TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

A Nobel Prize Algorithm: The Stable Marriage

Algorithms, Data and Security
A.Y. 2024/25

Valeria Cardellini

Global Governance, 3rd year
Science and Technology Major

Matching problems

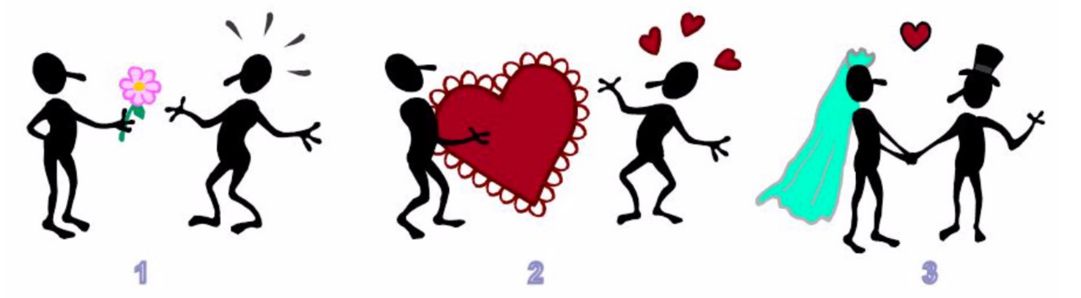
- Matching professors and courses
- Matching rooms and courses
- Matching students and internships
- Matching users and servers

- And other examples when we aim to find a matching that is optimal from individual perspective

A Nobel prize algorithm

- Lloyd Shapley (1923-2016): an American mathematician and Nobel Prize-winning economist
  – Contributed to mathematical economics and especially game theory
  – Won 2012 Nobel Memorial Prize in Economic Sciences with Alvin Roth "for the theory of stable allocations and the practice of market design"
  – One contribution is the **Gale-Shapley algorithm**

The stable marriage problem



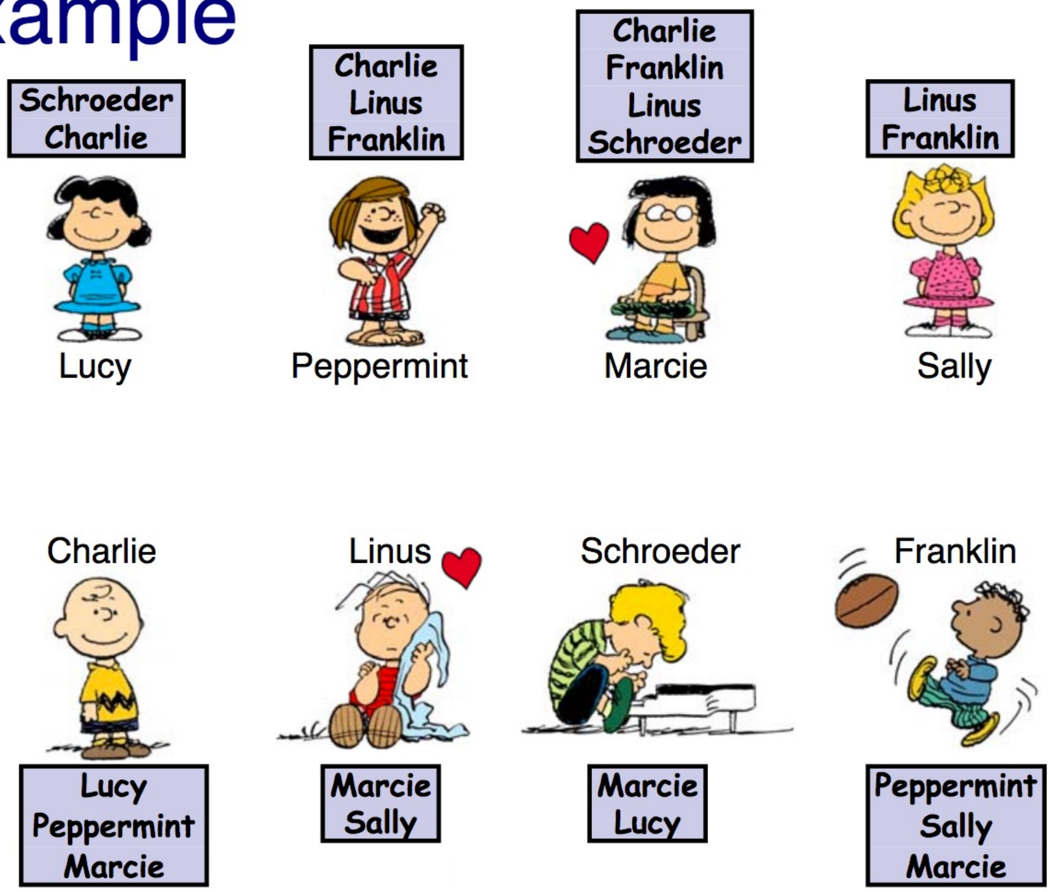
- A well-known problem of matching the elements of two sets of equal size
- There are *n* men and *n* women, which are unmarried. Each has ranked all members of the opposite sex in order of preference
- Also known as stable matching problem

The stable marriage problem

- Does there exist and can we find a stable matching (stable opposite-sex marriage)?
  - That is, a matching of men and women, such that there is no pair of a man and a woman who both prefer each other to their current partner in the matching

Note: we will not go over original description, which was not gender neutral

Example



Blocking pairs and stable marriage

- Let's see some definition
- Given a matching *M* of men and women, a pair (*m*, *w*), where *m* is a man and *w* is a woman, is a **blocking pair** iff
  - *m* and *w* are not partners in M
  - but *m* prefers *w* over his partner in *M* and *w* prefers *m* over her partner in *M*
- Matching *M* is **stable** iff it has no blocking pairs

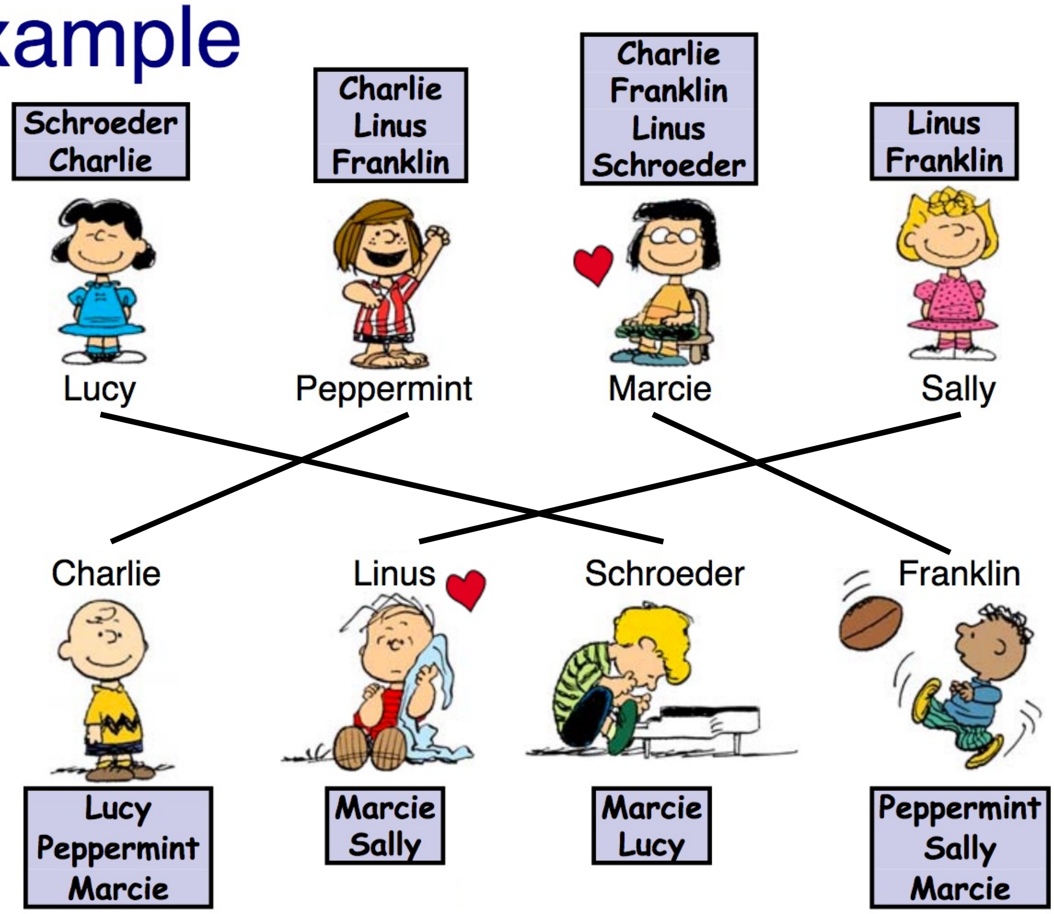
Example

- Alex: Betty Ann Cindy
- Bert: Ann Cindy Betty
- Carl: Ann Cindy Betty

- Ann: Bert Alex Carl
- Betty: Alex Carl Bert
- Cindy: Bert Alex Carl

- Matching (Alex, Ann), (Bert, Betty), (Carl, Cindy) is not stable because Alex and Betty prefer each other over their current partner
  - Alex and Betty are a blocking pair (i.e., a "rogue couple")
- Stable matching: (Alex, Betty), (Bert, Ann), (Carl, Cindy)
  - Because a matching is stable if it has no blocking pair

Example



Stable marriage!

Original application

- Origin: assignment of med-school students to hospitals (for internships) for National Resident Matching Program in USA
  – Students list hospitals in order of preference
  – Hospitals list students in order of preference

- Gale-Shapley paper (1962): college admission and opposite-sex marriage
  – Students list colleges in order of preference
  – Colleges list students in order of preference

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

1. Introduction. The problem with which we shall be concerned relates to the following typical situation: A college is considering a set of n applicants of which it can admit a quota of only q. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the q best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept.

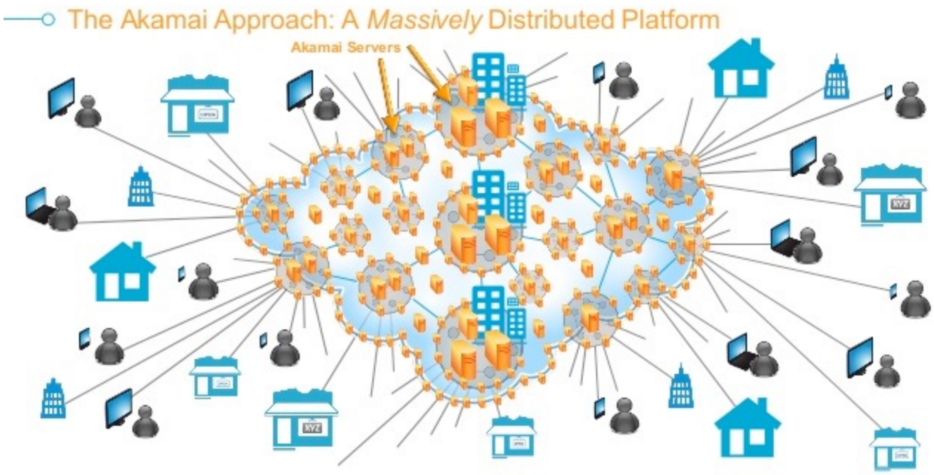
Practical applications

- A wide variety of practical applications, e.g.,:
  - Matching resident doctors to hospitals
  - Matching students to colleges
  - Matching sailors to ships
  - Matching kidneys and other organs with patients who require transplant
  - Matching users to servers in a large distributed Internet service (Akamai)

- More generally to any two-sided market
  - A market with 2 distinct groups of participants each with their own preferences

Break: what is Akamai?

- The largest content delivery network
  https://www.akamai.com/
- Distributes much of world's content on web: delivers 15-30% of Internet traffic every day
  - More than 350000 servers distributed in 134 countries



Akamai and stable marriage

- Users: preferences based on latency and packet loss
- Web servers: preferences based on costs of bandwidth and co-location
- Goal: assign billions of users to web servers, every 10 seconds

Algorithmic Nuggets in Content Delivery

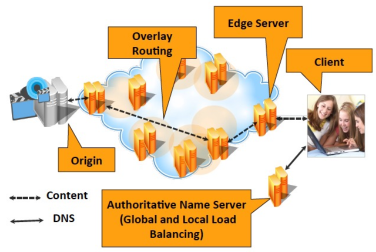
Bruce M. Maggs
Duke and Akamai
bmm@cs.duke.edu

Ramesh K. Sitaraman
UMass, Amherst and Akamai
ramesh@cs.umass.edu

This article is an editorial note submitted to CCR. It has NOT been peer reviewed.
The authors take full responsibility for this article's technical content. Comments can be posted through CCR Online.

ABSTRACT

This paper "peeks under the covers" at the subsystems that provide the basic functionality of a leading content delivery network. Based on our experiences in building one of the largest distributed systems in the world, we illustrate how sophisticated algorithmic research has been adapted to balance the load between and within server clusters, manage the caches on servers, select paths through an overlay routing network, and elect leaders in various contexts. In each instance, we first explain the theory underlying the algorithms, then introduce practical considerations not captured by the theoretical models, and finally describe what is implemented in practice. Through these examples, we highlight the role of algorithmic research in the design of complex networked systems. The paper also illustrates the close synergy that exists between research and industry where research ideas cross over into products and product requirements drive future research.



Simple-minded approach 1

- "**Greedy**" approach:
  - Greedy for women: First woman chooses, then second woman chooses …
  - Greedy for men: First man chooses, then second man chooses …
- Claim: it does not work

Simple-minded approach 1: example

Preference list for men

- Alex: Ann Betty Cindy
- Bert: Ann Betty Cindy
- Carl: Betty Ann Cindy

Preference list for women

- Ann: Carl Bert Alex
- Betty: Carl Bert Alex
- Cindy: Bert Carl Alex

Alex: Ann Betty Cindy

Bert: ~~Ann~~ Betty Cindy

Carl: ~~Betty~~ ~~Ann~~ Cindy

Ann: Carl Bert Alex

Betty: Carl Bert Alex

Cindy: Bert Carl Alex

- Greedy matching for men (Alex, Ann), (Bert, Betty), (Carl, Cindy) is not stable
  - Ann and Carl prefer each other over assigned partner

Simple-minded approach 1: example

Preference list for men

- Alex: Ann Betty Cindy
- Bert: Ann Betty Cindy
- Carl: Betty Ann Cindy

Preference list for women

- Ann: Carl Bert Alex
- Betty: Carl Bert Alex
- Cindy: Bert Carl Alex

Alex: Ann Betty Cindy

Bert: Ann Betty Cindy

Carl: Betty Ann Cindy

Ann: Carl Bert Alex

Betty: ~~Carl~~ Bert Alex

Cindy: ~~Bert~~ ~~Carl~~ Alex

- Greedy matching for women (Ann, Carl), (Betty, Bert), (Cindy, Alex) is not stable
  - Betty and Carl prefer each other over assigned partner

What is a greedy algorithm?

- A greedy algorithm is a simple, intuitive algorithm that is used to solve optimization problems
- It makes the optimal choice at each stage as it attempts to find the overall optimal way to solve the entire problem

Simple-minded approach 2

- "**Local search**" approach
  - Call it Soap-Series-Algorithm

    While there is a blocking pair

    Do switch the blocking pair

- Claim: it can go on forever (so it gives no solution for some input)

Simple-minded approach 2: example

- Consider the following preference list

w_1	m_2	m_3	m_1
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

m_1	w_1	w_3	w_2
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

Simple-minded approach 2: example

- Consider the following preference list
- Start from this matching

w_1	m_2	m_3	m_1
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

m_1	w_1	w_3	w_2
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

- w_1 & m_3 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_1 and m_3

w_1	m_2	m_3	m_1	m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3	m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2	m_3	w_1	w_2	w_3

- w_2 & m_1 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_2 and m_1

w_1	m_2	m_3	m_1	m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3	m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2	m_3	w_1	w_2	w_3

- w_3 & m_1 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_3 and m_1

w_1	m_2	m_3	m_1	m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3	m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2	m_3	w_1	w_2	w_3

- w_1 & m_2 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_1 and m_2

w_1	m_2	m_3	m_1	m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3	m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2	m_3	w_1	w_2	w_3

- w_2 & m_2 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_2 and m_2

w_1	m_2	m_3	m_1		m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3		m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2		m_3	w_1	w_2	w_3

- w_3 & m_3 run away together: switch

Simple-minded approach 2: example

- Consider the following preference list
- Match w_3 and m_3

w_1	m_2	m_3	m_1		m_1	w_1	w_3	w_2
w_2	m_1	m_2	m_3		m_2	w_2	w_1	w_3
w_3	m_3	m_1	m_2		m_3	w_1	w_2	w_3

- This was exactly the starting point!

Simple-minded approach 2

- “Local search” approach does not need to terminate

  While there is a blocking pair

  Do switch the blocking pair

- Can go on forever
- We need something else

Local search algorithms

- Local search algorithms move from solution to solution in the space of candidate solutions (the *search space*) by applying local changes, until a solution deemed optimal is found or a time bound is elapsed

Gale-Shapley algorithm

- Finds always a stable matching
  - Input: list of men and women and their preference list
  - Output: stable matching

- Note again: we will not go over original description, which was not gender neutral
  - But still not completely politically correct ☺
    - We will see the women optimal solution (and later the men optimal solution)
  - The algorithm has bias towards one of the two genders

Gale-Shapley algorithm

- Fix some ordering on the women
- Repeat until everyone is matched
  - Let A be the first unmatched woman in the ordering
  - Find man B such that B is the most desirable man in A’s list such that B is unmatched, or B is currently matched to woman C and A is more preferable to B than C
  - Match A and B; possible this turns C to be unmatched

Gale-Shapley algorithm: example

Input:

Preference list for men
- Alex: Ann Betty Cindy
- Bert: Ann Betty Cindy
- Carl: Betty Ann Cindy

Preference list for women
- Ann: Carl Bert Alex
- Betty: Carl Bert Alex
- Cindy: Bert Carl Alex

Let's apply Gale-Shapley algorithm

Ordering: Ann, Betty, Cindy

Pick Ann, match (Ann, Carl)

Ordering: ~~Ann~~, Betty, Cindy

Pick Betty, match (Betty, Carl) unmatching Ann

Ordering: Ann, ~~Betty~~, Cindy

Pick Ann, match (Ann, Bert)

Ordering: ~~Ann~~, ~~Betty~~, Cindy

Pick Cindy, match (Cindy, Alex)

Ordering: ~~Ann~~, ~~Betty~~, ~~Cindy~~

Stable matching: (Ann, Bert), (Betty, Carl), (Cindy, Alex)

Gale-Shapley algorithm

- Does this algorithm terminate?
- How fast?
- Does this algorithm give a stable matching?

Termination and number of steps

- Once a man is matched, he never becomes unmatched (his partner can change)
  - Note that women can alternate between being free and being engaged
- When the partner of a man changes, this is to a more preferable partner for him: at most n-1 times
- Every iteration, either an unmatched man becomes matched, or a matched man changes partner: at most n^2 matchings
  - n iterations, in each iteration at most n-1 matchings
- Note that n^2 is the total size of the input
  - n women and n men and their preferences

Stability of final matching

- Proof *by contradiction*
- Suppose final matching is not stable
- Take:
  - Alex is matched to Ann
  - Bert is matched to Betty
  - Alex prefers Betty to Ann
  - Betty prefers Alex to Bert
- So: Betty is before Ann in Alex's preference list, but Alex is not matched to Betty. Two cases:

1. When Betty considers Alex, he has a partner (say Carola) preferable to Betty: Carola is also preferable to Ann, but in the algorithm men can get only more preferable partners, contradiction.
2. When Betty considers Alex, he is free, but Betty is later replaced by someone preferable to Betty. Again, Alex can never end up with Ann.

Women optimal stable matching

- Theorem
  - All possible executions of Gale-Shapley algorithm give the same stable matching
  - In this matching, the women have the best partner they can have in any stable matching
  - In this matching, the men have the worst partner they can have in any stable matching

Men optimal stable matching

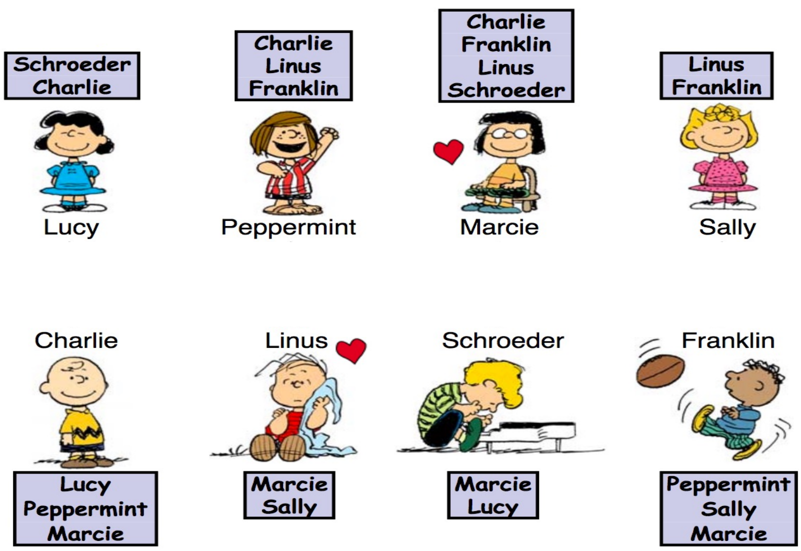
- For gender balance ☺
- In slide 29, let's interchange the roles of women and men so to achieve *men optimal* stable matching
  - In this matching, the men have the best partner they can have in any stable matching
  - In this matching, the women have the worst partner they can have in any stable matching

Take-away

- A stable matching always exists
- The simple-minded soap-opera algorithm does not necessarily terminate
  - "Start with any matching, and make switches when a pair prefers each other to their current partner"
- A stable matching can be found efficiently (in *linear time* in the input size) using Gale-Shapley algorithm
- Controversy: women-optimal Gale-Shapley algorithm is better for women (hospitals in the original application)

Exercise 1

- Find a stable marriage for the Peanuts



Exercise 2

- Set of women W = {w_1, w_2, w_3}
- Set of men M= {m_1, m_2, m_3}
- Preference lists:
  - m_1: w_1 w_2 w_3 (i.e., m_1 prefers w_1 to w_2 to w_3)
  - m_2: w_2 w_1 w_3
  - m_3: w_3 w_2 w_1
  - w_1: m_1 m_2 m_3
  - w_2: m_3 m_1 m_2
  - w_3: m_2 m_1 m_3
- Find women optimal stable marriage and men optimal stable marriage
  - Do they differ?
  - Does the order of unmatched women/men have any impact on the stable matching you have found?

References

- Shapley and Gale, College admissions and the stability of marriage, *The American Mathematical Monthly*, 69(1):9–15, 1962. https://www.eecs.harvard.edu/cs286r/courses/fall09/papers/galeshapley.pdf
- Wikipedia, Stable marriage problem https://en.wikipedia.org/wiki/Stable_marriage_problem
- The Royal Swedish Academy of Science, Stable matching: Theory, evidence, and practical design, 2012. https://www.nobelprize.org/uploads/2018/06/popular-economicsciences2012.pdf
- Meyer, Stable Matching Video, MIT 6.042J Mathematics for Computer Science, 2015. https://www.youtube.com/watch?v=RE5PmdGNgj0