frac{1}{7}$

A & B are inverses. Multiply them.

$$\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $A \cdot A^{-1} = I$

Using matrix equations

You can take a system of equations and write it with matrices!!!

$3x + 2y = 11$

$2x + y = 8$

Becomes

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

Coefficient matrix **Variable matrix** **Answer matrix**

https://matrixcalc.org/