CALCULUS – ACADEMIC YEAR 2024/2025

- In 1900, life expectancy of a newborn child was 46 years, and by 2000 it had grown to 77 years. Determine how much was the life expectancy of a newborn in 1950 according to the two following models:

  (i) assuming that in the last century the life expectancy increased linearly with time: $E(t) = a\,t + b$

  (ii) assuming that in the last century the life expectancy increased exponentially with time: $E(t) = a\,e^{bt}$

  where t denotes the number of years after 1900.

- The magnitude of an earthquake is measured according to the Richter scale as

$$R = \frac{2}{3}\log_{10}\left(\frac{E}{E_0}\right)$$

  where E is the energy released by the earthquake (measured in joules) and E_0 is a reference constant.

  The 2011 Fukushima earthquake measured $9,0$ in the Richter scale, while the earthquake recently occurred in Turkey and Siria was 7.8 in the Richter scale. Is the energy released by the recent earthquake more or less than 50% of the energy released in Fukushima?

  Suppose that an earthquake releases 30% of the energy released in Fukushima; how much would such an earthquake measure in the Richter scale?

- Find the domain of definition of the following functions:

$$f(x) = \sqrt[4]{3x - \frac{2}{x-1}}$$

$$f(x) = \frac{1}{2x - \sqrt{x}}$$

$$f(x) = \frac{x}{2 - \log(x-1)}$$

- A certain population increased in the last 5 years according to the logistic growth

$$P(t) = \frac{a}{1 + 3e^{-b\,t}} \qquad \text{thousands of people per year}$$

  Five years from now the population was 60% of the present value. How much more time will be needed in order to double the population of five years ago ?

- In a particular year, a company determines that the profit P obtained from the sales in week t $(1 \le t \le 52)$ was given by

$$P(t) = 75 + 55\cos\left(\frac{\pi(t-15)}{26}\right)$$

  (i) What was the largest weekly profit and when did it occur ?

  (ii) What was the smallest weekly profit and when did it occur ?

  (iii) Were the sales more favorable in the first half or in the second half of the year ?

- Sketch a graph (approximate) of the following functions:

$$f(x) = 3\cos(\frac{x}{2}) \quad ; \quad f(x) = 2^{-x} \quad ; \quad f(x) = \frac{1}{x-2} \quad ; \quad f(x) = \log_{10}(x^3)$$

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  where t denotes the number of years after 1900.

i) $E(t) = at + b$ $\qquad$ t = NUMBER OF YEARS AFTER 1900

$t=0 \longrightarrow 1900$

$t=100 \longrightarrow 2000$

$$\boxed{\begin{aligned} E(0) &= 46 \\ E(100) &= 77 \end{aligned}}$$

$E(0) \to$ SUBSTITUTE $t=0$ IN THE FORMULA

$$\begin{cases} \cancel{a\cdot 0} + b = 46 \\ a\cdot 100 + b = 77 \end{cases}$$ SYSTEM IN THE VARIABLES (a,b)

$$\begin{cases} b = 46 \\ 100a + 46 = 77 \end{cases} \longrightarrow \begin{cases} b = 46 \\ 100a = 77 - 46 \end{cases}$$

$$\begin{cases} b = 46 \\ \dfrac{\cancel{100}\,a}{\cancel{100}} = \dfrac{31}{100} \end{cases} \longrightarrow \begin{cases} b = 46 \\ a = \dfrac{31}{100} \end{cases}$$

$$\boxed{E(t) = \frac{31}{100}t + 46}$$

$1950 \to 50$ Years after 1900

$\to t = 50$

$$E(50) = \frac{31}{\cancel{100}_{2}}\,\cancel{50}^{1} + 46 = \boxed{61,5}$$

$E(t) = a e^{bt}$

$$\boxed{\begin{aligned} E(0) &= 46 \\ E(100) &= 77 \end{aligned}}$$

$$\begin{cases} a e^{b\cdot 0} = 46 \\ a e^{b\cdot 100} = 77 \end{cases}$$

$$\begin{cases} a \cancel{e^0} = 46 \\ a e^{100b} = 77 \end{cases} \longrightarrow \begin{cases} a = 46 \\ \frac{46 e^{100b}}{46} = \frac{77}{46} \end{cases}$$

$$e^{100b} = \frac{77}{46}$$

$$\cancel{\ln} \cancel{e}^{100b} = \ln \frac{77}{46}$$

$$(\cancel{\sqrt{x}})^{\cancel{2}}$$

$$\frac{\cancel{100}b}{\cancel{100}} = \left(\ln \frac{77}{46}\right) \cdot \frac{1}{100} \qquad b = \frac{1}{100} \ln \frac{77}{46}$$

CAREFUL!
YOU CAN APPLY $\ln$ IF BOTH LHS AND RHS ARE STRICTLY POSITIVE

$$E(t) = 46\, e^{\frac{1}{100}\left(\ln \frac{77}{46}\right) t}$$

$$\boxed{e^{\ln x} = x}$$

$$x^{a\cdot b} = (x^a)^b$$

$$E(t) = 46 \left(\cancel{e}^{\cancel{\ln} \frac{77}{46}}\right)^{\frac{t}{100}}$$

$$E(t) = 46 \left(\frac{77}{46}\right)^{t/100}$$

$E(0) = 46$

$$46 \cdot \left(\frac{\cancel{77}}{\cancel{46}}\right)^{\cancel{0}} = 46$$

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$E(100) = 77$

$$46 \cdot \left(\frac{77}{46}\right)^{\frac{\cancel{100}\,1}{\cancel{100}}} = \cancel{46} \cdot \frac{77}{\cancel{46}} = 77$$

$$E(50) = 46 \cdot \left(\frac{77}{46}\right)^{\frac{\cancel{50}\,1}{\cancel{100}\,2}} = 46 \cdot \left(\frac{77}{46}\right)^{\frac{1}{2}}$$

REMARK: $\boxed{x^{\frac{m}{n}} = \sqrt[n]{x^m}}$

$$E(50) = 46 \cdot \sqrt{\frac{77}{46}} = (\sqrt{46})^{\cancel{2}} \frac{\sqrt{77}}{\cancel{\sqrt{46}}} = \sqrt{46 \cdot 77}$$

- A certain population increased in the last 5 years according to the logistic growth

$$P(t) = \frac{a}{1+3e^{-bt}} \qquad \text{thousands of people per year}$$

Five years ~~from now~~ AGO the population was 60% of the present value. How much more time will be needed in order to double the population of five years ago ?

$t = 12.345$

SUBTRACT 5

$t = 7.345$ FROM NOW

$\boxed{t = 8}$

$t=0 \longrightarrow$ FIVE YEARS FROM NOW

$t=5 \longrightarrow$ NOW

$P(0)$ POPULATION 5 YEARS AGO

$P(5)$ POPULATION NOW

$$\boxed{P(0) = \frac{\cancel{60}^{\,3}}{\cancel{100}_{\,5}} P(5)}$$

$$\boxed{\underline{P6}: \text{ Find } t \text{ s.t. } P(t) = 2P(0)}$$

$$\underbrace{\frac{a}{1+3e^{-b\cdot 0}}}_{P(0)} = \underbrace{\frac{3}{5}}_{60\%} \cdot \underbrace{\frac{a}{1+3e^{-b\cdot 5}}}_{P(5)}$$

$$\underbrace{\frac{a}{1+3e^{-bt}}}_{P(t)} = 2 \cdot \underbrace{\frac{a}{1+3e^{-b\cdot 0}}}_{P(0)}$$

$$\longrightarrow \begin{cases} \frac{1}{\cancel{a}} \cdot \frac{\cancel{a}}{1+3} = \frac{3\cancel{a}}{5(1+3e^{-5b})} \cdot \frac{1}{\cancel{a}} \\ \frac{1}{\cancel{a}} \frac{\cancel{a}}{1+3e^{-bt}} = \frac{2\cancel{a}}{1+3} \cdot \frac{1}{\cancel{a}} \end{cases}$$

SYSTEM OF 2 EQUATIONS IN 3 UNKNOWNS (a, b AND t)

$$\begin{cases} \dfrac{1}{4} = \dfrac{3}{5(1+3e^{-5b})} \\[2ex] \dfrac{1}{1+3e^{-6t}} = \dfrac{\cancel{2}}{\cancel{4}}\ \dfrac{1}{2} \end{cases}$$

$$\frac{\cancel{b}}{\cancel{a}} \cdot \frac{\cancel{a}}{\cancel{b}} = \frac{c}{d} \cdot \frac{b}{a} \longrightarrow \frac{b}{a} = \frac{d}{c}$$

$$\frac{d}{c} \cdot 1 = \frac{\cancel{c}}{\cancel{d}} \cdot \frac{b}{a} \cdot \frac{\cancel{d}}{\cancel{c}} \qquad \frac{d}{c} = \frac{b}{a}$$

$$\begin{cases} 4 = \dfrac{5+15e^{-5b}}{3} \\[2ex] 1+3e^{-6t} = 2 \end{cases}$$

$\longrightarrow$ WE START FROM HERE, COMPUTING b

$$3 \cdot 4 = \frac{5+15e^{-5b}}{\cancel{3}}\ \cancel{3} \longrightarrow 12 = 5 + 15e^{-5b} \longrightarrow -15e^{-5b} = 5-12$$

$$\frac{\cancel{-15}\,e^{-5b}}{\cancel{15}} = \frac{+7}{15} \longrightarrow e^{-5b} = \frac{7}{15} \longrightarrow \cancel{\ln}\,\cancel{e}^{-5b} = \ln\frac{7}{15}$$

$$\longrightarrow \frac{\cancel{-5}\,b}{\cancel{-5}} = \ln\frac{7}{15} \cdot \left(-\frac{1}{5}\right) \longrightarrow b = -\frac{1}{5}\ln\frac{7}{15}$$

$$1+3e^{-\left(-\frac{1}{5}\ln\frac{7}{15}\right)t}=2$$

$$1+3e^{\frac{t}{5}\ln\frac{7}{15}}=2$$

$$\frac{\cancel{3}e^{t/5\ln 7/15}}{\cancel{3}}=\frac{2-1}{3}$$

$$e^{\frac{t}{5}\ln\frac{7}{15}}=\frac{1}{3}$$

$$\cancel{\ln}\,\cancel{e}^{\frac{t}{5}\ln\frac{7}{15}}=\ln\frac{1}{3}$$

$$\frac{\cancel{5}}{\cancel{\ln\frac{7}{15}}}\cdot\frac{t}{\cancel{5}}\cancel{\ln\frac{7}{15}}=\ln\frac{1}{3}\cdot\frac{5}{\ln\frac{7}{15}}$$

$$\boxed{t=\frac{5\ln 1/3}{\ln 7/15}} \rightsquigarrow t=12.345$$

$$t=\frac{5(\cancel{\ln 1}-\ln 3)}{\ln 7-\ln 15}=\frac{5\ln 3}{\ln 15-\ln 7}=5\log_{\frac{15}{7}}3$$

- Find the domain of definition of the following functions:

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DOMAIN :

1) DENOMINATORS $\neq 0$

2) ARGUMENT OF AN EVEN INDEX ROOT ≥ 0

3) ARGUMENT OF THE LOGARITHM > 0

$$f(x) = \sqrt[4]{3x - \frac{2}{x-1}}$$

$$\begin{cases} x - 1 \neq 0 \\ \frac{3x}{1} - \frac{2}{x-1} \geq 0 \end{cases} \longrightarrow \begin{cases} x \neq 1 \\ \frac{3x(x-1) - 2}{x-1} \geq 0 \end{cases}$$

$$\boxed{\frac{3x^2 - 3x - 2}{x-1} \geq 0}$$

IN THE INEQUALITIES, WE CAN MULTIPLY:

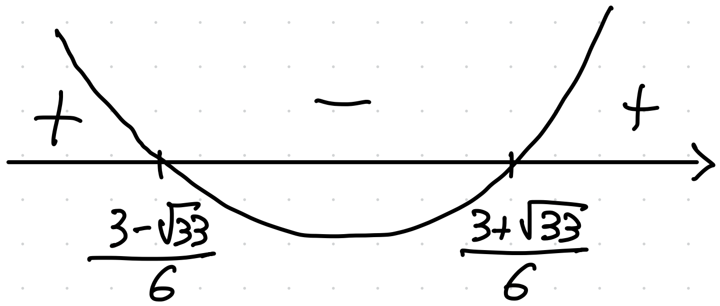
-) BY POSITIVE QUANTITIES

-) BY NEGATIVE QUANTITIES, CHANGING THE VERSE OF THE INEQUALITY

$N: 3x^2 - 3x - 2 \geq 0$

$$3x^2-3x-2=0$$

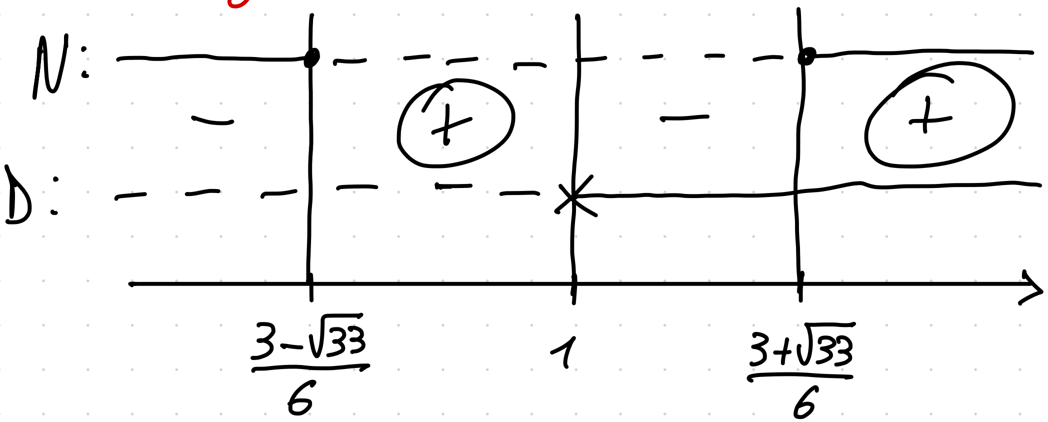
$$x_{1,2}=\frac{3\pm\sqrt{9-4\cdot 3\cdot(-2)}}{2\cdot 3}=\frac{3\pm\sqrt{33}}{6}$$



$$x\le\frac{3-\sqrt{33}}{6}\ \ \vee\ \ x\ge\frac{3+\sqrt{33}}{6}$$

$\rightarrow\ \frac{3+\sqrt{9}}{6}=1$

D: $x-1>0$ $\qquad x>1$



——— = POSITIVE VALUES

– – – – = NEGATIVE VALUES

• INCLUDED (= ALLOWED)

× NOT INCLUDED

$$\boxed{\frac{3-\sqrt{33}}{6}\le x<1\ \ \vee\ \ x\ge\frac{3+\sqrt{33}}{6}}$$